AN ELLIPSOIDAL STATE ESTIMATION ALGORITHM FOR NONLINEAR SYSTEMS SUBJECT TO BOUNDED DISTURBANCES

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Abstract

This contribution presents a recursive algorithm for state estimation of nonlinear systems using ellipsoidal bounds on the process and observation noises. A novel approach based on state bounding techniques and on the classical Extended Kalman Filter with switching gain is proposed. A particular parameterization of the algorithm is introduced to increase performances and to characterize the set of state estimates compatible with the noises bounds. Simulation results on a £fth-order two-phase nonlinear model of an induction motor are also given.

1 Introduction

State estimation of stochastic dynamical systems has been extensively studied during the last decades and the problem is usually solved by assuming white and Gaussian noises on model and measurements. However, when the statistical properties of the noises are unknown or not satis£ed, an alternative approach consists in considering that only bounds on the possible magnitude of the disturbances are available, the so-called set-membership estimation was £rst introduced in [1, 2] using ellipsoidal bounding techniques. The aim is to determine a set of state estimation vectors compatible with the bounds on the process disturbance and measurement noise. Since these pioneer works, a vast literature is dedicated to this subject in the context of parameter identi£cation [3] or state estimation [4, 5, 6, 7]. However, to our knowledge, very few works have been developed when the model is nonlinear like most of real-life problems.

The goal of this paper is to outline a robust recursive algorithm based on the classical Extended Kalman Filter for state estimation of nonlinear discrete-time systems with unknown but bounded disturbances corrupting both the dynamical equation and the output vector. The proposed algorithm can be decomposed into two steps : *time updating*, inspired from an ellipsoidal state bounding method developed in [7] and *observation updating* that uses a switching estimation Kalman-like gain matrix. The latter step may be seen as a generalization of a parameter estimation algorithm for multi-output nonlinear systems introduced in [8]. Particular emphasis is given to the design of weighting matrices that ensure consistency of the estimated states with the input-output data and the noise constraints, and improve convergence properties. Suf£cient conditions for the decrease of a crucial parameter related to the size of the set of interest are established. Finally, the effectiveness of the proposed algorithm is demonstrated through a numerical example.

2 Notations and Problem Formulation

In this paper, we will use some standard notations :

- An ellipsoid in $\mathbb{R}^s$, where $s \in \mathbb{N}^*$, is de£ned as follows
$$\mathcal{E}(c,P) := \{x \in \mathbb{R}^s |\, (x-c)^T P^{-1}(x-c) \leq 1\}$$
where $c \in \mathbb{R}^s$ is the center of this ellipsoid and $P \in \mathbb{R}^{s\times s}$ is a symmetric positive de£nite matrix that de£nes its shape, size and orientation in the $\mathbb{R}^s$ space.
- We also de£ne the exterior of the ellipsoid $\mathcal{E}(c,P)$, as
$$\overline{\mathcal{E}}(c,P) := \{x \in \mathbb{R}^s |\, x \notin \mathcal{E}(c,P)\} = \{x \in \mathbb{R}^s |\, (x-c)^T P^{-1}(x-c) > 1\}$$
- $\|x\| = (x^T x)^{\frac{1}{2}}$ is the Euclidean norm of the vector x;
- $\|x\|_W = (x^T W x)^{\frac{1}{2}}$ is the weighted Euclidean norm of the vector x (W is a symmetric positive de£nite matrix of appropriate dimension);
- $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are the minimum and maximum eigenvalues of the square symmetric matrix M;
- $\|A\| = \max\limits_{x\neq 0} \dfrac{\|Ax\|}{\|x\|}$ is the 2-norm of the matrix A. We also have $\|A\| = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)}$;
- $\mathrm{tr}(A) = \sum_i \lambda_i(A)$ is the trace of the square matrix A;
- the symbol $:=$ means that the RHS is de£ned to be equal to the LHS.

Let us consider the following discrete-time nonlinear system written in the state space :

$$x_k^* = f(x_{k-1}^*, u_{k-1}) + G_{k-1} w_{k-1} \quad (1a)$$
$$y_k = h(x_k^*, u_k) + v_k \quad (1b)$$

where $x_k^* \in \mathbb{R}^n$ is the unknown state vector to be estimated, $u_{k-1} \in \mathbb{R}^m$ is a known control vector, $y_k \in \mathbb{R}^p$ is a measurable system output vector, $w_{k-1} \in \mathbb{R}^q$ ($q \geq n$) and $v_k \in \mathbb{R}^p$ are unobservable bounded noise vectors with unknown statistical characteristics that may include the modelling inaccuracies, the discretization errors or the computer round-off errors. v_k is more likely to represent the measurement noise and w_{k-1} can be viewed as unknown but bounded inputs. $G_{k-1} \in \mathbb{R}^{n\times q}$ is a noise matrix. The only property veri£ed by v_k and w_{k-1} are

$$v_k \in \mathcal{E}(0, V_k) \Longleftrightarrow v_k^T V_k^{-1} v_k \leq 1, \quad \forall k \in \mathbb{N}^* \quad (2a)$$
$$w_{k-1} \in \mathcal{E}(0, W_{k-1}) \Longleftrightarrow w_{k-1}^T W_{k-1}^{-1} w_{k-1} \leq 1, \ \forall k \in \mathbb{N}^* \quad (2b)$$

where $W_{k-1} \in \mathbb{R}^{q\times q}$ and $V_k \in \mathbb{R}^{p\times p}$ are known symmetric positive de£nite matrices that specify the size and the orientation of the ellipsoids containing all possible values of the noise vectors w_{k-1} and v_k respectively. W_{k-1} and V_k re¤ect known upper bounds on the unknown noise covariance matrices. These ellipsoids must obviously not be too large in comparison with the state and output vectors.

Let $\hat{x}_k \in \mathbb{R}^{n\times n}$ be the state estimate at time k. Our aim in the sequel is summarized by the following items

i. Design an estimation algorithm for the system $(1)-(2)$ that constrains the output error vector $y_k - h(\hat{x}_k, u_k)$ to reach the interior of the ellipsoid (2a) enclosing all possible values of the disturbance vectors v_k, *i.e.*, that ensures $\lim_{k\to\infty}(y_k - h(\hat{x}_k,u_k))^T V_k^{-1}(y_k - h(\hat{x}_k,u_k)) = 1$. By this way, the decrease of the estimation error $\widetilde{x}_k = x_k^* - \hat{x}_k$ will be favored;
ii. Quantify the set that contains the true state x_k^* as closely as possible;
iii. Formulate the suf£cient conditions that ensure the decrease of some parameter characterizing the size of the ellipsoid that contains the true state vector.

3 Time Update

The time updating stage consists in calculating the prediction state vector, called $\hat{x}_{k/k-1}$, obtained by the use of the available informations at the previous step time $k-1$, i.e., the estimate $\hat{x}_{k-1}$ and the control u_{k-1} :

$$\hat{x}_{k/k-1} = f(\hat{x}_{k-1}, u_{k-1})$$

In the other hand, the true state x_k^* evolves obeying to the plant dynamics described by (1a) affected by the unknown noise w_{k-1}.

In this section, we'll recall some useful results that will allow us to enclose the set containing the prediction state vector $\hat{x}_{k/k-1}$ into an ellipsoid.

Consider two ellipsoids $\mathcal{E}(c_1,P_1)$ and $\mathcal{E}(c_2,P_2)$ in $\mathbb{R}^n$. Their sum de£ned as $\mathcal{E}(c_1,P_1) \oplus \mathcal{E}(c_2,P_2) := \{x \in \mathbb{R}^n, |\, x = x_1 + x_2 : x_1 \in \mathcal{E}(c_1,P_1),\ x_2 \in \mathcal{E}(c_2,P_2)\}$ is not, in general, a regular set. The following lemma de£nes ellipsoids that contain the set $\mathcal{E}(c_1,P_1) \oplus \mathcal{E}(c_2,P_2)$.

Lemma 1. *[2] The ellipsoid $\mathcal{E}(c,P)$ where*

$$c = c_1 + c_2 \tag{3a}$$
$$P(\nu) = {P_1}/{\nu} + {P_2}/{(1-\nu)} \tag{3b}$$

contains the sum $\mathcal{E}(c_1,P_1)+\mathcal{E}(c_2,P_2)$ for all $\nu \in]0,1[$. ■

Owing to this lemma, we obtain a family $\mathcal{P}_\nu(c_1,P_1,c_2,P_2) := \{\mathcal{E}(c,P(\nu)|\, c = c_1 + c_2,\ P(\nu) = {P_1}/{\nu} + {P_2}/{(1-\nu)},\ 0<\nu<1\}$ of ellipsoids parameterized by ν among which, we should £nd the optimal one, that is, the one of the smallest size with respect to some criteria. Two kinds of measure of the size of an ellipsoid $\mathcal{E}(c,P)$ (size of the matrix P) are often considered in the literature. The £rst one, $f_1(P)$ is a function of its volume and the other one, $f_2(P)$ is related to the sum of squared semi-lengths of its axes :

$$f_1(P) = \ln\det P \tag{4a}$$
$$f_2(P) = \operatorname{tr} P \tag{4b}$$

Theorem 1. *[7] The functions f_1 and f_2 de£ned in* (4a) *and* (4b) *are strictly convex and the optimal ellipsoid $\mathcal{E}(c^*,P^*)$ bounding the set $\mathcal{E}(c_1,P_1)+\mathcal{E}(c_2,P_2)$ that minimizes either $f_1(P(\nu))$ or $f_2(P(\nu))$ is unique and belongs to $\mathcal{P}_\nu(c_1,P_1,c_2,P_2)$ and is such that*

$$c^* = c_1 + c_2$$
$$P^* = P(\nu^*)$$

where

$$\nu^* = \arg\min_{0<\nu<1} f_1(P(\nu))$$

or

$$\nu^* = \arg\min_{0<\nu<1} f_2(P(\nu)) \tag{5}$$

Furthermore, the minimization problem (5) *has explicit solution for*

$$\nu^* = \arg\min_{0<\nu<1} \operatorname{tr}\left(\frac{P_1}{\nu} + \frac{P_2}{1-\nu}\right) = \frac{\sqrt{\operatorname{tr} P_1}}{\sqrt{\operatorname{tr} P_1} + \sqrt{\operatorname{tr} P_2}} \tag{6}$$

■

For the proof of the theorem 1, we refer the reader to [7]. The optimization of determinant criterion (4a) has no explicit solution. For this reason, we will consider the trace criterion as the measure of the size of an ellipsoid, in the rest of the paper. Hereafter, let us introduce the following hypothesis

(**H1**) The nonlinear function $f(x,u_k)$ is differentiable with respect to x and its Jacobian matrix computed at $x=\xi$ is bounded for all bounded ξ.

(**H2**) The nonlinear function $f(x,u_k)$ is twice differentiable with respect to x and its n Hermitian matrices computed at $x=\xi$ are bounded for all bounded ξ.

We can now state the following lemma

Lemma 2. *Assuming (H1)–(H2), if $x_{k-1}^* \in \mathcal{E}(0,\sigma_{k-1}^2 P_{k-1})$ and if the ellipsoid $\mathcal{E}(0,\sigma_{k-1}^2 P_{k-1})$ is bounded, then there exists $\varepsilon_{k-1} \in \mathbb{R}_+^*$ such that $x_k^* \in \mathcal{E}(\hat{x}_{k/k-1}, \sigma_{k/k-1}^2 P_{k/k-1}(\mu))$ with*

$$\hat{x}_{k/k-1} = f(\hat{x}_{k-1}, u_{k-1}) \tag{7}$$
$$P_{k/k-1}(\mu) = {(F_{k-1}+\varepsilon_{k-1}I_n)\,P_{k-1}\,(F_{k-1}+\varepsilon_{k-1}I_n)^T}/{\mu} + {G_{k-1}W_{k-1}G_{k-1}^T}/{(\sigma_{k-1}^2(1-\mu))} \tag{8}$$
$$\sigma_{k/k-1}^2 = \sigma_{k-1}^2 \tag{9}$$

for all $0<\mu<1$ and the value of μ that minimizes the size of the ellipsoid $\mathcal{E}(\hat{x}_{k/k-1}, \sigma_{k/k-1}^2 P_{k/k-1}(\mu))$ according to (4b) *is given by*

$$\mu_{k-1}^* = \left(\operatorname{tr}(F_{k-1}+\varepsilon_{k-1}I_n)\,P_{k-1}\,(F_{k-1}+\varepsilon_{k-1}I_n)^T\right)^{\frac{1}{2}} \times \left[\left(\operatorname{tr}(F_{k-1}+\varepsilon_{k-1}I_n)\,P_{k-1}\,(F_{k-1}+\varepsilon_{k-1}I_n)^T\right)^{\frac{1}{2}} + \sigma_{k-1}^{-1}\left(\operatorname{tr} G_{k-1}W_{k-1}G_{k-1}^T\right)^{\frac{1}{2}}\right]^{-1} \tag{10}$$

with $F_{k-1} \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the vector f :

$$F_{k-1} := F(\hat{x}_{k-1}, u_{k-1}) = \frac{\partial f}{\partial x}(\hat{x}_{k-1}, u_{k-1}) \tag{11}$$

and

$$\varepsilon_{k-1} := \frac{1}{2}\max_{\xi,\psi\in\mathcal{E}(\hat{x}_{k-1},\sigma_{k-1}^2 P_{k-1})} \rho(\xi,\psi) \tag{12}$$

where

$$\rho(\xi,\psi) := \lambda_{\max}\begin{pmatrix}(\psi-\hat{x}_{k-1})^T\mathcal{H}_1(\xi,u_{k-1})\\(\psi-\hat{x}_{k-1})^T\mathcal{H}_2(\xi,u_{k-1})\\\vdots\\(\psi-\hat{x}_{k-1})^T\mathcal{H}_n(\xi,u_{k-1})\end{pmatrix}$$

and $\mathcal{H}_i(\xi,u_{k-1})$ is the $n\times n$ Hermitian matrix of the ith component, $f_i(x,u_{k-1})$ ($i\in\{1,2,\dots,n\}$), of the vector $f(x,u_{k-1})$ at $x=\xi\in\mathcal{E}(\hat{x}_{k-1},\sigma_{k-1}^2P_{k-1})$:

$$\mathcal{H}_i(\xi,u_{k-1}) := \left(\frac{\partial}{\partial x}\frac{\partial f_i}{\partial x}(\xi,u_{k-1})\right)^T \tag{13}$$

■

Proof.

First, we introduce the estimation and prediction error vectors

$$\widetilde{x}_{k-1} := x^*_{k-1}-\hat{x}_{k-1} \tag{14}$$

$$\begin{aligned}\widetilde{x}_{k/k-1} &:= x^*_k-\hat{x}_{k/k-1}=x^*_k-f(\hat{x}_{k-1},u_{k-1})\\&= f(x^*_{k-1},u_{k-1})+G_{k-1}w_{k-1}-f(\hat{x}_{k-1},u_{k-1})\\&= F_{k-1}\widetilde{x}_{k-1}+\varphi_{k-1}+G_{k-1}w_{k-1}\end{aligned} \tag{15}$$

where φ_{k-1} is a residual vector resulting from the first order linearization of the function f around $\hat{x}_{k-1}$:

$$\begin{aligned}\varphi_{k-1} &:= \varphi(x^*_{k-1},\hat{x}_{k-1},u_{k-1})\\&= f(x^*_{k-1},u_{k-1})-f(\hat{x}_{k-1},u_k)-F_{k-1}\widetilde{x}_{k-1}.\end{aligned}$$

The ith component ($i\in\{1,2,\dots,n\}$) of the linearization error vector φ_{k-1} can be written as

$$\varphi_{i_{k-1}}(x^*_{k-1},\hat{x}_{k-1},u_{k-1})=\tfrac{1}{2}\widetilde{x}^T_{k-1}\mathcal{H}_i(\xi,u_{k-1})\widetilde{x}_{k-1}$$

for some $\xi\in\mathcal{E}(\hat{x}_{k-1},\sigma^2_{k-1}P_{k-1})$, where $\mathcal{H}_i(\xi,u_{k-1})$ is defined in (13). This allows us to introduce a matrix $L_{k-1}\in\mathbb{R}^{n\times n}$ such that

$$\varphi_{k-1} = L_{k-1}\widetilde{x}_{k-1}$$

thus

$$f(x^*_{k-1},u_{k-1})-f(\hat{x}_{k-1},u_{k-1})=(F_{k-1}+L_{k-1})\widetilde{x}_{k-1} \tag{16}$$

and (15) becomes

$$\widetilde{x}_{k/k-1} = (F_{k-1}+L_{k-1})\widetilde{x}_{k-1}+G_{k-1}w_{k-1}$$

where

$$L_{k-1}=L(\xi,x^*_{k-1},\hat{x}_{k-1},u_{k-1}) := \frac{1}{2}\begin{pmatrix}\widetilde{x}^T_{k-1}\mathcal{H}_1(\xi,u_{k-1})\\\widetilde{x}^T_{k-1}\mathcal{H}_2(\xi,u_{k-1})\\\vdots\\\widetilde{x}^T_{k-1}\mathcal{H}_n(\xi,u_{k-1})\end{pmatrix}$$

is an unknown matrix, where $\xi\in\mathcal{E}(\hat{x}_{k-1},\sigma^2_{k-1}P_{k-1})$. At time $k-1$:

$$(x^*_{k-1}-\hat{x}_{k-1})^TP^{-1}_{k-1}(x^*_{k-1}-\hat{x}_{k-1})\le\sigma^2_{k-1}. \tag{17}$$

Taking into account only the informations available at the step time $k-1$, the ellipsoid containing the state vector x^*_k at time k, is $\mathcal{E}(\hat{x}_{k/k-1},\sigma^2_{k/k-1}P_{k/k-1})$ where we have to determine the relations between $P_{k/k-1}$ and P_{k-1} and between $\sigma_{k/k-1}$ and σ_{k-1}. On one hand, from (17) we have

$$\widetilde{x}^T_{k-1}(F_{k-1}+L_{k-1})^T\left[(F_{k-1}+L_{k-1})P_{k-1}(F_{k-1}+L_{k-1})^T\right]^{-1}\times(F_{k-1}+L_{k-1})\widetilde{x}_{k-1}\le\sigma^2_{k-1}. \tag{18}$$

Let

$$\varepsilon_{k-1} = \max_{\xi,x^*_{k-1}\in\mathcal{E}(\hat{x}_{k-1},\sigma^2_{k-1}P_{k-1})}\|L(\xi,x^*_{k-1},\hat{x}_{k-1},u_{k-1})\| \tag{19}$$

ε_{k-1} is bounded because the Hermitian matrices $\mathcal{H}_i(x)$ are bounded and the ellipsoid $\mathcal{E}(\hat{x}_{k-1},\sigma^2_{k-1}P_{k-1})$ is so. We can rewrite (19) as

$$\varepsilon_{k-1} = \max_{\xi,x^*_{k-1}\in\mathcal{E}(\hat{x}_{k-1},\sigma^2_{k-1}P_{k-1})}\max_{x\in\mathbb{R}^n}\left(\frac{x^TL^T_{k-1}L_{k-1}x}{x^Tx}\right)^{\frac{1}{2}} \tag{20}$$

(20) implies that for all $x\in\mathbb{R}^n$

$$\begin{aligned}&x^T\left((F_{k-1}+L_{k-1})P_{k-1}(F_{k-1}+L_{k-1})^T\right)x\\&\le x^T\left((F_{k-1}+\varepsilon_{k-1}I_n)P_{k-1}(F_{k-1}+\varepsilon_{k-1}I_n)^T\right)x.\end{aligned} \tag{21}$$

By the use of (21), (18), (16) and (7), we can write that

$$f(x^*_{k-1},u_{k-1})\in\mathcal{E}\left(\hat{x}_{k/k-1},\sigma^2_{k-1}\Phi_{k-1}\right).$$

where $\Phi_{k-1}=(F_{k-1}+\varepsilon_{k-1}I_n)P_{k-1}(F_{k-1}+\varepsilon_{k-1}I_n)^T$. Posing $\Gamma_{k-1}=G_{k-1}W_{k-1}G^T_{k-1}$, from (2b), we also have

$$G_{k-1}w_{k-1}\in\ \mathcal{E}(0,\Gamma_{k-1}).$$

Now, we have to express the ellipsoid enclosing the set $\mathcal{S}_{k/k-1}$ that contains all possible values of x^*_k :

$$\begin{aligned}\mathcal{S}_{k/k-1} := \{x\in\mathbb{R}^n|\ &x=x_1+x_2,\ x_1\in\mathcal{E}(0,\Gamma_{k-1}),\\&x_2\in\mathcal{E}\left(\hat{x}_{k/k-1},\sigma^2_{k-1}\Phi_{k-1}\right)\}.\end{aligned}$$

For this purpose, we use the *Lemma* 1 and the *Theorem* 1. Thus, using (1a), we can write :

$$\left(x^*_k-\hat{x}_{k/k-1}\right)^T\left(\frac{\sigma^2_{k-1}\Phi_{k-1}}{\mu}+\frac{\Gamma_{k-1}}{1-\mu}\right)^{-1}\left(x^*_k-\hat{x}_{k/k-1}\right)\le 1.$$

This proves (7)−(9) of the *Lemma* 2. The obtention of (10) is straightforward by using (5)−(6). ❑

4 Observation Update

In this section, we update the state prediction $\hat{x}_{k/k-1}$ by taking into account the measurement information at step k in order to obtain the estimate $\hat{x}_k$. We assume that

(H3) The nonlinear function $h(z,u_k)$ is differentiable with respect to z and its Jacobian matrix computed at $z=\zeta$ is bounded for all bounded ζ.

For the estimation of the state vector x^*_k of the system (1a)−(1b), we use the Extended Kalman Filter structure derived from [9] :

$$\hat{x}_k=\hat{x}_{k/k-1}+K_k\delta_k \tag{22}$$

$$K_k=P_{k/k-1}H^T_k(H_kP_{k/k-1}H^T_k+\lambda\Lambda^{-1}_k)^{-1} \tag{23}$$

$$P_k=\tfrac{1}{\lambda}(I_n-K_kH_k)P_{k/k-1} \tag{24}$$

where

$$\delta_k=y_k-h(\hat{x}_{k/k-1},u_k), \tag{25}$$

$$H_k=\tfrac{\partial h}{\partial x}(\hat{x}_{k/k-1},u_k) \tag{26}$$

and $0<\lambda<1$ is a forgetting factor (that could be time varying) to be fixed by the user and Λ_k is a weighing matrix that we have to define and on which all the algorithm strategy is built. Notice that, the prediction parameters $\hat{x}_{k/k-1}$ and $P_{k/k-1}$ are computed in previous section. $\hat{x}_{k/k-1}$ is given by (7) and $P_{k/k-1}=P_{k/k-1}(\mu^*_{k-1})$ where $P_{k/k-1}(\mu)$ and μ^*_{k-1} are defined in (8) and (10).

Lemma 3. *Assuming (H1)-(H3), if x^*_{k-1} belongs to a bounded ellipsoid $\mathcal{E}(0,\sigma^2_{k/k-1}P_{k/k-1})$, then*

i. $\exists\epsilon_k\in\mathbb{R}^*_+$: $\forall x^*_k\in\mathcal{E}(0,\sigma^2_{k/k-1}P_{k/k-1})$,

$$\left\|h(x^*_k,u_k)-h(\hat{x}_{k/k-1},u_k)-H_k\widetilde{x}_{k/k-1}\right\|\le\epsilon_k \tag{27}$$

ii. $x^*_k\in\mathcal{E}(\hat{x}_k,\sigma^2_kP_k)$

where $\hat{x}_k$, P_k are defined in (22), (24) *and σ_k is given by*

$$\sigma^2_k=\lambda\sigma^2_{k/k-1}+\|\Lambda_kR_k\|-\lambda\delta^T_k\left(H_kP_{k/k-1}H^T_k+\lambda\Lambda^{-1}_k\right)^{-1}\delta_k \tag{28}$$

and where $R_k\in\mathbb{R}^{p\times p}$ is given by

$$R_k=\left(1+\epsilon_k\sqrt{p}/\sqrt{\operatorname{tr}V_k}\right)V_k+\left(1+\sqrt{\operatorname{tr}V_k}/(\epsilon_k\sqrt{p})\right)\epsilon^2_kI_p \tag{29}$$

■

Proof.

i. As $\mathcal{E}(0,\sigma^2_{k/k-1}P_{k/k-1})$ is bounded, we can always find a time-varying scalar ϵ_k which is all the smaller as $\mathcal{E}(0,\sigma^2_{k/k-1}P_{k/k-1})$ is.

ii. Let us consider the following Lyapunov function

$$\mathcal{V}_k := \widetilde{x}^T_kP^{-1}_k\widetilde{x}_k \tag{30}$$

where $\widetilde{x}_k$ is defined in (14). Using (24) and (23) and after some routine algebra, the following relations are obvious

$$K_k = P_k H_k^T \Lambda_k \quad (31)$$
$$P_k^{-1} = \lambda P_{k/k-1}^{-1} + H_k^T \Lambda_k H_k. \quad (32)$$

Substituting the equation (31) in (22) and the latter in (30) yields

$$\mathcal{V}_k = \left(\tilde{x}_{k/k-1} - P_k H_k^T \Lambda_k \delta_k\right)^T P_k^{-1} \left(\tilde{x}_{k/k-1} - P_k H_k^T \Lambda_k \delta_k\right) (33)$$

Using (32) and (25), (33) becomes

$$\mathcal{V}_k = \lambda \mathcal{V}_{k/k-1} + \delta_k^T \Lambda_k \left(H_k P_k H_k^T - \Lambda_k^{-1}\right) \Lambda_k \delta_k + \left(\delta_k - H_k \widetilde{x}_{k/k-1}\right)^T \Lambda_k \left(\delta_k - H_k \widetilde{x}_{k/k-1}\right) \quad (34)$$

where $\mathcal{V}_{k/k-1} = \widetilde{x}_{k/k-1}^T P_{k/k-1}^{-1} \widetilde{x}_{k/k-1}$. By the aid of (14), (25) and the output equation (1b), it comes that

$$\delta_k - H_k \widetilde{x}_{k/k-1} = v_k + \chi_k \quad (35)$$

where χ_k is a residual vector resulting from the £rst order linearization of the function h around $\hat{x}_{k/k-1}$:

$$\chi_k := h(x_k^*, u_k) - h(\hat{x}_{k/k-1}, u_k) - H_k \widetilde{x}_{k/k-1}.$$

Using (24), (23) and the matrix inversion lemma, we obtain

$$H_k P_k H_k^T - \Lambda_k^{-1} = -\lambda \Lambda_k^{-1} \left(H_k P_{k/k-1} H_k^T + \lambda \Lambda_k^{-1}\right)^{-1} \Lambda_k^{-1} \quad (36)$$

By inserting (35) and (36) in (34), we £nd

$$\mathcal{V}_k = \lambda \mathcal{V}_{k/k-1} - \lambda \delta_k^T \left(H_k P_{k/k-1} H_k^T + \lambda \Lambda_k^{-1}\right)^{-1} \delta_k + \left(v_k + \chi_k\right)^T \Lambda_k \left(v_k + \chi_k\right). \quad (37)$$

Still applying the *Theorem* 1 to the ellipsoids de£ned in (2a) and (27) that enclose the measurement and the linearization error vectors respectively, we can write

$$\left(v_k + \chi_k\right)^T \left(V_k/\nu_k + \epsilon_k^2 I_p/(1-\nu_k)\right)^{-1} \left(v_k + \chi_k\right) \le 1$$

with

$$\nu_k = \sqrt{\operatorname{tr} V_k} / \left(\sqrt{\operatorname{tr} V_k} + \epsilon_k \sqrt{p}\right)$$

here, we obtain the expression (29) of R_k such that

$$\left(v_k + \chi_k\right)^T R_k^{-1} \left(v_k + \chi_k\right) \le 1. \quad (38)$$

We are able, at this stage, to de£ne σ_k :

$$\sigma_k^2 := \sup_{\substack{\tilde{x}_{k-1} \in \mathcal{E}(0, \sigma_{k-1}^2 P_{k-1}) \\ w_{k-1} \in \mathcal{E}(0, W_{k-1}),\ v_k \in \mathcal{E}(0, V_k) \\ \varphi_{k-1} \in \mathcal{E}(0, \varepsilon_{k-1}^2 \sigma_{k-1}^2 P_{k-1}),\ \chi_k \in \mathcal{E}(0, \epsilon_k^2 I_p)}} \mathcal{V}_k \quad (39)$$

Using the de£nitions of the ellipsoids $\mathcal{E}(0, \sigma_{k-1}^2 P_{k-1})$ and $\mathcal{E}(0, \sigma_{k/k-1}^2 P_{k/k-1})$ containing $\widetilde{x}_{k-1}$ and $\widetilde{x}_{k/k-1}$ respectively, and (39), we can write the following

$$\sup_{\substack{\tilde{x}_{k-1} \in \mathcal{E}(0, \sigma_{k-1}^2 P_{k-1}),\ w_{k-1} \in \mathcal{E}(0, W_{k-1}) \\ \varphi_{k-1} \in \mathcal{E}(0, \varepsilon_{k-1}^2 \sigma_{k-1}^2 P_{k-1})}} \mathcal{V}_{k/k-1} = \sigma_{k-1}^2 \quad (40)$$

Afterwards, by the aid of (37), (39)-(40), we £nd the following recursion law for σ_k^2

$$\begin{aligned}\sigma_k^2 &= \max_{\mathcal{V}_{k/k-1} \le \sigma_{k/k-1}^2, v_k \in \mathcal{E}(0, V_k), \chi_k \in \mathcal{E}(0, \epsilon_k^2 I_p)} \mathcal{V}_k \\ &= \lambda \sigma_{k/k-1}^2 - \lambda \delta_k^T \Lambda_k \left(H_k P_{k/k-1} H_k^T \Lambda_k + \lambda I\right)^{-1} \delta_k \\ &+ \max_{v_k + \chi_k \in \mathcal{E}(0, R_k)} \left(v_k + \chi_k\right)^T \Lambda_k \left(v_k + \chi_k\right). \end{aligned} \quad (41)$$

Now, setting

$$v_k + \chi_k = \bar{R}_k \bar{r}_k \quad (42)$$

where $\bar{R}_k \bar{R}_k^T = R_k$, it comes from (38) and (42) that

$$\left(v_k + \chi_k\right)^T R_k^{-1} \left(v_k + \chi_k\right) = \bar{r}_k^T \bar{r}_k \le 1$$

and

$$\begin{aligned}\max_{v_k + \chi_k \in \mathcal{E}(0, R_k)} \left(v_k + \chi_k\right)^T \Lambda_k \left(v_k + \chi_k\right) &= \max_{\|\bar{r}_k\| \le 1} \bar{r}_k^T \bar{R}_k^T \Lambda_k \bar{R}_k \bar{r}_k \\ = \max_{\|\bar{r}_k\| = 1} \bar{r}_k^T \bar{r}_k \frac{\bar{r}_k^T \bar{R}_k^T \Lambda_k \bar{R}_k \bar{r}_k}{\bar{r}_k^T \bar{r}_k} &= \left\|\bar{R}_k^T \Lambda_k \bar{R}_k\right\| = \|\Lambda_k R_k\|. \end{aligned} \quad (43)$$

Finally, the substitution of (43) in (41) yields to (28). ❑

5 Main Result

First, we consider the system (1)−(2), where the functions f and h are linearized around the state estimate $\hat{x}_k$ and the state prediction $\hat{x}_{k/k-1}$ respectively :

$$x_k^* = F_{k-1} x_{k-1}^* + G_{k-1} w_{k-1} \quad (43a)$$
$$y_k = H_k x_k^* + v_k \quad (43b)$$

and we de£ne the state transition matrix $\phi_{(r,s)}$ with $r > s$ of the system (43a)−(43b) as follows

$$\phi_{(r,s)} = F_{r-1} F_{r-2} \ldots F_s, \quad \phi_{(s,s)} = I_n$$

Now, let us introduce the following additional assumptions :

(H4) The initial true state vector x_0^* belongs to a known bounded suf£ciently small ellipsoid :

$$x_0^* \in \mathcal{E}(\hat{x}_0, \sigma_0^2 P_0) \iff (x_0^* - \hat{x}_0)^T P_0^{-1} (x_0^* - \hat{x}_0) \le \sigma_0^2$$

(H5) The system (43a)−(43b) is N-locally observable, *i.e.*, there exists a £nite integer $N > 0$ and two positive real numbers α and β such that for all $k \ge 1$

$$\alpha I_n \le \sum_{i=k-N}^{k} \phi_{(i,k-N)}^T H_i^T \Lambda_i H_i \phi_{(i,k-N)} \le \beta I_n$$

for all $\hat{x}_k$, $\hat{x}_{k/k-1} \in \mathcal{X}_k$ (a neighborhood of x_k^*) and for all M-tuple of input vectors $U_{(k-N,k-1)} = (u_{k-N}, u_{k-N+1} \ldots, u_{k-1}) \in \mathcal{U}$ ($k \in \mathbb{N}^*$), where $\mathcal{X}$ and $\mathcal{U}$ are compact subsets in $\mathbb{R}^n$ and $\mathbb{R}^{m \times N}$ respectively.

(H5) is used to get rid of the conditions that we made at the *Lemmas* 2 and 3 about the boundedness of $\mathcal{E}(0, \sigma_{k-1}^2 P_{k-1})$ and $\mathcal{E}(0, \sigma_{k/k-1}^2 P_{k/k-1})$. We also need **(H4)** in order to avoid high linearization errors and consequently big values for ε_{k-1} and ϵ_k de£ned in (12) and (27). Before we enunciate our main result, let us introduce the following de£nitions

De£nition 1. *The stay-time of a vector $z_k \in \mathbb{R}^r$ in a subset $\mathcal{S}$ of $\mathbb{R}^r$ is the interval $T = \{l, l+1, \ldots, m-1, m\}$, $(l < m)$ of consecutive samples, such that $z_k \in \mathcal{S}$ for all $k \in T$.*

De£nition 2. *All the stay-times of the vector z_k in the set $\mathcal{S}$ are £nite (or z_k have no in£nite stay-time in $\mathcal{S}$), if and only if, for each integer k_0 for which $z_{k_0} \in \mathcal{S}$, there exists a £nite integer τ such that $z_{k_0+\tau} \notin \mathcal{S}$.*

Now, we decompose the $\mathbb{R}^p$ space spanned by the vector δ_k into three regions :

$$\mathcal{D}_1^\delta := \bar{\mathcal{E}}(0, R_k) \cap \bar{\mathcal{E}}(0, Q_k) \quad (44a)$$
$$\mathcal{D}_2^\delta := \bar{\mathcal{E}}(0, R_k) \cap \mathcal{E}(0, Q_k) \quad (44b)$$
$$\mathcal{D}_3^\delta := \mathcal{E}(0, R_k) \quad (44c)$$

where $Q_k = \|\delta_k\|_{R_k^{-1}} \left\| \left(H_k P_{k/k-1} H_k^T\right)^{-1} R_k \right\| H_k P_{k/k-1} H_k^T$. We also decompose the ellipsoid $\mathcal{E}(0, R_k)$ enclosing all the vector sum of the measurement noise v_k and the linearization error χ_k of the function h around $\hat{x}_{k/k-1}$, into two regions :

$$\mathcal{D}_1^{v+\chi} := \mathcal{E}(0, R_k) \cap \mathcal{E}(0, S_k) \quad (45a)$$
$$\mathcal{D}_2^{v+\chi} := \mathcal{E}(0, R_k) \cap \bar{\mathcal{E}}(0, S_k) \quad (45b)$$

where $S_k = \|\delta_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} H_k P_{k/k-1} H_k^T / \|\delta_k\|_{R_k^{-1}}$.

Theorem 2. *If (H1) and (H3)−(H5) are satis£ed, then the state estimation algorithm for the system (1) − (2) de£ned by (7)−(10), (11), (22)−(26) and (28)−(29), where ε_{k-1} and ϵ_k are positive real numbers such that*

$\forall \xi \in \mathcal{E}(\hat{x}_{k-1}, \sigma^2_{k-1}P_{k-1})$, $\forall \zeta \in \mathcal{E}(\hat{x}_{k/k-1}, \sigma^2_{k/k-1}P_{k/k-1})$, $\|f(\xi, u_{k-1}) - f(\hat{x}_{k-1}, u_{k-1}) - F_{k-1}(\xi - \hat{x}_{k-1})\| \le \varepsilon_{k-1}\|\xi - \hat{x}_{k-1}\|$ *and* $\left\|h(\zeta, u_k) - h(\hat{x}_{k/k-1}, u_k) - H_k(\zeta - \hat{x}_{k/k-1})\right\| \le \epsilon_k$ *guaranties that*

i. the ellipsoid $\mathcal{E}(\hat{x}_k, \sigma^2_k P_k)$ *contains all possible values of* x^*_k *for all* $k \in \mathbb{N}^*$*;*

Furthermore, if the weighting matrix Λ_k *of the estimation gain matrix* (23) *is de£ned by*

$$\Lambda_k = \begin{cases} \lambda \left(\|\delta_k\|_{R_k^{-1}} - 1\right) \left(H_k P_{k/k-1} H_k^T\right)^{-1} & \text{if } \|\delta_k\|_{R_k^{-1}} > 1 \\ & \text{and } H_k P_{k/k-1} H_k^T > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

then, the above mentioned algorithm also guaranties that

ii. the output error $y_k - H_k\hat{x}_k$ *of the linearized measurement equation* (43b) *belongs to the ellipsoid* $\mathcal{E}(0, R_k)$ *that contains the vector sum of the measurement noise and the linearization error vectors :*

$$(y_k - H_k\hat{x}_k)^T R_k^{-1} (y_k - H_k\hat{x}_k) \le 1, \quad (47)$$

$$\lim_{\|\delta_k\| \to 0} R_k = V_k; \quad (48)$$

iii. $\forall w_{k-1} \in \mathcal{E}(0, W_{k-1})$, $\forall v_k \in \mathcal{E}(0, V_k)$ *: if* $\delta_k \in \mathcal{D}^\delta_1$*, then* $\sigma^2_k < \lambda\sigma^2_{k-1}$ *and if* $\delta_k \in \mathcal{D}^\delta_1$*, then* $\sigma^2_k = \lambda\sigma^2_{k-1}$*;*

iv. if $\delta_k \in \mathcal{D}^\delta_2$*, then* $\exists \bar{\sigma}_k \in \mathbb{R}^*_+$, $\bar{\sigma}^2_k < \lambda\bar{\sigma}^2_{k-1}$ *and* $x^*_k \in \mathcal{E}(\hat{x}_k, \bar{\sigma}^2_k P_k) \subseteq \mathcal{E}(\hat{x}_k, \sigma^2_k P_k)$ *provided that* $v_k + \chi_k \in \mathcal{D}^{v+\chi}_1$*;*

v. $\forall w_{k-1} \in \mathcal{E}(0, W_{k-1})$, $\forall v_k \in \mathcal{E}(0, V_k)$*, the ellipsoid* $\mathcal{E}(\hat{x}_k, \sigma^2_k P_k)$ *is bounded for all* $k \in \mathbb{N}^*$ *provided that the innovation vector* δ_k *have no in£nite stay-time in* $\mathcal{D}^\delta_2$*.*

where $\mathcal{D}^\delta_{i\in\{1,2,3\}}$ *are respectively de£ned in* (44a), (44b) *and* (44c) *and* $\mathcal{D}^{v+\chi}_{j\in\{1,2\}}$ *are de£ned in* (45a) *and* (45b). ■

Proof.

i. This property comes from *Lemmas* 2 and 3 and the hypothesis (**H1**)$-$(**H5**).

ii. We rewrite the linearized output error after some linear manipulations using (24) and (23) :

$$y_k - H_k\hat{x}_k = \lambda \left(H_k P_{k/k-1} H_k^T \Lambda_k + \lambda I_p\right)^{-1} \delta_k \quad (49)$$

and now, consider the weighted norm of the linearized output error intervening in (47), in which we replace (49) and (46) :

$$(y_k - H_k\hat{x}_k)^T R_k^{-1} (y_k - H_k\hat{x}_k) = \begin{cases} \|\delta_k\|^2_{R_k^{-1}} & \text{if } \|\delta_k\|_{R_k^{-1}} \le 1, \\ 1 & \text{otherwise.} \end{cases}$$

and the result (47) follows. As for (48), it comes straightforwardly from the expression of R_k in (29) and the ability of the upper bound of the linearization error ϵ_k to go to zero when the estimation error goes to zero that is, when the output error goes to zero (because the system is assumed to be observable).

iii. From (44a), we have $\delta_k \in \mathcal{D}^\delta_1 \Leftrightarrow \|\delta_k\|_{R_k^{-1}} \le 1$

and $\|\delta_k\|_{\left(H_k P_{k/k-1} H_k^T\right)_k^{-1}} > \|\delta_k\|_{R_k^{-1}} \left\|\left(H_k P_{k/k-1} H_k^T\right)^{-1} R_k\right\|$.

If $\|\delta_k\|_{R_k^{-1}} > 1$, σ_k de£ned in (28) is rewritten by the mean of (9) and (46) as follows

$$\sigma^2_k = \lambda\sigma^2_{k-1} + \lambda \left(\|\delta_k\|_{R_k^{-1}} - 1\right) \times \left(\left\|\left(H_k P_{k/k-1} H_k^T\right)^{-1} R_k\right\| - \|\delta_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} / \|\delta_k\|_{R_k^{-1}}\right).$$

So, it is clear that when $\delta_k \in \mathcal{D}^\delta_1$, $\sigma^2_k < \lambda\sigma^2_{k-1}$. It is also clear that if $\delta_k \in \mathcal{D}^\delta_3$, that is if $\|\delta_k\|_{R_k^{-1}} \le 1$, then, using (9) and (46) again, we have $\sigma^2_k = \lambda\sigma^2_{k-1}$.

iv. For studying the case when $\delta_k \in \mathcal{D}^\delta_2$ we have to reconsider the de£nition of σ^2_k given in (39). In this case, we can not maximize $\mathcal{V}_k$ with respect to the vector $v_k + \chi_k$ when it spans all the ellipsoid $\mathcal{E}(0, R_k)$ because $\sigma^2_k - \lambda\sigma^2_{k-1}$ will be positive. The subset of the ellipsoid $\mathcal{E}(0, R_k)$ containing $v_k + \chi_k$, in which $\bar{\sigma}^2_k - \lambda\bar{\sigma}^2_{k-1} < 0$ is, therefore, of interest. We can, subsequently, rede£ne $\bar{\sigma}_k$ as the maximum of $\mathcal{V}_k$ on this subset. First, let us rewrite $\mathcal{V}_k$ by replacing (46) in (37) for $\|\delta_k\|_{R_k^{-1}} > 1$:

$$\mathcal{V}_k = \lambda\mathcal{V}_{k/k-1} + \lambda \left(\|\delta_k\|_{R_k^{-1}} - 1\right) \times \left(\|v_k + \chi_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} - \|\delta_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} / \|\delta_k\|_{R_k^{-1}}\right).$$

Now, following the same reasoning we had when we deduced the expression (28) of σ^2_k, we obtain the following recursion law for $\bar{\sigma}^2_k$

$$\bar{\sigma}^2_k - \lambda\bar{\sigma}^2_{k-1} = \mathcal{V}_k - \lambda\mathcal{V}_{k/k-1}.$$

Next, putting $r_k = v_k + \chi_k$, from (45a), it is clear that a suf£cient condition for $\bar{\sigma}^2_k - \lambda\bar{\sigma}^2_{k-1} < 0$ is $\|r_k\|_{R_k^{-1}} \le 1$ and $\|\delta_k\|_{R_k^{-1}} \|r_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} / \|\delta_k\|^2_{\left(H_k P_{k/k-1} H_k^T\right)^{-1}} \le 1$, which amounts to $r_k \in \mathcal{D}^{v+\chi}_1$. On the other hand, as the vector $v_k + \chi_k$ actually belongs to a set smaller than $\mathcal{E}(0, R_k)$ ($\mathcal{D}^{v+\chi}_1 \subset \mathcal{E}(0, R_k)$), the following holds

$$\bar{\sigma}^2_k = \max_{\tilde{x}_{k/k-1} \in \mathcal{E}(0, \bar{\sigma}^2_{k-1} P_{k/k-1}),\ v_k + \chi_k \in \mathcal{D}^{v+\chi}_1} \mathcal{V}_k \le \max_{\tilde{x}_{k/k-1} \in \mathcal{E}(0, \sigma^2_{k-1} P_{k/k-1}),\ v_k + \chi_k \in \mathcal{E}(0, R_k)} \mathcal{V}_k = \sigma^2_k$$

and consequently $\mathcal{E}(0, \bar{\sigma}^2_k P_k) \subset \mathcal{E}(0, \sigma^2_k P_k)$.

v. The hypothesis (**H5**) guaranties that the matrix P_k is bounded [10], that is there exists two positive scalars $\underline{p}$ and $\overline{p}$ such that

$$\underline{p} I_n \le P_k \le \overline{p} I_n$$

And by virtue of iii of *Theorem* 2, we have also the decrease of σ^2_k when $\delta_k \in \mathcal{D}^\delta_1 \cup \mathcal{D}^\delta_3$. The only chance for σ^2_k to become unbounded occurs if δ_k stays during an in£nite time in $\mathcal{D}^\delta_2$. Otherwise, the ellipsoid $\mathcal{E}(\hat{x}_k, \sigma^2_k P_k)$ is bounded for all k and all noise vectors. This complets the proof of the theorem. ❑

6 Illustrative Example

The numerical example that we consider in this section is a £fth-order two-phase nonlinear model of an induction motor which was already the subject of a large number of applications, especially in control designs (see [11] and the references inside). It could be mentioned that, unlike most of the works on induction motors where the rotor speed is assumed to be known, only the stator currents are needed to provide an estimate of both rotor ¤uxes and angular speed.

Using an Euler discretization of step size h, the complete discrete-time model in stator £xed (a, b) reference frame is given by :

$$x^*_{1k+1} = x^*_{1k} + h(-\gamma x^*_{1k} + \frac{K}{T_r} x^*_{3k} + Kp x^*_{5k} x^*_{4k} + \frac{1}{\sigma L_s} u_{1k}) + w_{1k}$$
$$x^*_{2k+1} = x^*_{2k} + h(-\gamma x^*_{2k} - Kp x^*_{5k} x^*_{3k} + \frac{K}{T_r} x^*_{4k} + \frac{1}{\sigma L_s} u_{2k}) + w_{2k}$$
$$x^*_{3k+1} = x^*_{3k} + h(\frac{M}{T_r} x^*_{1k} - \frac{1}{T_r} x^*_{3k} - p x^*_{5k} x^*_{4k}) + w_{3k}$$
$$x^*_{4k+1} = x^*_{4k} + h(\frac{M}{T_r} x^*_{2k} + p x^*_{5k} x^*_{3k} - \frac{1}{T_r} x^*_{4k}) + w_{4k}$$
$$x^*_{5k+1} = x^*_{5k} + h(\frac{pM}{JL_r}(x^*_{3k} x^*_{2k} - x^*_{4k} x^*_{1k}) - \frac{T_L}{J}) + w_{5k}$$
$$y_{1k} = x^*_{1k} + v_{1k}, \quad y_{2k} = x^*_{2k} + v_{2k}$$

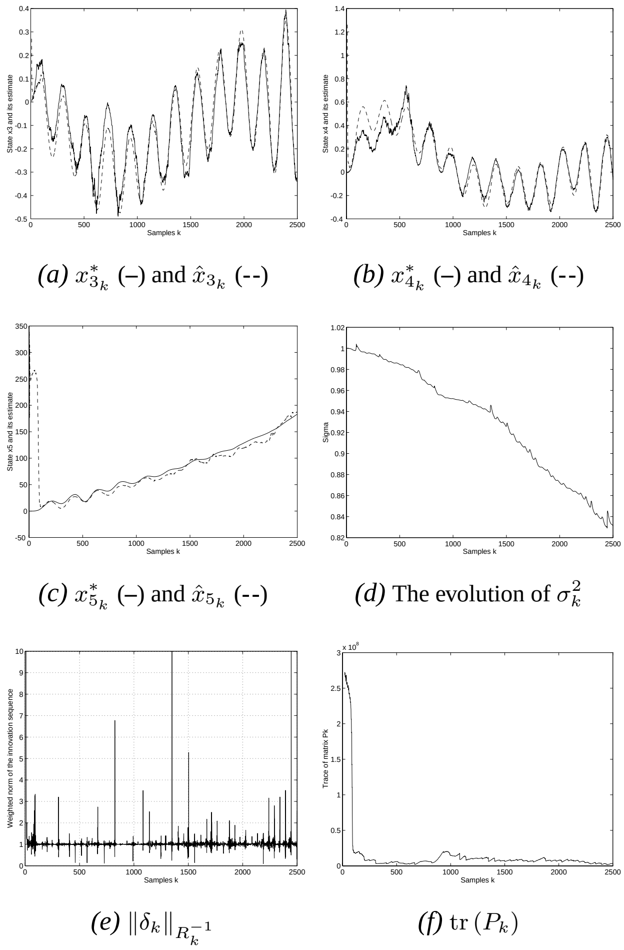
where $x^*_k = \begin{pmatrix} x^*_{1k} & x^*_{2k} & x^*_{3k} & x^*_{4k} & x^*_{5k} \end{pmatrix}^T = \begin{pmatrix} i_{sak} & i_{sbk} & \phi_{rak} & \phi_{rbk} & \omega_k \end{pmatrix}^T$ represents the stator currents, the rotor ¤uxes and the angular speed respectively,

$u_k^T = \begin{pmatrix} u_{1k} & u_{2k} \end{pmatrix} = \begin{pmatrix} u_{sak} & u_{sbk} \end{pmatrix}$ is the stator voltages control vector, p is the number of pair of poles, T_L is the load torque and h is the sampling period. The parameters T_r, σ, K and γ are de£ned as $T_r = \frac{L_r}{R_{rN}}$, $\sigma = 1 - \frac{M^2}{L_s L_r}$, $K = \frac{M}{\sigma L_s L_r}$, $\gamma = \frac{R_s}{\sigma L_s} + \frac{R_r M^2}{\sigma L_s L_r^2}$ where R_s, R_{rN} denote stator and rotor resistances; L_s, L_r are stator and rotor inductances and J is the rotor moment of inertia.

Simulations are performed using the same numerical values as in [11] : $R_s = 0.18\ \Omega$, $R_{rN} = 0.15\ \Omega$, $L_s = 0.0699$ H, $L_r = 0.0699$ H, $M = 0.068$ H, $J = 0.0586$ kgm^2, $p = 1$, $T_L = 0$ Nm.

The input signals are : $u_{sak} = 220\cos(314kh)$ and $u_{sbk} = 220\sin(314kh)$. The noises vectors w_{k-1} and v_k are generated in such a way as to verify $w_{k-1}^T W_{k-1}^{-1} w_{k-1} \leq 1$ and $v_k^T V_k^{-1} v_k$, where $W_{k-1} = 0.05 \underset{i\in\{1,\dots,5\}}{\text{diag}} (x_{ik-1}^2)$ and $V_k = 0.05 \underset{j\in\{1,2\}}{\text{diag}} (y_{jk}^2)$. The bounds on the linearization errors are chosen as $\varepsilon_k = \epsilon_k = 10^{-3}\left(\|\delta_k\|_{R_k^{-1}} - 1\right)^2 / \|\delta_k\|_{R_k^{-1}}$. The forgetting factor λ is £xed to 1. The initial conditions are : $\hat{x}_0 = (200\ 200\ 50\ 50\ 300)^T$, $P_0 = 10^6$ and $\sigma_0 = 1$ while the actual initial state vector is : $x_0^* = (0\ 0\ 0\ 0\ 0)^T$.

Figures (1(a))-(1(f)) show clearly the satisfying performances of the proposed observer to track the true state with unknown bounded noises, without the need of the rotor speed measurement and even with bad initialisations (the transients were skipped). We can see, for instance, that σ_k^2 is mostly decreasing on the estimation horizon. Its infrequent and very small growths however is due to the few and brief presence of δ_k in the set $\mathcal{D}_2^\delta$. We also notice that the weighted norm of the innovation sequence is very often close to 1 so the objective of the algorithm is achieved in a way.



(a) $x_{3_k}^*$ (–) and $\hat{x}_{3_k}$ (--)   *(b)* $x_{4_k}^*$ (–) and $\hat{x}_{4_k}$ (--)

(c) $x_{5_k}^*$ (–) and $\hat{x}_{5_k}$ (--)   *(d)* The evolution of σ_k^2

(e) $\|\delta_k\|_{R_k^{-1}}$ *(f)* $\mathrm{tr}\,(P_k)$

Figure 1: The simulaiton results

7 Conclusion

A recursive state bounding technique for nonlinear systems has been presented. The objective of this algorithm was to determine, at each sample time, an ellipsoid that encloses the true state and which is compatible with the bounds on the noises and the linearization errors. As Kalman £ltering, the algorithm has been decomposed into *time updating* and *observation updating* steps. During the time update stage, an ellipsoid that encloses, as "tightly" as possible, the vector sum of two ellipsoids, one containing the true state of the previous sample and the other, the state noises. The observation update step consists in calculating the state estimate taking into consideration the current measurement. It was shown how to design some weighting matrix such that the output error could be as closer as possible to the ellipsoid containing the measurement noise. Convergence problems have been highlighted and suf£cient conditions for acceptable tracking performances has been given.

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