ON A SYNTHESIS METHOD FOR ROBUST PID CONTROLLERS FOR A CLASS OF UNCERTAINTIES

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Abstract

PID controller design is considered where optimal controller parameters are found with constraint on maximum sensitivity and robustness with regard to cone bounded static nonlinearity acting on the plant.

1 Introduction

Many mainstream control synthesis methods result in controllers of order related to the order of the plant. But often it is beneficial to design controllers with a restricted structure. There are many reasons for this. Their performance is often close to optimal performance while they often remain substantially less complex. One of the most common controllers is the PID controller. Its stronghold has been the process industry but it can be encountered elsewhere as well. The synthesis procedure presented in this article, is an extension to synthesis procedures presented in [1, 6] which are collected in [5]. There, a design procedure for PI(D) controllers was presented which minimizes the effect of a load disturbance by maximizing integral gain while making sure that the closed loop system is stable. Furthermore, it is guaranteed that the Nyquist curve of the loop transfer function is outside a circle with center $-C_s$ and radius R_s. This constraint can be expressed with the equations

$$\begin{aligned} &\text{maximize} && k_i && \\ &\text{subject to} && F(k,k_i,k_d,w) \geq R_s^2 && \forall \omega > 0 \end{aligned} \tag{1}$$

where F is the function:

$$F(k,k_i,k_d,\omega) = |C_s + C(i\omega)G(i\omega)|^2 \tag{2}$$

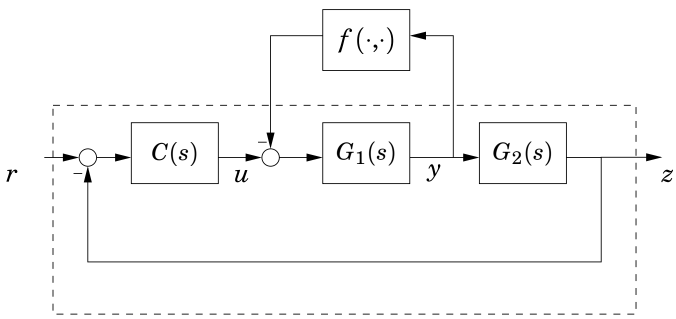

Figure 1: Block diagram showing nonlinearity, plant and controller.

and $G(s)$ is a linear time invariant plant and $C(s)$ is the PID controller parameterized as

$$C(s) = k + \frac{k_i}{s} + k_d s \tag{3}$$

In the PI case, a number of robustness conditions could be fulfilled by choosing C_s and R_s differently. By choosing $C_s = 1$ and $R_s = 1/M_s$, the resulting controller will guarantee a maximum sensitivity function ($S(s) = 1/(1+G(s)C(s))$) equal to M_s. Controllers with constraints on the maximum complementary sensitivity function ($T(s) = 1 - S(s)$) could also be designed or a combination of the these constraints.

In [5] the recommended criterion was to constrain the maximum of the sensitivity function and that is what will be used here as well. M_s is therefore a design parameter with value typically between 1.4-2. Selecting M_s in this interval guarantees a well damped closed loop system. In the PID case additional constraint are presented. These will be reviewed later.

The extension presented in this article guarantees asymptotic stability of equilibrium points of the system when a cone bounded nonlinearity is present in feedback with part of the plant, as shown in Fig. 1, while keeping the constraint presented in relation (2).

By combining these two conditions a controller is obtained which is robust toward the uncertainty description presented as well as keeping the good qualities of a controller with limited maximum sensitivity.

The result given in Theorem 1 in this paper have a higher degree of generality than that in [9] while the parametric relations remain simpler. Equivalent results were presented in [8].

Robust PID control design has been considered by many authors. In [2] a synthesis method for H_∞ optimal PID controllers is proposed. There, a linear programming characterization of all admissible PID controllers was presented.

The design procedure in this paper results in constraints in the controller parameter space. Similar approaches can be found in [4, 7].

The layout of the paper is the following: in section 2 conditions for stability are given for the uncertainties in question. In section 3 the synthesis problem and the underlying optimization and design problems are discussed. In section 4 the design procedure is applied on an example while in section 5 an industrial application of this method is described. Finally, conclusions are drawn in section 6.

2 Conditions and Theorems about stability

Consider the transfer functions $G(s) = G_1(s)G_2(s)$, with

$$\begin{aligned} G(i\omega) &= a(\omega)+ib(\omega) = r(\omega)e^{i\phi(\omega)} \\ G_1(i\omega) &= a_1(\omega)+ib_1(\omega) = r_1(\omega)e^{i\phi_1(\omega)} \end{aligned}$$

Consider for the beginning the case when the controller is a PI, i.e.:

$$C(s) = k + \frac{k_i}{s}.$$

Define the transfer function from u to y as

$$P(s) \triangleq \frac{G_1(s)}{1+C(s)G(s)} \tag{4}$$

Conditions for stability of the system in Figure 1 can be obtained by applying the circle criterion, for the transfer function $P(s)$ as it is connected in a loop with the nonlinearity. In order to state the main results, some intermediary steps will be helpful.

Lemma 1 *Consider $P(s)$ as in (4) and $\alpha, \beta \in \mathbb{R}$. Then*

$$\Re\left\{\frac{1+\beta P(s)}{1+\alpha P(s)}\right\} > 0$$

if and only if

$$\begin{aligned} H(k,k_i,\omega) \;\triangleq\; & p_1(\omega)k^2 + p_2(\omega)k + q_1(\omega)k_i^2 \\ & + q_2(\omega)k_i + h(\omega) > 0 \end{aligned} \tag{5}$$

with

$$\begin{aligned} p_1(\omega) &= r(\omega)^2 \\ p_2(\omega) &= (a(\omega)a_1(\omega)+b(\omega)b_1(\omega))(\alpha+\beta)+2a(\omega) \\ q_1(\omega) &= r(\omega)^2 \\ q_2(\omega) &= \frac{1}{\omega}\left((a(\omega)b_1(\omega)-b(\omega)a_1(\omega))(\alpha+\beta)+2b(\omega)\right) \\ h(\omega) &= (\alpha+\beta)a_1(\omega)+\alpha\beta r_1(\omega)^2+1 \end{aligned} \tag{6}$$

Proof:

$$\Re\left\{\frac{1+\beta P(s)}{1+\alpha P(s)}\right\} = \Re\left\{\frac{1+C(s)G(s)+\beta G_1(s)}{1+C(s)G(s)+\alpha G_1(s)}\right\} \tag{7}$$

By elementary but tedious computations, (7) will result in an ellipse in the $k-k_i$ plane as in (5) with parameters as in (6). $\diamond$

Consider a minimal realization of the linear system in Fig. 1. The total system can be written as

$$\begin{aligned} \dot{x} &= Ax - B(f(y,t)) + B_r r \\ y &= Cx \\ z &= C_z x \end{aligned} \tag{8}$$

Assume now that $r = 0$. The following theorem gives conditions of absolute stability of the origin.

Theorem 1 *Consider $f(y,t)$ piecewise continuous in t and locally Lipschitz in y such that:*

$$\alpha y^2 \le y f(y,t) \le \beta y^2, \quad \forall y \in \mathbb{R}, \forall t \ge 0 \tag{9}$$

and $H(k,k_i,\omega)$ defined by (5). Assume $P(s)$ and $\frac{1}{1+\alpha P(s)}$ Hurwitz transfer functions, with $P(s)$ given by (4). If

$$H(k,k_i,\omega) > 0 \tag{10}$$

then the origin is absolutely stable.

Proof:
The circle criterion (see Theorem 10.1 in [3]) provides sufficient conditions for absolute stability of the system. The condition that:

$$\Re\left\{\frac{1+\beta P(s)}{1+\alpha P(s)}\right\} > 0$$

is ensured by Lemma 1, completing the proof. $\diamond$

Some of the assumptions in the theorem above should be commented in the context of the proposed synthesis method. The assumption that $P(s)$ is Hurwitz, is equivalent to the stability conditions for the linear system. These conditions will be obviously fulfilled by the controller resulting from the design procedure. The assumption that $\frac{1}{1+\alpha P(s)}$ is Hurwitz has a geometric interpretation when α is positive. Since $P(s)$ is Hurwitz, it states that the Nyquist curve is not allowed to go round the point $-1/\alpha, j0$ and therefore not around the circle in the circle criterion. The consequences in $k-k_i$ plane will be that the optimizer should stay below the ellipses family generated by the stability constraints for the nonlinear system.

The next theorem proves the asymptotic stability of equilibrium points for constant r.

Theorem 2 *Assume $P(s)$ and $\frac{1}{1+\alpha P(s)}$ are Hurwitz transfer functions.*

Consider $f(y,t)$ piecewise continuous in t and locally Lipschitz in y such that:

$$\alpha \leq \frac{f(y_1,t)-f(y_2,t)}{y_1-y_2} \leq \beta, \quad \forall y_1, y_2 \in \mathbb{R}, \forall t \geq 0 \tag{11}$$

Consider $H(k,k_i,\omega)$ defined by (5). If

$$H(k,k_i,\omega) > 0$$

then the equilibrium point of system (8) corresponding to a constant r is globally uniformly asymptotically stable.

Proof:
If x_r is the equilibrium point of (8) for some $r \in \mathbb{R}$ then it fulfills the equation:

$$0 = Ax_r - Bf(y_r,t) + B_r r \qquad y_r = Cx_r \tag{12}$$

Furthermore, consider the Lyapunov function:

$$V(x,u) = (x-x_r)^T P(x-x_r)$$

For ease of writing, denote $\tilde{x} = x - x_r$ and drop the t as argument of f then

$$\dot{V}(x,r) = \tilde{x}^T(A^TP+PA)\tilde{x} - 2\tilde{x}^T PB(f(y)-f(y_r)) \tag{13}$$

By (11) for f yields :

$$\begin{aligned} &(f(y)-f(y_r)-\alpha(y-y_r))\cdot \\ &(f(y)-f(y_r)-\beta(y-y_r)) \quad \leq \quad 0 \end{aligned} \tag{14}$$

Then subtracting (14) from (13) gives:

$$\begin{aligned} \dot{V}(x,r) \quad &\leq \quad \tilde{x}^T(A^TP+PA-2\alpha\beta C^TC)\tilde{x} \\ &+ \quad 2\tilde{x}^T\left((\alpha+\beta)C^T-PB\right)(f(y)-f(y_r)) \\ &- \quad 2(f(y)-f(y_r))^2 \\ &= \quad \begin{bmatrix} \tilde{x} \\ f(y)-f(y_r) \end{bmatrix}^T M \begin{bmatrix} \tilde{x} \\ f(y)-f(y_r) \end{bmatrix} \end{aligned} \tag{15}$$

where

$$M = \begin{bmatrix} A^TP+PA-2\alpha\beta C^TC & (\alpha+\beta)C^T-PB \\ (\alpha+\beta)C-B^TP & -2 \end{bmatrix}.$$

Then $\dot{V}(x,r) < 0$ if the following LMI is feasible for a $P > 0$:

$$\begin{bmatrix} A^TP+PA-2\alpha\beta C^TC & (\alpha+\beta)C^T-PB \\ (\alpha+\beta)C-B^TP & -2 \end{bmatrix} < 0$$

which by Schur complement is equivalent to:

$$\begin{aligned} &2(A^TP+PA)-(\alpha+\beta)\left(C^TB^TP-PBC\right) \\ &\qquad +PBB^TP^T+(\alpha-\beta)^2C^TC < 0 \end{aligned} \tag{16}$$

Consider now the linear system:

$$\frac{1+\beta P(s)}{1+\alpha P(s)} \tag{17}$$

Given the minimal realization of $P(s)$ in Eq. (8), then a minimal realization of (17) is given by:

$$\begin{aligned} \dot{x} &= (A-\alpha BC)x + Bu^* \\ y^* &= C(\beta-\alpha)x + u^* \end{aligned} \tag{18}$$

By the Kalman-Yakubovich-Popov Lemma, system (18) is SPR if and only if exists $P > 0$ such that:

$$\begin{bmatrix} (A-\alpha BC)^TP+P(A-\alpha BC) & PB-(\beta-\alpha)C^T \\ B^TP-(\beta-\alpha)C & -2 \end{bmatrix} < 0$$

which by Schur complement is equivalent to (16). Thus $\dot{V}(x,r) < 0$ if (17) is SPR. This is equivalent to the fact that $\frac{1}{1+\alpha P(s)}$ is Hurwitz and that

$$\Re\left\{\frac{1+\beta P(s)}{1+\alpha P(s)}\right\} > 0. \tag{19}$$

By Lemma 1 condition (19) is equivalent to (5) which completes the proof.$\diamond$

Remark

Eq. (10) is for a fixed frequency equivalent to

$$g(k,k_i) > 0 \tag{20}$$

where $g(k,k_i)$ has the form

$$g(k,k_i) = \frac{(k-k_0)^2}{A} + \frac{(k_i-k_{i0})^2}{B} - 1 \tag{21}$$

which describes an ellipse in the $k-k_i$ plane, which is a similar condition to that obtained in [1].

Remark

Notice that Theorem 2 assumes the existence of solutions to Eq. (12).

Remark

In the PID case, k_i is replaced with $k_i - \omega^2 k_d$. The condition for stability remain the same but the synthesis problem is more complex. PD controller design can be approached the same way.

Remark

Condition (11) means that the slope with respect to y, of the nonlinearity $f(y,t)$ is limited below and above by α and β respectively, for any y and t.

3 Optimization

Using the results presented in the previous section the synthesis problem can be stated as the following optimization problem.

$$\begin{array}{rll} \max & k_i & \\ \text{subject to} & F(k,k_i,k_d,\omega_1) \geq R_s^2,\ \forall \omega_1 > 0 & (22) \\ & H(k,k_i,k_d,\omega_2) \geq 0,\ \forall \omega_2 > 0 & (23) \\ & k > 0, k_i > 0, k_d > 0 & (24) \end{array}$$

Constraint (24) guarantees that the controller will not have a unstable zero. The two frequency dependent inequalities, define the exterior of two ellipses for a fixed frequency. For $0 < \omega < \infty$ these ellipses generate envelopes that define the boundaries of the set of parameters which satisfy the constraints. The inequality (22) guarantees that the maximum sensitivity will be limited. This on the other hand guarantees that $P(s)$ is Hurwitz.

The constraints can be visualized by plotting the ellipses for a tight griding of frequencies. The point with the largest k_i below the ellipses is then the solution to the syntheses problem. This graphical approach suitable when PI design is considered but is more difficult when a PID controller is the goal since then the ellipses would have to be plotted for a griding of k_d values. Then a numerical optimization procedure can give the desired result. For most numerical optimization procedures it is important to have good starting value. The problem of finding good starting values is related to determining if the problem has any feasible solution. But a quick view of the constraints for a few values of k_d should be sufficient to obtain good starting values and see if the problem is feasible.

So far the synthesis procedure that has been presented would need much manual intervention. It is of interest to automate the synthesis procedure so that only the process and the parameters characterizing the uncertainties would need to be specified. This is in principle to automate the checking of feasibility and finding a good start value for the numerical optimization procedure. Unfortunately this problem can be as complex as solving the optimization problem itself.

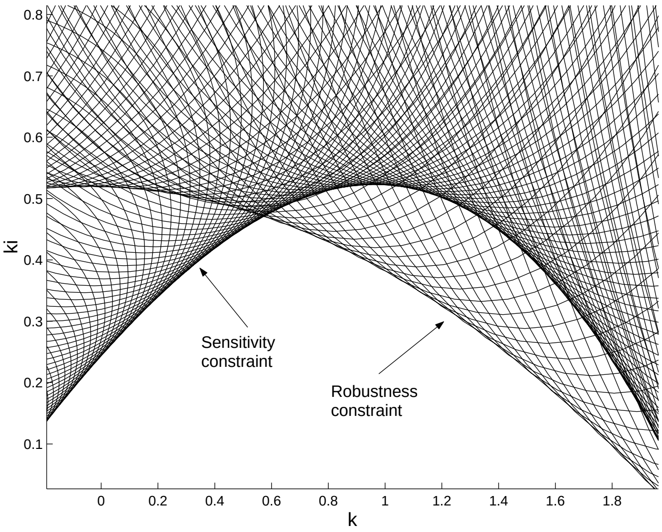

Figure 2: Constraints in the k-k_i plane for Ex.1.

4 Example

An example for PI controller design will now be given. The maximum sensitivity was chosen to be $M_s = 1.7$.

Example 1 *Consider the system in Figure 1 with*

$$G_1(s) = \frac{1}{(s+1)^3} \qquad G_2(s) = 1$$

and $f(y_1,t)$ a static, possibly time varying nonlinearity which is bounded according to Eq.(11) with $\alpha = 1$, $\beta = 4$. The two constraints, Eqs. (22) and (23) will give rise to constraint surfaces as shown in Fig. 2. Thus, in this case, the "optimum" considering only the sensitivity constraint will not guarantee robust stability of the system against a cone bounded uncertainty as considered above. Choosing the maximum k_i that falls bellow both constraint surfaces and a corresponding k, Theorem 1 guarantees asymptotic stability of the origin. The Nyquist plot of the loop transfer function and the transfer function defined by Eq. (4), shown in Figs. 3 respectively 4, confirm that the constraints are not violated. If $f(y_1,t)$ described an uncertain gain the transfer function would be given by

$$G(s) = \frac{1}{(s+1)^3 + \Delta}$$

where $\Delta \in [1,4]$.

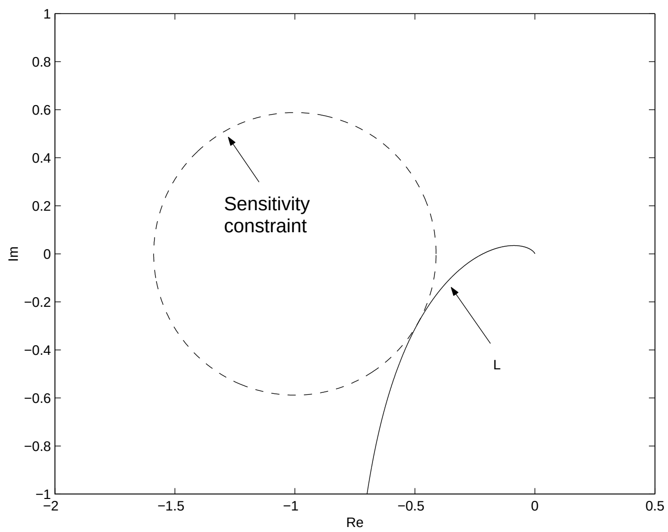

Figure 3: Nyquist plot of the loop transfer function for Ex.1.

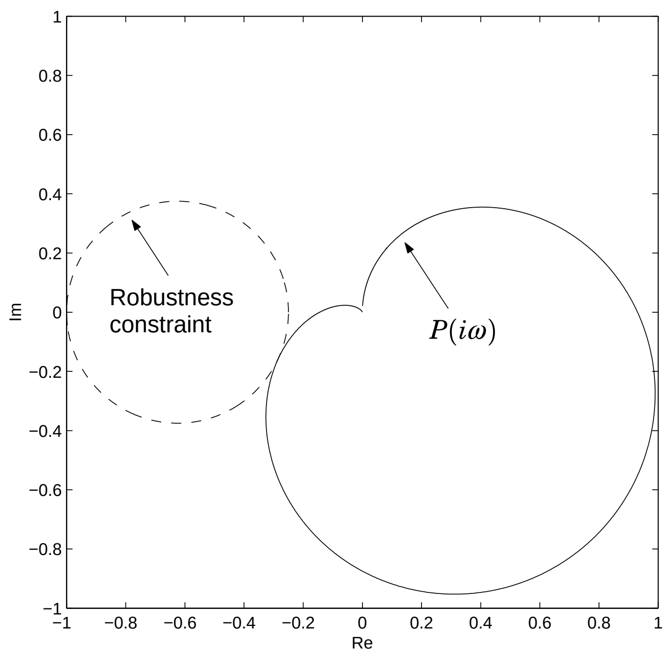

Figure 4: Nyquist plot of $P(s)$ for Ex.1.

5 Controller Synthesis for an Anti-Lock Braking System

In this section an application will be presented of the above synthesis method for an Anti-Lock Braking System (ABS). The synthesis method will be used to design local PI controllers for a gain scheduled scheme as presented in [10]. There a design model for a quarter-car (that is a wheel with a mass) was presented. A simplified model of the plant can be written as:

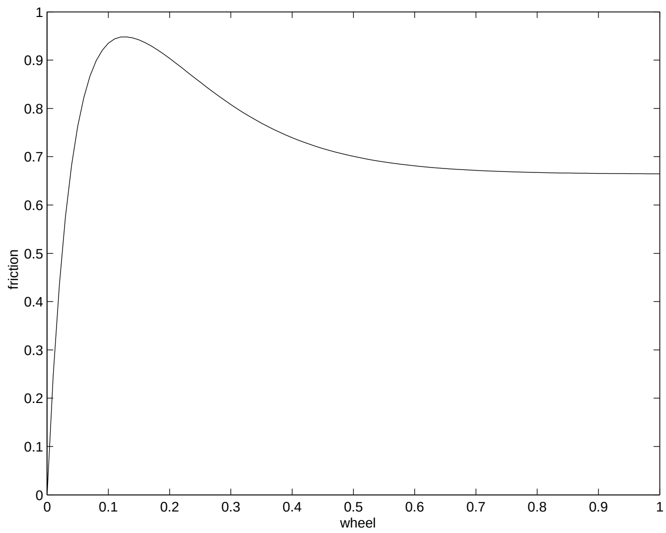

Figure 5: Tire friction curve

$$\dot{\lambda}(t)v = -\beta\mu(\lambda(t)) + \alpha u(t-T) \qquad (25)$$

where,

- v is the velocity of the car, and its variation are much slower in comparison to other variables in the system
- λ is the tire slip defined as $\lambda = \frac{v-\omega r}{v}$ where ωr is the linear velocity of the wheel
- α, β are constants depending on the car's parameters
- $\mu(\lambda)$ is the friction coefficient between the tire and surface
- u braking force
- T is time delay due to communication and sampling

The control input then is the brake force u and the controlled variable being the slip λ.
A typical dependence between the tire friction coefficient (μ) and tire slip (λ) is shown in Figure 5. By linearizing this nonlinear system in different operating points, the slope of the friction curve will affect the pole of the plant. Furthermore, scaling the controller by the car's velocity (v) the system can be put in the form shown in Figure 1 where $f(\cdot,\cdot)$ is proportional to the slope of the friction curve in the linearization point and inverse proportional to the velocity.

The design problem is to obtain a robust controller with respect to variations in one pole and the gain of the plant. The gain scheduled controller is chosen depending on the current position on the friction curve. Additional robustness is needed due to the highly uncertain nature of a tire-friction curve. Note that since it is possible to have negative slope in the friction curve, the plant can become unstable. Furthermore, to obtain a better performance, the actuator dynamics have also been considered. These are incorporated in $G_2(s)$ and can be easily handled by the proposed design method. The resulting gain scheduled controller had two different local controllers, scheduled with respect to the current value of the tire slip.

Within the European project H2C, experiments have been carried out in cooperation with DaimlerChrysler, on a Mercedes E Class test vehicle. Experimental results are shown in Figure 6. The first subplot shows

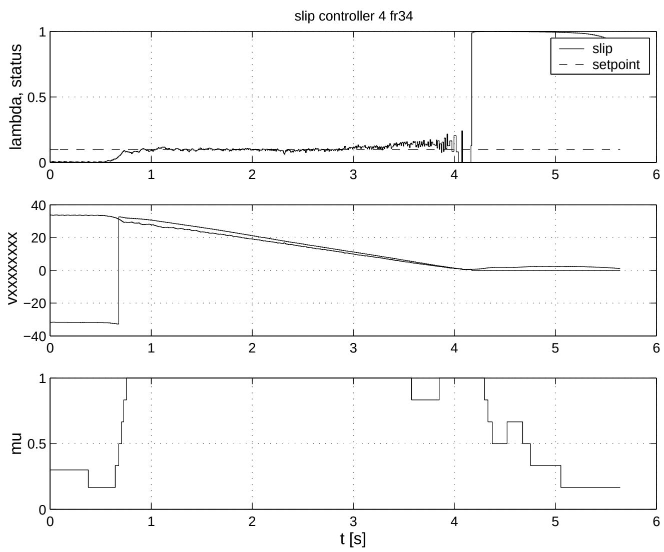

Figure 6: Experimental results for right back-wheel

the controlled slip, while the second presents the linear velocity of the wheel (ωr) respectively the car's velocity (v). The third subplot shows an estimate of the tire friction coefficient. The scenario is that of an emergency braking, while a reference tire-slip is to be maintained. Notice in the first plot that the tire-slip is smoothly controlled up to a velocity close zero.

6 Conclusions

The synthesis method presented deals with design of robust PID controllers. The uncertainty of the plant is described by a cone bounded nonlinearity which is in feedback with part of the plant. To obtain a good controller, maximum sensitivity is limited as well. The synthesis method presented requires much manual intervention but it is the belief of the authors it can be automated significantly. Experimental results have been presented from an industrial application where the design method was successfully used.

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