ON THE ROBUSTNESS OF GENERALIZED PI CONTROL WITH RESPECT TO PARAMETRIC UNCERTAINTIES

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Keywords: Generalized PI Control, Robustness, Underactuated Mechanical Systems, Lyapunov Stability.

Abstract

In this note we study the robustness of Generalized PI (GPI) control with respect to parametric uncertainties. We present two cases of study: a second order linear system and the inertia wheel pendulum. We propose a step by step procedure which may be used in each particular application.

1 Introduction.

Generalized PI (GPI) control has been recently proposed for linear systems [2, 3, 10]. Some applications to nonlinear systems have been reported by the authors [5, 6, 8]. Several good experimental results have been reported for either linear and nonlinear systems [9, 4, 7]. Further, a mixed control scheme combining GPI and sliding modes has also been proposed [11]. GPI control is an interesting control scheme because $1)$ allows state reconstruction instead of either state measurement or state asymptotic estimation, $2)$ its integral character introduces robustness with respect to additive external disturbances. However, the important question about its robustness with respect to parametric uncertainties has remained unsolved. In this paper we present an effort, if modest, to show that such GPI control robustness properties exist. An important feature of GPI control is state reconstruction by means of the direct integration of the plant dynamic model. This suggests that a very exact knowledge of the model parameters is necessary. However, some simulation studies have shown good performance when parametric uncertainties are present [6]. Moreover, the good experimental results that have been published [9, 4, 7] represent an important evidence of the robustness of GPI control. To the best knowledge of the authors, the first analysis about GPI control robustness with respect to parametric uncertainties has been presented in [6]. In that work the application of GPI control to position regulation in flexible joint robots has been presented. Robustness has been shown when uncertainties are present in some specific parameters. The common thread among the selected parameters is that they do not appear as factor of the input. The main reason for this restriction is that any uncertainty in parameters multiplying the input introduces additional terms depending on the input. This is an inconvenient because the input depends on the reconstructed states which introduce, again, terms depending on the input and its time derivatives because of the parametric uncertainties. Hence, these terms are difficult to handle in the resulting closed loop dynamics. On the other hand, the second author has recently found that, in the GPI control of linear systems, the input can be synthesized as a linear filter. This suggests that the input and some of its time derivatives can be expressed in terms of the state derivatives. Hence, by substitution we can write the closed loop dynamics in terms of the state variables and the desired trayectory. Finally, the stability of the closed loop system can by studied using the Lyapunov method. We study the robustness of GPI control with respect to parametric uncertainties through two cases of study: a second order linear system and the inertia wheel pendulum, a nonlinear subactuated mechanical system. We propose to use a similar procedure in each particular application in order to try to prove robustness with respect to parametric uncertainties when GPI control is used. This paper is organized as follows. In section 2 the robustness analysis procedure that we propose is applied to a second order linear system while the same procedure is applied to the inertia wheel pendulum in section 3. In section 4 we present some simulation results whereas in section 5 some concluding remarks are given. Finally, a remark on notation: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ represent, respectively, the Euclidean norm and the $\infty-$norm, whereas $|\cdot|$ represents the absolute value of a scalar number.

2 A second order linear system.

Consider the following second order SISO linear system:

$$\ddot{x} = -ax - b\dot{x} + cu, \quad y = x \tag{1}$$

where $c \neq 0$, a and b are constant scalars, $y, u \in R$ are the output and the input, respectively. Suppose that parameters a, b, c are not exactly known and $\bar{a}$, $\bar{b}$, $\bar{c}$ are their respective estimates. We desire to track a time varying trajectory, $x_d(t)$, using an output feedback control scheme. Hence, the following GPI control law is proposed:

$$u = \frac{1}{\bar{c}}\Big[\bar{a}x + \bar{b}\hat{\dot{x}} + \ddot{x}_d - k_1(\hat{\dot{x}} - \dot{x}_d) - k_0(x - x_d) - -k_i\int_0^t (x - x_d)ds - k_{ii}\int_0^t\int_0^s (x - x_d)drds\Big] \quad (2)$$

$$\hat{\dot{x}} = -\bar{a}\int_0^t xds - \bar{b}x + \bar{c}\int_0^t uds \quad (3)$$

The relationship beetwen the actual output derivative, $\dot{x}$, and the reconstructed output derivative, $\hat{\dot{x}}$, is given as:

$$\dot{x} = \hat{\dot{x}} + (\bar{a} - a)\int_0^t xds + (\bar{b} - b)x + (c - \bar{c})\int_0^t uds + \dot{x}(0) \quad (4)$$

In what follows we describe step by step the procedure that we propose to show the robustness of GPI control.

Step 1 *Replace the control law in the plant.*

Using (2) in (1) yields:

$$\frac{\bar{c}}{c}\ddot{x} = -\frac{a\bar{c}}{c}x - \frac{b\bar{c}}{c}\dot{x} + \bar{a}x + \bar{b}\hat{\dot{x}} + \ddot{x}_d - k_1(\hat{\dot{x}} - \dot{x}_d) - - k_0(x - x_d) - k_i\int_0^t (x - x_d)ds - -k_{ii}\int_0^t\int_0^s (x - x_d)drds \quad (5)$$

Step 2 *Find a linear perturbed differential equation in terms of the tracking error. Consider that parametric uncertainties are small.*

Suppose that the estimated values $\bar{a}$, $\bar{b}$, $\bar{c}$ are close to the actual values a, b, c, respectively. Hence:

$$\frac{\bar{c}}{c} = 1 + \epsilon \quad (6)$$

where ϵ is a real number close to zero. Thus, we can use (4), (5) and (6) to write:

$$e^{(4)} + k_1e^{(3)} + k_0\ddot{e} + k_i\dot{e} + k_{ii}e = -\epsilon x^{(4)} + + (\bar{a} - \frac{a\bar{c}}{c})\ddot{x} + (\bar{b} - \frac{b\bar{c}}{c})x^{(3)} - [(\bar{a} - a)\dot{x} + (\bar{b} - b)\ddot{x} + + (c - \bar{c})\dot{u}](\bar{b} - k_1) \quad (7)$$

where we have defferentiated twice and defined the tracking error as $e = x - x_d$.

Step 3 *Use the plant model to reduce the order of the perturbation.*

Differentiating (1) twice, replacing $x^{(4)}$, from the resulting expression, in (7) and recalling (6) yields:

$$e^{(4)} + k_1e^{(3)} + k_0\ddot{e} + k_i\dot{e} + k_{ii}e = = (\bar{b} - b)x^{(3)} + [-(\bar{b} - b)(\bar{b} - k_1) + (\bar{a} - a)]\ddot{x} - -(\bar{a} - a)(\bar{b} - k_1)\dot{x} - (\bar{c} - c)[\ddot{u} - (\bar{b} - k_1)\dot{u}] \quad (8)$$

Step 4 *Find a realization of the control signal in terms of a linear filter.*

Using (2) and (3) gives:

$$\bar{c}[\ddot{u} - (\bar{b} - k_1)\dot{u}] = (\bar{a} - \bar{b}^2 + \bar{b}k_1)\ddot{x} + (-\bar{a}\bar{b} + \bar{a}k_1)\dot{x} + +x_d^{(4)} + k_1x_d^{(3)} - k_0\ddot{e} - k_i\dot{e} - k_{ii}e \quad (9)$$

where we have differentiated twice.

Step 5 *Using the linear filter relization of the input, find the closed loop dynamics.*

Replacing (9) in (8) yields:

$$\dot{E} = AE + Bf, \qquad E = [e^{(3)}, \ddot{e}, \dot{e}, e]^T \quad (10)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_{ii} & k_i & k_0 & k_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f = -\frac{\bar{c} - c}{\bar{c}}[x_d^{(4)} + k_1x_d^{(3)} - k_0\ddot{e} - k_i\dot{e} - k_{ii}e] + + [-(\bar{b} - b)(\bar{b} - k_1) + (\bar{a} - a) - -\frac{\bar{c} - c}{\bar{c}}(\bar{a} + \bar{b}(k_1 - \bar{b}))]\ddot{x} + + [-(\bar{a} - a)(\bar{b} - k_1) - \frac{\bar{c} - c}{\bar{c}}\bar{a}(k_1 - \bar{b})]\dot{x} + +(\bar{b} - b)x^{(3)}$$

Step 6 *Use the robustness of exponentially stable systems to show stability of the entire closed loop system.*

Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}E^TPE \quad (11)$$

where P is a constant, symmetric, positive definite matrix. Differentiating once:

$$\dot{V} = \frac{1}{2}E^T(PA + A^TP)E + E^TPBf \quad (12)$$

Note:

$$x^{(k)} = e^{(k)} + x_d^{(k)}, \quad k = 0, 1, 2, 3, 4. \quad (13)$$

hence, f can be bounded as:

$$|f| \leq |\Big[\frac{\bar{c} - c}{\bar{c}}\left[\max\{k_0, k_i, k_{ii}\} - (\bar{a} + \bar{b}(k_1 - \bar{b}))\right] +(\bar{b} - b)[1 - (\bar{b} - k_1)] + (\bar{a} - a)\Big]|\|E\|_2 + + |[-(\bar{a} - a)(\bar{b} - k_1) - \frac{\bar{c} - c}{\bar{c}}\bar{a}(k_1 - \bar{b})]|\|E\|_2 + + |\frac{\bar{c} - c}{\bar{c}}||[x_d^{(4)} + k_1x_d^{(3)}]| + |(\bar{b} - b)||x_d^{(3)}| + + |[-(\bar{b} - b)(\bar{b} - k_1) + (\bar{a} - a) - -\frac{\bar{c} - c}{\bar{c}}(\bar{a} + \bar{b}(k_1 - \bar{b}))]||\ddot{x}_d| + + |[-(\bar{a} - a)(\bar{b} - k_1) - \frac{\bar{c} - c}{\bar{c}}\bar{a}(k_1 - \bar{b})]||\dot{x}_d| \quad (14)$$

Thus, $\dot{V}$ can be bounded as:

$$\begin{aligned}\dot{V} &\leq \frac{1}{2}E^T(PA + A^TP)E + \kappa_1\left\|E\right\|_2^2 + \\ &+ \kappa_2\left\|E\right\|_2\left\|[x_d^{(4)}, x_d^{(3)}, \ddot{x}_d, \dot{x}_d]\right\|_\infty \qquad (15)\end{aligned}$$

where κ_1 and κ_2 are positive constants which are smaller for smaller parametric uncertainties. Suppose that the gains k_{ii}, k_i, k_0 and k_1 are chosen such that matrix A given in (10) is Hurwitz, then there exists a constant, symmetric, positive definite matrix Q such that:

$$-Q = \frac{1}{2}(PA + A^TP) \qquad (16)$$

Hence, we can write:

$$\dot{V} \leq -(c_3-\kappa_1)\left\|E\right\|_2^2+\kappa_2\left\|E\right\|_2\left\|[x_d^{(4)}, x_d^{(3)}, \ddot{x}_d, \dot{x}_d]\right\|_\infty \quad (17)$$

where $c_3 > 0$ is the minimum eigenvalue of matrix Q and $(c_3 - \kappa_1) > 0$ for small enough parametric uncertainties. On the other hand, under the same condition κ_2 is small too. If $\left\|[x_d^{(4)}, x_d^{(3)}, \ddot{x}_d, \dot{x}_d]\right\|_\infty$ is bounded then there exists a ball centered at $E = 0$ such that outside it the quadratic term dominates the linear term in (17), and $\dot{V} < 0$ outside that ball. This guarantees $\left\|E\right\|_2$ is bounded. If, additionally, the desired trajectory converges to some constant, i.e., if $x_d^{(k)}(t) \to 0$, $k = 1, 2, 3, 4$ then $x \to x_d(t)$, for any initial condition. Thus, we can claim that the proposed GPI control scheme is robust with respect to small parametric uncertainties.

3 The inertia wheel pendulum.

Consider the inertia wheel pendulum dynamic model [1] (see fig. 1):

$$D(q)\ddot{q} + g(q) = u + T_d \qquad (18)$$

$$D(q) = \begin{pmatrix} m_1l_{c1}^2 + m_2l_1^2 + I_1 + I_2 & I_2 \\ I_2 & I_2 \end{pmatrix},$$

$$D(q) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \quad g(q) = \begin{pmatrix} \bar{m}g\sin(q_1) \\ 0 \end{pmatrix},$$

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad u = \begin{pmatrix} 0 \\ \tau \end{pmatrix}, \quad T_d = \begin{pmatrix} \tau_L \\ 0 \end{pmatrix}$$

See [1] for a complete description of this model . In particular, we have that: $d_{11} > 0$, and $d_{11}d_{22} - d_{12}d_{21} > 0$. Note that the torque disturbance τ_L appears at the pendulum joint. In [7] the following controller is proposed for swinging up and balancing of the inertia wheel pendulum:

$$\tau = \frac{d_{11}d_{22} - d_{12}d_{21}}{-d_{12}}\left[\frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}\bar{m}g\sin(q_1) + a_3\right]$$

$$a_3 = \ddot{q}_{1d} - k_1(\hat{\dot{q}}_1 - \dot{q}_{1d}) - k_2(q_1 - q_{1d}) - $$
$$-k_3\int_0^t (q_1 - q_{1d})ds - k_4\int_0^t\int_0^s (q_1 - q_{1d})dzds \qquad (19)$$

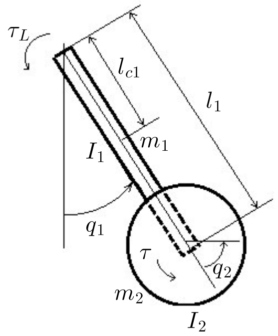

Figure 1: The inertia wheel pendulum.

where $\hat{\dot{q}}_1$ is given as:

$$\hat{\dot{q}}_1 = \frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}\int_0^t \left[-\bar{m}g\sin(q_1) - \frac{d_{12}}{d_{22}}\tau\right] ds \quad (20)$$

The gains k_1, k_2, k_3, k_4 are chosen such that the polynomial $p(s) = s^4 + k_1s^3 + k_2s^2 + k_3s + k_4$ is Hurwitz. It is supposed that the torque disturbance τ_L is constant and is only applied during short periods of time. It is important to remark that the relationship beetwen the reconstructed velocity and the actual velocity is given as:

$$\dot{q}_1 = \hat{\dot{q}}_1 + \dot{q}_1(0) + \frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}\int_0^t \tau_L ds \qquad (21)$$

See [7] for further details on this controller.
In the present paper we suppose that no plant parameter is known exactly, aside from the gravity constant g. Let δ_{11}, δ_{22}, δ_{12}, δ_{21} and μ be the estimates of d_{11}, d_{22}, d_{12}, d_{21} and $\bar{m}$, respectively. Hence, control law (19) is written as:

$$\tau = \frac{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}{-\delta_{12}}\left[\frac{\delta_{22}}{\delta_{11}\delta_{22} - \delta_{12}d_{21}}\mu g\sin(q_1) + a_4\right] \qquad (22)$$

$$\begin{aligned} a_4 &= \ddot{q}_{1d} - k_1(\hat{\dot{\varphi}}_1 - \dot{q}_{1d}) - k_2(q_1 - q_{1d}) - \\ &- k_3\int_0^t (q_1 - q_{1d})ds - k_4\int_0^t\int_0^s (q_1 - q_{1d})dzds \quad (23)\end{aligned}$$

$$\hat{\dot{\varphi}}_1 = \int_0^t [-\bar{\eta}_1\sin(q_1) - \bar{\eta}_2\tau]ds \qquad (24)$$

$$\begin{aligned}\dot{q}_1 &= \hat{\dot{\varphi}}_1 + \dot{q}_1(0) + n_3\int_0^t \tau_L ds + \\ &+ (\bar{\eta}_1 - n_1)\int_0^t \sin(q_1)ds + (\bar{\eta}_2 - n_2)\int_0^t \tau ds \qquad (25)\end{aligned}$$

$$\bar{\eta}_1 = \frac{\delta_{22}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}\mu g$$

$$\bar{\eta}_2 = \frac{\delta_{12}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}$$

$$
\begin{aligned}
n_1 &= \frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}\bar{m}g \\
n_2 &= \frac{d_{12}}{d_{11}d_{22} - d_{12}d_{21}} \\
n_3 &= \frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}
\end{aligned}
$$

Note that $\bar{\eta}_1$ and $\bar{\eta}_2$ are the estimates of n_1 and n_2, respectively, and their values are close for small parameter uncertainties. From the second row of (18) we get $\ddot{q}_2$ and replace it in the first row of the same equation. From the resulting expression we get $\ddot{q}_1$:

$$\ddot{q}_1 = -n_1 \sin(q_1) - n_2\tau - n_3\tau_L \tag{26}$$

The control law (22) can be written as:

$$\tau = \frac{\delta_{22}}{-\delta_{12}}\mu g \sin(q_1) + \frac{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}{-\delta_{12}}a_4 \tag{27}$$

Step 1 *Replace the control law in the plant.*

Using (26) and (27) yields:

$$
\begin{aligned}
\ddot{q}_1 &= a \sin(q_1) + ba_4 - c\tau_L \qquad (28) \\
a &= -\frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}\bar{m}g + \frac{d_{12}}{d_{11}d_{22} - d_{12}d_{21}}\frac{\delta_{22}}{\delta_{12}}\mu g \\
b &= \frac{n_2}{\bar{\eta}_2} \\
c &= n_3
\end{aligned}
$$

Step 2 *Find a linear perturbed differential equation in terms of the tracking error. Consider that parametric uncertainties are small.*

Note that, if parameter uncertainties are small then a is a small real number close to zero and b is a positive real number close to the unity. Hence, we can write:

$$\frac{1}{b} = 1 + \epsilon = \frac{\bar{\eta}_2}{n_2} \tag{29}$$

where ϵ is a real number close to zero. Thus, using (25), (23) and (29) we can write (28) as:

$$
\begin{aligned}
&\ddot{e} + k_1\dot{e} + k_2 e + k_3\int_0^t e ds + k_4 \int_0^t \int_0^s e dr ds = \\
= \; & -\epsilon\ddot{q}_1 + \frac{a}{b}\sin(q_1) - \frac{c}{b}\tau_L + \\
+ \; & k_1 \left[\dot{q}_1(0) + n_3\int_0^t \tau_L ds + (\bar{\eta}_1 - n_1)\int_0^t \sin(q_1) ds + \right. \\
& \left. +(\bar{\eta}_2 - n_2)\int_0^t \tau ds\right] \qquad (30)
\end{aligned}
$$

where we have defined the tracking error as $e = q_1 - q_{1d}$.

Step 3 *Use the plant model to reduce the order of the perturbation.*

Replacing $\ddot{q}_1$, from (26), in (30) and differentiating twice:

$$
\begin{aligned}
& e^{(4)} + k_1 e^{(3)} + k_2\ddot{e} + k_3\dot{e} + k_4 e = \\
& = (\epsilon n_1 + \frac{a}{b})\frac{d^2}{dt^2}\sin(q_1) + \\
& + k_1(\bar{\eta}_1 - n_1)\frac{d}{dt}\sin(q_1) + (\bar{\eta}_2 - n_2)[\ddot{\tau} + k_1\dot{\tau}] \qquad (31)
\end{aligned}
$$

where (29) has been used to show $\epsilon n_2 = \bar{\eta}_2 - n_2$ and we recall τ_L is constant.

Step 4 *Find a realization of the control signal in terms of a linear filter.*

We can use (22), (23), (24) and differentiate twice to obtain:

$$
\begin{aligned}
\bar{\eta}_2(\ddot{\tau} + k_1\dot{\tau}) &= -\bar{\eta}_1\frac{d^2}{dt^2}\sin(q_1) - q_{1d}^{(4)} - k_1 q_{1d}^{(3)} + \\
& + k_2\ddot{e} + k_3\dot{e} + k_4 e - k_1\bar{\eta}_1\frac{d}{dt}\sin(q_1) \qquad (32)
\end{aligned}
$$

Step 5 *Using the linear filter relization of the input, find the closed loop dynamics.*

Replacing (32) in (31):

$$
\begin{aligned}
\dot{\Sigma} &= \Lambda\Sigma + \Gamma h, \qquad \Sigma = [e^{(3)}, \ddot{e}, \dot{e}, e]^T \qquad (33) \\
\Lambda &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_4 & k_3 & k_2 & k_1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
h &= [(\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1](\ddot{q}_1\cos(q_1) - \dot{q}_1^2\sin(q_1)) + \\
&+ [k_1(\bar{\eta}_1 - n_1) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}k_1\bar{\eta}_1]\dot{q}_1\cos(q_1) + \\
&+ \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}[-q_{1d}^{(4)} - k_1 q_{1d}^{(3)} + k_2\ddot{e} + k_3\dot{e} + k_4 e]
\end{aligned}
$$

Step 6 *Use the robustness of exponentially stable systems to show stability of the entire closed loop system.*

We propose the following Lyapunov function candidate:

$$V = \Sigma^T P \Sigma \tag{34}$$

where P is a constant, symmetric, positive definite matrix. Differentiating once:

$$\dot{V} = \frac{1}{2}\Sigma^T(P\Lambda + \Lambda^T P)\Sigma + \Sigma^T P\Gamma h \tag{35}$$

Note that:

$$q_1^{(k)} = e^{(k)} + q_{1d}^{(k)}, \quad k = 0, 1, 2, 3, 4. \tag{36}$$

Hence, the absolute value of h can be bounded as:

$$| \, h \, | \; \leq \; \Bigg[| \, \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2} \, | \max\{k_2, k_3, k_4\} +$$

$$
\begin{aligned}
&+ \quad | \ (\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1 \ | + \\
&+ \quad | \ k_1(\bar{\eta}_1 - n_1) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2} k_1 \bar{\eta}_1 \ | \Big] \ \|\Sigma\|_2 + \\
&+ \quad | \ \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2} \ || \ [q_{1d}^{(4)} + k_1 q_{1d}^{(3)}] \ | + \\
&+ \quad | \ [(\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1] \ || \ \ddot{q}_{1d} \ | + \\
&+ \quad | \ [k_1(\bar{\eta}_1 - n_1) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2} k_1 \bar{\eta}_1] \ || \ \dot{q}_{1d} \ | + \\
&+ \quad | \ [(\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1] \ | \ \|\Sigma\|_2^2 + \\
&+ \quad | \ [(\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1] \ || \ \dot{q}_{1d}^2 \ | + \\
&+ \quad 2 \ | \ [(\epsilon n_1 + \frac{a}{b}) - \frac{\bar{\eta}_2 - n_2}{\bar{\eta}_2}\bar{\eta}_1] \ || \ \dot{q}_{1d} \ | \ \|\Sigma\|_2
\end{aligned}
$$

Note that, if the gains k_1, k_2, k_3 and k_4 are chosen such that matrix Λ is Hurwitz then there exists a constant, symmetric and positive definite matrix Q, such that:

$$\frac{1}{2}(P\Lambda + \Lambda^T P) = -Q \tag{37}$$

Hence, we can write:

$$
\begin{aligned}
\dot{V} \quad &\leq \quad -\alpha\|\Sigma\|_2^2 + \beta_4\|\Sigma\|_2 \|[\dot{q}_{1d}, \ddot{q}_{1d}, q_{1d}^{(3)}, q_{1d}^{(4)}]\|_\infty + \\
&+ \quad \beta_5\|\Sigma\|_2 \ | \ \dot{q}_{1d}^2 \ | \qquad (38) \\
\alpha \quad &= \quad (c_3 - \beta_1 - \beta_2\|\Sigma\|_2 - \beta_3 \ | \ \dot{q}_{1d} \ |)
\end{aligned}
$$

where $c_3 > 0$ is the minimum eigenvalue of matrix Q, constants β_1, β_2, β_3, β_4, β_5 are smaller for smaller parameter uncertainties. Thus $\alpha > 0$ for small parameter uncertainties, a small region $\|\Sigma\|_2 < \delta$, and slow desired trajectories $|\dot{q}_{1d}| < \delta_d$, for small positive constants δ and δ_d. Then, if $\|[\dot{q}_{1d}, \ddot{q}_{1d}, q_{1d}^{(3)}, q_{1d}^{(4)}]\|_\infty$ is bounded, there exists a ball centered in $\Sigma = 0$ such that, outside it, the quadratic term (with respect to $\|\Sigma\|_2$) in (38) dominates the linear terms and $\dot{V} < 0$. Thus, boundedness of Σ is ensured. This means that the desired trajectory is followed closely by the actual output. Moreover, if the desired trajectory becomes constant the tracking error e converges to zero. Finally, we can proceed as in [7], using the fact that the input can be obtained as a linear filter, to show that the zero dynamics:

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 = \tau \tag{39}$$

is stable. Hence, if τ_L is zero the velocity of the wheel, $\dot{q}_2$, converges to a constant whereas if τ_L is a nonzero constant then the velocity of the wheel grows with a constant rate to compensate for such disturbance. As it was pointed out in [7], torque disturbances are allowed to appear only during finite periods of time. Thus, we can claim that the proposed GPI control scheme is robust with respect to small parametric uncertainties.

4 Simulation results.

In fig. 2 we present some simulation results when we use the second order linear system (1) together with the GPI control law (2). We used $c = 2$, $b = 3$, $a = 4$, $\bar{c} = 2c$, $\bar{b} = 2b$, $\bar{a} = 2a$, $k_1 = 40$, $k_0 = 600$, $k_i = 4000$, $k_{ii} = 10000$. The trajectory to be tracked, x_d, is generated using a step signal, of value 2, filtered using a third order linear system with the following characteristic polynomial $(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + f)$, with $\zeta = 0.1$, $\omega_n = 8$ and $f = 7$. The dashed line represents the desired trajectory x_d. Note that, as expected, the system output follows closely the desired trajectory and convergence is obtained when the latter becomes constant. In fig. 3 we present the simulation results otained when we use the control law (27) to control the inertia wheel pendulum (18). We used $\tau_L = 0$, $k_1 = 80$, $k_2 = 2400$, $k_3 = 32000$, $k_4 = 160000$, $\mu = 1.9\bar{m}$, $\delta_{11} = 1.9d_{11}$, $\delta_{12} = 1.9d_{12}$, $\delta_{21} = 1.9d_{21}$, $\delta_{22} = 1.9d_{22}$. For a description of the desired trajectory as well as the value of the parameters of the inertia wheel pendulum we refer the reader to [7]. Note that the desired trajectory as well as the real trajectory are very close because of the large value of the controller parameters used. On the other hand, convergence is obtained when the desired trajectory becomes constant, as expected. Also note that the wheel velocity $\dot{q}_2$ remains bounded and converges to a constant. We present the evolution of the variable E_U instead of the applied torque τ. To see the relationship between these variables we refer the reader, again, to [7].

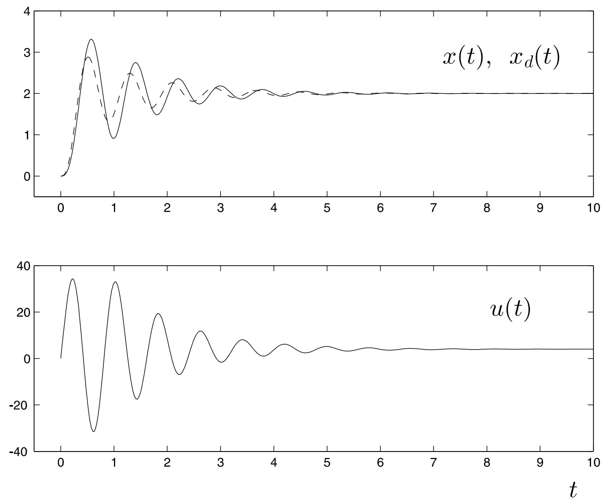

Figure 2: Second order linear system response.

5 Conclusions.

We have proposed an analysis procedure to show robustness of GPI control with respect to parametric uncertainties. This study is important because it shows that one of the disadvantages attributed to GPI control, i.e., the lack of robustness when parametric uncertainties are present, does not exist or, at least, it is not as dangerous as it was considered. We present our result by means of two cases of study. Although a general procedure for the n order case is not presented we think that a similar procedure may be used in each particular application to try to show robustness. Our stability results ensure that the desired trajectory is closely followed by the actual output and that con-

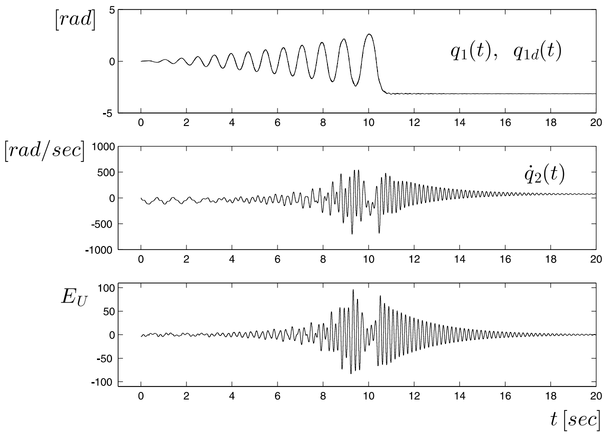

Figure 3: Inertia wheel pendulum response.

vergence to zero of the tracking error is achieved if the desired trajectory converges to a constant.

6 Acknowledgements

This work has been financed by CINVESTAV-IPN and CONACYT-México, under grant 32681-A. V.M.Hernández work is supported by a scholarship from the Universidad Autónoma de Querétaro, México.

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