A DESIGN PROCEDURE FOR ROBUST H_2 CONTROL USING A MULTIPLIER APPROACH

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Abstract

This paper presents a method for the design and tuning of robust H_2 output feedback controllers for uncertain plants, based on Finsler's Lemma. The conservatism of design is considerably reduced by a proposed iterative scaling procedure, referred to as K-Θ iteration. The proposed method can deal with non-square uncertainty models, and it is shown that an approach proposed previously based on the S-procedure is a special case of this technique. Application to the ACC benchmark problem shows that the proposed technique outperforms previously published solutions.

1 Introduction

The problem addressed in this paper is robust design of H_2 optimal controllers for uncertain systems: given a family of admissible plant models, find the controller that minimizes the worst-case H_2 norm over all admissible models.

The robust H_2 problem has been studied in [9] and [13]. In [11] bounds were proposed on the worst-case H_2 norm of a system subject to norm-bounded, time-varying uncertainties. In [6] it was shown that the computation of bounds on the H_2 performance can be reduced to a convex optimization problem involving linear matrix inequalities (LMI), which can be solved via efficient convex optimization techniques.

A major difficulty in designing robust H_2 optimal controllers, is the fact that the bounds on the worst-case H_2 norm that can be used for controller synthesis tend to be rather conservative. In [4] an iterative procedure was proposed that can be used to reduce this conservatism. A drawback of this procedure is that it applies only to systems where parametric uncertainty is limited to the state matrix A of the plant state space model. Moreover, the matrix Δ used in an LFT representation of model uncertainty is restricted to be square. A more realistic situation is the design of a robust H_2 controller when both state matrix A and input matrix B are uncertain and non-square Δ are permitted.

The contribution of this paper is twofold: 1. we present an iterative procedure for computing the controller that minimizes the worst-case H_2 norm for a plant with uncertain A and B matrices. Instead of using the S-procedure as in [4] to eliminate the uncertainty from the plant model, a less conservative technique based on Finsler's lemma is used. The proof of the result presented in this paper was inspired by the use of Finsler's lemma to derive analysis results given in [10]. Using a standard change of variables technique [3], an approach similar to D-K iteration for robust H_2 controller design - referred to as $K-\Theta$ iteration - is proposed, where the objective is to minimize the H_2 norm alternatingly over the scaling matrix Θ and over the controller.

A second contribution is to show that a design technique proposed in [4] and extended in [5], which is based on the S-procedure, is shown to be a special case of this more general approach.

The proposed method is applied to a well-known benchmark problem with a plant that has parametric uncertainty in the A and B matrices of its state space model; it is shown to outperform previously published solutions.

The paper is organized as follows. In section 2 a brief review of robust H_2 control is given, and the proposed design procedure is introduced. In section 3 the design procedure is illustrated by application to the ACC benchmark problem. Conclusions are drawn in section 4. The proof of the main result is given in the Appendix.

2 Robust H_2 Control and $K-\Theta$ Iteration

In this paper we consider the design of linear time-invariant controllers for a plant with state space realization

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where elements of the matrices A and B are not known exactly but only guaranteed to take values within specified intervals. For the purpose of controller design, this uncertain plant model is embedded in a generalized plant P with state space realization

$$\begin{aligned} \dot{x} &= A_0x + B_1w_1 + B_2w_2 + B_0u \\ z_1 &= C_1x + D_1u \\ z_2 &= C_2x + D_{2u}u \\ y &= Cx + D_{2w}w_2 \end{aligned} \tag{2}$$

The plant is shown in Figure 1. The physical plant model (1) is represented by the matrices (A_0, B_0, C), where A_0 and B_0

stand for the nominal values of the uncertain matrices A and B. Perturbations of the nominal plant matrices ($A_0,\ B_0$) are expressed via fictitious inputs through B_1, fictitious outputs through C_1 and a fictitious feedthrough term D_1: Introducing feedback $w_1 = \Delta z_1$, where the matrix Δ represents perturbations and is assumed to satisfy $\|\Delta\| < 1$, leads to the representation

$$\dot{x} = (A_0 + B_1\Delta C_1)x + B_2 w_2 + (B_0 + B_1\Delta D_1)u, \quad y = Cx \tag{3}$$

of the physical plant with parametric uncertainty in A and B. The input w_2 is a white noise process with unit variance. If the matrices $C_2,\ D_{2u},\ B_2$ and D_{2w} are chosen as

$$C_2 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \qquad D_{2u} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \tag{4}$$

$$B_2 = [Q_e^{1/2}\ \ 0], \qquad D_{2w} = [0\ \ R_e^{1/2}] \tag{5}$$

then

$$J = E\|z_2(t)\|_2^2 = E\left[\lim_{t\to\infty} \frac{1}{T}\int_0^T z_2^T z_2\, dt\right] \tag{6}$$

represents a LQG cost function with the usual weight matrices $Q,\ R$ and noise covariances $Q_e,\ R_e$.

The problem considered in this paper is to find a strictly proper controller $K(s)$ with state space realisation

$$\begin{aligned} \dot{\zeta}(t) &= A_K\zeta(t) + B_K y(t) \\ u(t) &= C_K\zeta(t) \end{aligned} \tag{7}$$

such that the LQG cost is guaranteed to be less than a given value $J \le \nu^2$ in all admissible operating conditions, i.e. for all $\|\Delta\| < 1$.

This problem can be expressed in the form of linear matrix inequalities as follows. Consider the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \bar{A}\begin{bmatrix} x \\ \zeta \end{bmatrix} + [\bar{B}_1\ \ \bar{B}_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix}\begin{bmatrix} x \\ \zeta \end{bmatrix}$$

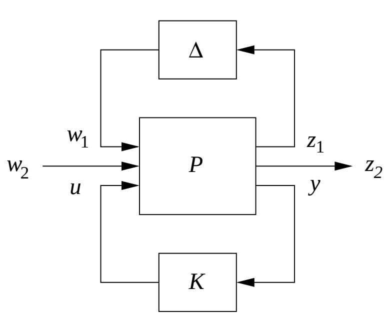


Figure 1: Generalized plant

where

$$\bar{A} = \begin{bmatrix} A_0 & B_0 C_K \\ B_K C & A_K \end{bmatrix},\ \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\ \bar{B}_2 = \begin{bmatrix} B_2 \\ B_K D_{2w} \end{bmatrix}$$

$$\bar{C}_1 = [C_1\ \ D_1 C_K], \quad \bar{C}_2 = [C_2\ \ D_{2u} C_K]$$

The design procedure proposed in this paper is based on the following result.

Theorem 2.1

In the control system in Figure 1, the performance index satisfies $J \le \nu^2$ for all $\|\Delta\| < 1$, if there exist a positive definite matrix P and a matrix W such that

$$\text{trace}\, W < \nu^2,$$

$$\begin{bmatrix} P\bar{A} + \bar{A}^T P & (*) & (*) & (*) \\ \bar{C}_2 & -I & 0 & 0 \\ \theta_{11}\bar{C}_1 & 0 & -\theta_{11} & 0 \\ \bar{B}_1^T P - \theta_{12}^T\bar{C}_1 & 0 & 0 & \theta_{22} \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} W & (*) \\ P\bar{B}_2 & P \end{bmatrix} > 0 \tag{9}$$

Here Θ is a symmetric scaling matrix that satisfies

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \Delta \end{bmatrix} \ge 0$$

for all $\|\Delta\| < 1$. The notation $M < 0$ ($M > 0$) means that the matrix M is negative (positive) definite. The matrices θ_{11} , θ_{12} , θ_{22} in (8) are partions of Θ and $\theta_{22} \le 0$. For a proof see Appendix A.

Conservatism of Design and K-Θ Iteration

The closed-loop matrices $\bar{A},\ \bar{B}_2$ and $\bar{C}_2$ in (8) depend on the controller matrices $A_K,\ \ B_K,\ \ C_K$. Due to the presence of the product terms $\bar{A}P$ and $\bar{C}_2 P$, (8) cannot be solved as an LMI problem for the controller, because it is nonlinear in the controller matrices and the matrix variable P. However, it is straightforward to use a linearizing change of variables, proposed in [3], to transform (8) into an LMI problem that can be solved for the controller with efficient LMI solvers - details are omitted here for lack of space.

On the other hand, the LMI conditions (8) and (9) are only sufficient conditions for the worst case bound on the performance. The resulting conservatism can be reduced by a suitable choice of the scaling matrix Θ. Unfortunately it is not possible to treat Θ as a matrix variable and solve an (8) as an LMI problem to find the scaling that yields the best worst-case performance. This is because the linearizing transformation introduces a term that is nonlinear in Θ and the controller variables. To overcome this problem, we propose the following iterative technique

K-step Assume $\theta_{11} = I$, $\theta_{12} = 0$, and solve

$$\min_{K(s),P,\theta_{22}} \ \text{trace}\, W$$

subject to the linearized form of (8)

Θ-step Using the controller obtained in the 'K-step', solve

$$\min_{P,\Theta} \text{trace}\, W \quad \text{subject to} \quad (8)$$

Go back to the 'K-step' and repeat with Θ obtained in the 'Θ-step' until no further drop in trace W is observed.

3 Robust Design Example: The ACC Benchmark Problem

This problem was proposed as a benchmark problem for robust control at the American Control Conference 1990 [16]. Two bodies with masses m_1 and m_2 are connected by a spring with stiffness k, as shown in Fig. 2. A state space model of the system is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

$$y = x_2 \tag{10}$$

We consider the following problem.

Design Problem: For a unit impulse disturbance exerted either on body 1 or 2, the controlled output x_2 must have a settling time of no more than 15 sec for the nominal system ($m_1 = m_2 = k = 1$; the settling time t_s is defined by $|x_2| < 0.1 \forall t \geq t_s$). The closed-loop system should be stable for $0.5 \leq k \leq 2.0$ and $0.5 \leq m_1, m_2 \leq 2.0$.

A systematic way of constructing the uncertainty representation in the state space model (3) was proposed in [15]. Here the following model is used to represent the above problem with $\|\Delta\| < 1$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.1179 & 2.132 & 0 & 0 \\ 2.1429 & -2.1067 & 0 & 0 \end{bmatrix} \qquad B_0 = \begin{bmatrix} 0 \\ 0 \\ 1.25 \\ 0 \end{bmatrix}$$

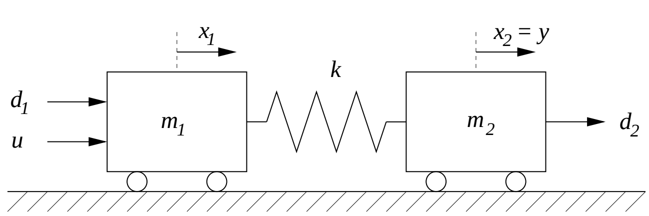


Figure 2: Two-mass-spring system

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.8831 & 0 \\ 0 & -1.8963 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -.991 & 1 & 0 & 0 \\ 1 & -.9772 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 \\ 0 \\ 0.3983 \end{bmatrix}$$

and the uncertain gain matrix Δ has the structure

$$\Delta = \begin{bmatrix} \delta_1 & 0 & \delta_3 \\ 0 & \delta_2 & 0 \end{bmatrix}$$

3.1 Robust H_2 Controller Design

Having determined matrices $(A_0, B_0, B_1, C_1, D_1)$, the uncertain plant can be represented in the form of (3). To facilitate the tuning of a robust controller for this plant, we replace the uncertainty-input matrix B_1 - which is a common left factor of the perturbations in A and B - by the matrix ρB_1, where $\rho > 0$ is a tuning parameter that can be used to scale the perturbation. The uncertainty representation (3) is therefore replaced by

$$A = A_0 + \rho B_1 \Delta C_1, \qquad B = B_0 + \rho B_1 \Delta D_1$$

With $\rho = 0$, this model represents the nominal plant, and larger values of ρ mean that a larger range of uncertain parameters is covered.

A quadratic performance index is also included in the model by choosing the matrices $C_2,\ D_{2u},\ B_2$ and D_{2w} in (2) according to (5), with

$$Q = qI, \quad Q_e = q_e I, \quad R = 1, \quad R_e = 1$$

This representation leaves the designer with three tuning parameters $q,\ q_e$ and ρ. Once values for these parameters have been chosen, the K-Θ iteration procedure presented in the previous section can be applied to compute the controller with the lowest worst-case H_2 cost for this problem.

Tuning Parameters

The influence of ρ on performance and robustness is shown in Figure 3. As expected, the price to be paid for improving robustness (increasing δ_a - the maximum allowed variation in all parameters k, m_1, m_2 for which the system is stable) is a loss of performance (i.e. larger values for t_s).

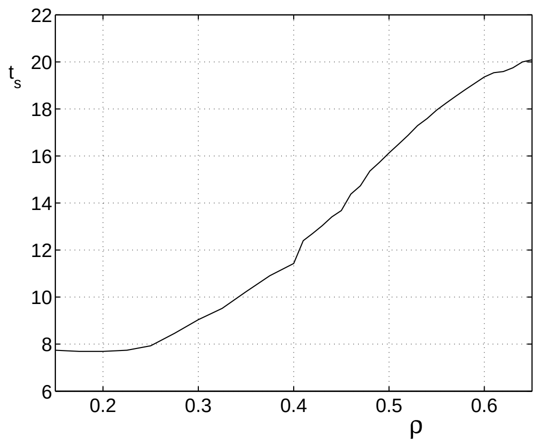
Following the design procedure outlined above, after a few iterations the following controller was obtained with tuning parameters $q = 0.02$, $q_e = 0.02$ and $\rho = 0.053$

$$K(s) = \frac{-0.074208(s + 0.1285)(s^2 + 4.114s + 10.6)}{(s^2 + 1.225s + 0.5846)(s^2 + 0.6182s + 2.337)} \tag{11}$$

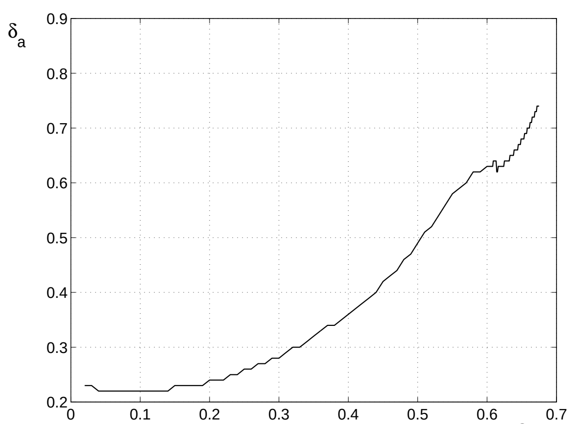
In [14], a scoring scheme was proposed to evaluate and compare the performance of different controllers for this benchmark problem. The performance measures achieved with the above controller are shown in Table 1, and compared with the

Design	Reference (equation)	PM (deg)	GM (dB)	t_s (sec)	u_{max}	$k_{min} - k_{max}$	p_m	Score
Requirement		30	6.0	15	1	0.5 - 2.0	0.30	
$K-\theta$ iteration	this paper (11)	31.12	6.62	14.71	0.51	0.37 - 9.9	0.531	8.6
Classical/H_2	[14] (19)	35	6.0	14.5	0.759	0.450 - 2.800	0.41	7.3
H_∞	[17] (40)	34	6.1	15.2	0.573	0.440 - 3.900	0.45	6.4
Pole placement	[7] next after (5)	24	3.7	28.9	0.549	0.230 - ∞	0.37	0.7
μ-synthesis	[1] (29-32)	27	2.8	14.1	0.953	0.580 - 2.500	0.37	-0.1
Minimax LQG	[8] (37)	19	3.4	18.1	0.691	0.690 - 1.400	0.18	-7.2
LTR	[2]	19	2.4	22	0.7	0.68 - 1.50	0.19	-8.5

Table 1: Controller performance



(a) t_s versus ρ (Performance)



(b) δ_a versus ρ (Robustness)

Figure 3: Influence of ρ on robustness and performance, $q = q_e = 0.1$

performance achieved in [14] and a collection of controllers presented in [12], including the three best designs. It is clear that the proposed robust H_2 design outperforms all other controllers. Moreover, the design procedure is simple and it would be straightforward to re-tune the controller to trade speed of response against robustness, or both against control effort.

4 Conclusion

A new method for iterative design of robust H_2 controllers has been proposed for plants with uncertain A and B matrices. A previously published result is shown to be a special case of this more general approach. The practical importance of the proposed approach is illustrated via a well known benchmark problem, where it outperforms previously published solutions.

5 Appendix A

This appendix presents a derivation of the LMI condition (8) for a worst case performance bound.

Consider the uncertain system:

$$\begin{aligned} \dot{x} &= \bar{A}x + B_1 w_1 + B_2 w_2 \\ z_1 &= \bar{C}_1 x \\ z_2 &= \bar{C}_2 x \end{aligned} \qquad (12)$$

and feedback

$$w_1 = \Delta z_1$$

where $\bar{A}$ is assumed stable, Δ is a real, possibly time-varying matrix that represents model uncertainty, and w_2 is a white noise process with unit covariance, as in section 2.

Initially, assume that $B_2 = 0$ and an initial state $x(0) = x_0$ is given. Consider the search for a Lyapunov function $V(x) = x^T P x$ such that for all trajectories $x(t)$

$$\frac{dV(x)}{dt} + z_2^T z_2 < 0 \quad \forall \Delta : \|\Delta\| < 1 \qquad (13)$$

It is straightforward to show that the existence of a matrix $P > 0$ satisfying the above, guarantees

$$J \leq \text{trace } P x_0 x_0^T, \quad \forall \Delta : \|\Delta\| < 1 \qquad (14)$$

This means the worst-case value of the performance index J is bounded, which implies robust stability.

Thus, the robust H_2 problem can be formulated as

$$\min \text{trace } P x_o x_o^T$$

subject to

$$\frac{dV}{dt} + z_2^T z_2 < 0 \tag{15}$$

$$w_1 = -\Delta z_1 \tag{16}$$

Conditions (15) and (16) can be rewritten respectively as

$$\begin{bmatrix} x \\ w_1 \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P + P\bar{A} + C_2^T C_2 & PB_1 \\ B_1^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} \Delta & -I \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix} = 0 \tag{18}$$

Define Γ as

$$\Gamma^T = \begin{bmatrix} \Delta & -I \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix}$$

Since $\Gamma^\perp \Gamma = 0$ and $\Gamma^T \Gamma^{\perp T} = 0$, and using equation (18), $\Gamma^\perp = \begin{bmatrix} x^T & w_1^T \end{bmatrix}$.

LMI (17) can be rewritten as

$$\Gamma^\perp \Omega \Gamma^{\perp T} < 0\, , \; \Omega = \begin{bmatrix} \bar{A}^T P + P\bar{A} + C_2^T C_2 & PB_1 \\ B_1^T P & 0 \end{bmatrix} \tag{19}$$

Using Finsler's lemma (19) holds, iff $\exists \alpha > 0$ such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + C_2^T C_2 & PB_1 \\ B_1^T P & 0 \end{bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix}^T \Phi \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix} < 0 \tag{20}$$

where:

$$\Phi = -\alpha \begin{bmatrix} \Delta^T \\ -I \end{bmatrix} \begin{bmatrix} \Delta & -I \end{bmatrix}$$

It is clear that $\Phi \leq 0$, thus there exists a constant matrix Θ such that

$$\Theta - \Phi \geq 0 \tag{21}$$

Using Finsler's lemma again the existence of a positive scalar α to satisfy LMI (21) is guaranteed if and only if the following LMI holds

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0 \tag{22}$$

Thus (20) is equivalent to

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + C_2^T C_2 & PB_1 \\ B_1^T P & 0 \end{bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix}^T \Theta \begin{bmatrix} C_1 & 0 \\ 0 & -I \end{bmatrix} < 0 \tag{23}$$

Using the Schur complement, (23) can be rewritten as

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} & C_2^T & C_1^T \theta_{11}^T & B_1 - C_1^T \theta_{12} \\ C_2 & -I & 0 & 0 \\ \theta_{11} C_1 & 0 & -\theta_{11} & 0 \\ B_1^T - \theta_{12}^T C_1 & 0 & 0 & \theta_{22} \end{bmatrix} < 0 \tag{24}$$

To complete the derivation we need to express (14) in LMI form. Now, removing the assumption $B_2 = 0$, the bound (14) on the worst-case performance can be interpreted as a worst-case bound on the energy of z_2 with initial condition $x_0 = B_2 w_2$, where w_2 is white noise with unit covariance as defined above. Taking the expectation yields

$$J = E\|z_2\|_2^2 \leq \operatorname{trace} B_2^T P B_2$$

With a slack matrix variable W, the above is equivalent to $J \leq \operatorname{trace} W$ and

$$\begin{bmatrix} W & B_2^T P \\ PB_2 & P \end{bmatrix} > 0 \tag{25}$$

This completes the prove.

A Special Case

Consider the special case where the scaling matrix Θ is given by

$$\Theta = \begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}$$

In that case (22) will always be satisfied because $S \geq \Delta^T S \Delta$. Now equation (24) simplifies to

$$\begin{bmatrix} A^T P + PA & (*) & (*) & (*) \\ C_2 & -I & 0 & 0 \\ SC_1 & 0 & -S & 0 \\ B_1^T & 0 & 0 & -S \end{bmatrix} < 0 \tag{26}$$

This last LMI is identical to the LMI condition (6) in [4]. In other words, the robust H_2 design approach proposed in [4] which was derived using the S-procedure, is a special case of the method proposed in this paper.

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