# GLOBAL TRACKING FOR A CLASS OF NONLINEAR SYSTEMS SUBJECT TO UNKNOWN SINUSOIDAL DISTURBANCES

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## Abstract

The widely studied class of minimum phase observable nonlinear systems with output dependent nonlinearities is considered. The problem of tracking any smooth bounded output reference signal is addressed, when the system is perturbed by additive disturbances generated by an unknown stable linear exosystem whose order is known. An output feedback solution is presented which achieves asymptotic tracking for any initial condition of the closed loop system and of the exosystem, without requiring any persistency of excitation condition.

## 1 Introduction

The class of nonlinear observable systems considered in this paper is modeled by the equations

$$\begin{aligned} \dot{x} &= A_c x + \phi(y) + bu + Gw, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \\ \dot{w} &= Sw, \quad w \in \mathbb{R}^r \\ y &= C_c x, \qquad y \in \mathbb{R} \end{aligned} \tag{1}$$

in which the triple $(A_c, b, C_c)$ is in observer canonical form

$$\begin{aligned} A_c &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ b_\rho \\ \vdots \\ b_n \end{bmatrix} \\ C_c &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{2}$$

with $b$ a Hurwitz vector of degree $\rho$ ($\rho$ is the relative degree), i.e. all the zeros of the polynomial $b_\rho s^{n-\rho} + \cdots + b_{n-1}s + b_n$ have negative real part (without loss of generality we assume $b_\rho = 1$); $\phi(y) = [\phi_1(y), \ldots, \phi_n(y)]^T$ is a known vector of smooth functions; the matrices $S$ and $G$ are unknown (only the dimension $r$ of the exosystem is supposed to be known) but are such that the overall system is observable, i.e. the pair

$$\begin{bmatrix} A_c & G \\ 0 & S \end{bmatrix}, \qquad \begin{bmatrix} C_c & 0 \end{bmatrix} \tag{3}$$

is observable and $S$ is a stable matrix. This hypothesis allows for additive sinusoidal disturbances: in this case, the eigenvalues of $S$ are on the imaginary axis.

This class of observable nonlinear systems has been widely studied when there are no disturbances ($w = 0$): geometric conditions for the existence of a state diffeomorphism transforming a system into (1) with $w = 0$ are given in [**?**]; it is shown in [**?**] how to track any given smooth bounded output reference signal $y_r$ with bounded derivatives $y_r^{(1)}, \ldots, y_r^{(\rho)}$ globally and exponentially by output feedback; it is shown in [**?**] how to achieve robust output feedback stabilization and set point regulation; it is shown in [**?**] how to design a robust regulation when the control which solves the regulator equations is immersible into a linear observable system. In [**?**] arbitrary disturbance attenuation by output feedback is obtained for a class of nonlinear systems with output dependent nonlinearities and additive disturbances. The problem of rejecting unknown sinusoidal disturbances has been recently addressed in several papers ([**?**, **?**, **?**, **?**]): linear stable systems are considered in [**?**]; minimum phase linear systems with unknown parameters are studied in [**?**]; in [**?**] the class of systems (1) is addressed with the aim of driving the output to zero; in [**?**] the same class with uncertain parameters is considered and the regulation problem is solved; the regulator problem for linear systems (not necessarily minimum phase) is solved in [**?**].

In this paper, we allow for additive disturbances $w$ in (1) generated by an unknown stable linear exosystem whose order is known and pose the tracking problem by output feedback; we would like to reject asymptotically the disturbance $w$ and to track asymptotically a given smooth bounded output reference signal while in [**?**] the reference output is zero and in [**?**] it is generated by an exosystem. We will show that by using appropriate filtered transformations the problem may be recasted into an equivalent one for which nonlinear adaptive techniques apply, so that output tracking can be achieved globally without requiring any persistency of excitation condition.

## 2 Main result

In this section the following problem is addressed and solved.

**Definition 2.1** *The global tracking problem with disturbance rejection is said to be solvable for system (1), if there exists an output feedback control such that for any given smooth bounded output reference signal $y_r(t)$ with bounded derivatives $y_r^{(1)(t)}, \ldots, y_r^{(\rho)}(t)$, for any initial condition and for any unknown pair of matrices $(S, G)$, all the closed loop signals are bounded and the tracking error $y(t) - y_r(t)$ tends asymp-*

*totically to zero.*

**Theorem 2.1** *Consider system (1). If:*

1) $b$ *is a Hurwitz vetcor of degree* $\rho$*, i.e. all the zeros of the polynomial* $b_\rho s^{n-\rho} + \cdots + b_{n-1}s + b_n$ *have negative real part (without loss of generality, we assume* $b_\rho = 1$*);*

2) $\phi(y)$ *is a known vector of smooth functions;*

3) *the unknown matrices* $S$ *and* $G$ *are such that the pair* $\begin{bmatrix} A_c & G \\ 0 & S \end{bmatrix}, \begin{bmatrix} C_c & 0 \end{bmatrix}$ *is observable;*

4) $S$ *is a stable matrix (i.e.* $w(t)$ *is bounded for any initial condition* $w(0)$*) with known order* $r$*;*

*then the global tracking problem with disturbance rejection is solvable.*

*Proof.* With reference to system (1), if the relative degree $\rho > 1$, we introduce the input-filtered transformation ($\bar{\mu} \in \mathbb{R}^{\rho-1}$)

$$\dot{\bar{\mu}} = \begin{bmatrix} -\bar{\lambda}_1 & 1 & \dots & 0 \\ 0 & -\bar{\lambda}_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -\bar{\lambda}_{\rho-1} \end{bmatrix} \bar{\mu} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad (4)$$

$$\bar{x} = x - \sum_{i=2}^{\rho} \bar{d}[i]\bar{\mu}_{i-1} \quad (5)$$

with

$$\begin{aligned} \bar{d}[\rho] &= b \\ \bar{d}[i-1] &= A_c\bar{d}[i] + \bar{\lambda}_{i-1}\bar{d}[i], \qquad \rho \geq i \geq 2 \end{aligned}$$

and $\bar{\lambda}_i$, $1 \leq i \leq \rho - 1$, arbitrary positive reals. We obtain from (1), (4) and (5)

$$\begin{aligned} \dot{\bar{x}} &= A_c\bar{x} + \bar{d}\bar{\mu}_1 + \phi(y) + Gw \\ y &= C_c\bar{x} \quad (6) \\ \dot{w} &= Sw \quad (7) \end{aligned}$$

where $\bar{d} = \bar{d}[1]$ is, by construction, a Hurwitz vector of degree 1 with $\bar{d}_1 = 1$, by virtue of assumption 1). If $\rho = 1$, we simply set $\bar{d} = b$, $\bar{\mu}_1 = u$, $\bar{x} = x$. Consider the output-filtered transformation

$$\begin{aligned} \zeta_1 &= \bar{x}_1 \\ \zeta_j &= \bar{x}_j - \xi_{j-1}, \qquad 2 \leq j \leq n \quad (8) \end{aligned}$$

$$\dot{\xi} = \begin{bmatrix} -\bar{d}_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{d}_{n-1} & 0 & \dots & 1 \\ -\bar{d}_n & 0 & \dots & 0 \end{bmatrix} \xi + \left[ \begin{array}{c|c} -\bar{d}_2 & \\ \vdots & I \\ -\bar{d}_n & \end{array} \right] \phi \quad (9)$$

and, by virtue of assumption 2), the new control variable

$$v = \bar{\mu}_1 + \phi_1(y) + \xi_1 \,. \quad (10)$$

From (7), (8), (9) and (10), we have

$$\begin{aligned} \dot{\zeta} &= A_c\zeta + \bar{d}v + Gw \\ y &= C_c\zeta \\ \dot{w} &= Sw \,. \quad (11) \end{aligned}$$

By virtue of hypothesis 3), system (11) is observable and there exists a linear change of coordinates

$$z = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \zeta \\ w \end{bmatrix} \triangleq T \begin{bmatrix} \zeta \\ w \end{bmatrix}, \quad z \in \mathbb{R}^{n+r} \quad (12)$$

transforming (11) into the observer canonical form

$$\begin{aligned} \dot{z} &= A_c z + ay + T_1\bar{d}v \triangleq A_c z + ay + dv \\ y &= C_c z \quad (13) \end{aligned}$$

in which the matrices $A_c$ and $C_c$ are now $(n+r) \times (n+r)$ and $1 \times (n+r)$, respectively. Since $z_1 = \zeta_1 = y$, it follows that the first row of $T_1$ is $T_{11} = [1, 0, \dots, 0]^T$, so that the first element of $T_1\bar{d} = d$ is $d_1 = 1$. The vectors $a$ and $d$ are unknown since $T_1$ depends on the unknown matrix $S$. Let us consider the filtered transformation ($\lambda_j$, $1 \leq j \leq n + r - 1$ are arbitrary positive reals, $\mu[i] \in \mathbb{R}^i$ )

$$\bar{z} = z - \sum_{i=2}^{n+r} \delta_i \sum_{j=2}^{i} g[j]\mu_{j-1}[i-1] \quad (14)$$

$$\begin{aligned} \dot{\mu}[i] &= \begin{bmatrix} -\lambda_1 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -\lambda_i \end{bmatrix} \mu[i] + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \\ y[i] &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \mu[i], \; 1 \leq i \leq n + r - 1 \quad (15) \end{aligned}$$

with

$$\begin{aligned} g[n+r] &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T \\ g[j-1] &= A_c g[j] + \lambda_{j-1} g[j], \qquad 2 \leq j \leq n + r \,. \end{aligned}$$

The constants $\delta_i$ are given by

$$\begin{bmatrix} 1 & \delta_2 & \cdots & \delta_{n+r} \end{bmatrix}^T = \begin{bmatrix} g[1] & \cdots & g[n+r] \end{bmatrix}^{-1} d$$

and are unknown since the vector $d$ is unknown. From (13), (14) and (15), we obtain

$$\begin{aligned} \dot{\bar{z}} &= A_c\bar{z} + ay + g(v + \sum_{i=2}^{n+r} \delta_i y[i-1]) \\ y &= C_c\bar{z} \quad (16) \end{aligned}$$

with $g = g[1] = [1, g_2, \dots, g_{n+r}]^T$ such that

$$\begin{aligned} & s^{n+r-1} + g_2 s^{n+r-2} + \cdots + g_{n+r-1}s + g_{n+r} \\ &= \prod_{i=1}^{n+r-1} (s + \lambda_i) \,. \quad (17) \end{aligned}$$

Finally, we consider the output-filtered transformation

$$\begin{aligned} \chi_1 &= \bar{z}_1 = y \\ \chi_j &= \bar{z}_j - \sum_{i=1}^{n+r} \xi_{j-1}[i] a_i, \quad 2 \le j \le n+r \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{\xi}[i] &= \begin{bmatrix} -\bar{g}_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{g}_{n+r-1} & 0 & \dots & 1 \\ -\bar{g}_{n+r} & 0 & \dots & 0 \end{bmatrix} \xi[i] \\ &\quad + \left[ \begin{array}{c|c} -\bar{g}_2 & \\ \vdots & I \\ -\bar{g}_{n+r} & \end{array} \right] e_i y \end{aligned} \tag{19}$$

with $e_i$ being the i-th column of the $(n+r-1) \times (n+r-1)$ identity matrix, which transforms (16) into

$$\begin{aligned} \dot{\chi} &= A_c \chi + g \left( v + \sum_{i=2}^{n+r} \delta_i y[i-1] + \sum_{i=1}^{n+r} a_i \xi_1[i] + a_1 y \right) \\ y &= C_c \chi . \end{aligned} \tag{20}$$

Make the linear change of coordinates

$$\begin{aligned} y &= \chi_1 \\ \eta_i &= \chi_{i+1} - g_{i+1}\chi_1, \qquad 1 \le i \le n+r-1 . \end{aligned} \tag{21}$$

In the new coordinates, we have

$$\begin{aligned} \dot{y} &= \eta_1 + g_2 y + a_1 y + v + \sum_{i=2}^{n+r} \delta_i y[i-1] + \sum_{i=1}^{n+r} a_i \xi_1[i] \\ \dot{\eta} &= \Gamma \eta + \bar{g} y \end{aligned} \tag{22}$$

with

$$\Gamma = \begin{bmatrix} -g_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_{n+r-1} & 0 & \dots & 1 \\ -g_{n+r} & 0 & \dots & 0 \end{bmatrix}$$

$$\bar{g} = \begin{bmatrix} g_3 - g_2^2 \\ \vdots \\ g_{n+r} - g_2 g_{n+r-1} \\ -g_{n+r} g_2 \end{bmatrix} .$$

We now consider the case $\rho = 1$ and define the following dynamic output feedback controller ($\tilde{y} = y - y_r$)

$$\begin{aligned} u &= v - \phi_1(y) - \xi_1 \\ v &= -k\tilde{y} + \dot{y}_r - \hat{\eta}_1 - g_2 y - \hat{a}_1 y - \sum_{i=2}^{n+r} \hat{\delta}_i y[i-1] \\ &\quad - \sum_{i=1}^{n+r} \hat{a}_i \xi_1[i] \\ \dot{\hat{\eta}} &= \Gamma \hat{\eta} + \bar{g} y \\ \dot{\hat{a}}_1 &= c_1 y \tilde{y} \\ \dot{\hat{a}}_i &= c_i \tilde{y} \xi_1[i], \quad 2 \le i \le n+r \\ \dot{\hat{\delta}}_i &= c_{n+r+i-1} y[i-1] \tilde{y}, \qquad 2 \le i \le n+r \end{aligned} \tag{23}$$

in which $c_i$, $1 \le i \le 2(n+r)-1$, are positive adaptation gains. The error dynamics are given by ($\tilde{\eta} = \eta - \hat{\eta}$, $\tilde{a}_i = a_i - \hat{a}_i$, $\tilde{\delta}_i = \delta_i - \hat{\delta}_i$)

$$\begin{aligned} \dot{\tilde{y}} &= -k\tilde{y} + \tilde{\eta}_1 + \tilde{a}_1 y + \sum_{i=2}^{n+r} (\tilde{a}_i \xi_1[i] + \tilde{\delta}_i y[i-1]) \\ \dot{\tilde{\eta}} &= \Gamma \tilde{\eta} \\ \dot{\tilde{a}}_1 &= -c_1 \tilde{y} y \\ \dot{\tilde{a}}_i &= -c_i \tilde{y} \xi_1[i], \quad 2 \le i \le n+r \\ \dot{\tilde{\delta}}_i &= -c_{n+r+i-1} y[i-1] \tilde{y}, \quad 2 \le i \le n+r . \end{aligned} \tag{24}$$

Consider the function

$$V = \frac{1}{2}\tilde{y}^2 + \epsilon \tilde{\eta}^T P \tilde{\eta} + \frac{1}{2} \sum_{i=1}^{n+r} \frac{\tilde{a}_i^2}{c_i} + \frac{1}{2} \sum_{i=2}^{n+r} \frac{\tilde{\delta}_i^2}{c_{n+r+i-1}} \tag{25}$$

with $P$ solution of $\Gamma^T P + P\Gamma = -Q < 0$ and $\epsilon > 0$. Its time derivative along (18) is such that

$$\dot{V} = -k\tilde{y}^2 + \tilde{y}\tilde{\eta}_1 - \epsilon \tilde{\eta}^T Q \tilde{\eta} \tag{26}$$

which implies by a proper choice of $\epsilon$ that for a suitable $c > 0$

$$\dot{V} \le -c \left\| \begin{bmatrix} \tilde{y} \\ \tilde{\eta} \end{bmatrix} \right\|^2 \tag{27}$$

so that $\tilde{y}$, $\tilde{\eta}$, $\tilde{a}_i$, $\tilde{\delta}_i$ are bounded. Therefore, $y(t)$ is bounded and, consequently, from (9), (19) and (22), $\xi(t)$, $\xi[i](t)$ and $\eta(t)$ are bounded. Moreover, $\hat{\eta}(t)$ is also bounded from (23). Since in system (1) $b$ is a Hurwitz vector of degree one and $y(t)$ is bounded, it follows that $x(t)$ is also bounded. This fact may be verified by making the change of coordinates

$$\begin{aligned} y &= x_1 \\ \bar{\eta}_i &= x_{i+1} - b_{i+1} x_1, \qquad 1 \le i \le n-1 \end{aligned} \tag{28}$$

which maps (1) into

$$\begin{aligned} \dot{y} &= \bar{\eta}_1 + b_2 y + \phi_1(y) + u + G_1^T w \\ \dot{\bar{\eta}} &= \begin{bmatrix} -b_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1} & 0 & \dots & 1 \\ -b_n & 0 & \dots & 0 \end{bmatrix} \bar{\eta} + y \begin{bmatrix} b_3 - b_2^2 \\ \vdots \\ b_n - b_2 b_{n-1} \\ -b_n b_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \phi_3(y) - \phi_2^2(y) \\ \vdots \\ \phi_n(y) - \phi_{n-1}(y)\phi_2(y) \\ -\phi_n(y)\phi_2(y) \end{bmatrix} \\ &\quad + \begin{bmatrix} G_3^T w - (G_2^T w)^2 \\ \vdots \\ G_n^T w - G_{n-1}^T w G_2^T w \\ -G_n^T w G_2^T w \end{bmatrix} \end{aligned} \tag{29}$$

with $G_i^T$ being the i-th row of matrix $G$. From the first equation in (29), we have

$$u = \frac{dy}{dt} - b_2 y - \phi_1(y) - G_1^T w - \bar{\eta}_1 \tag{30}$$

and from (10) and (15) (recall that we are considering $\rho = 1$)

$$\sum_{j=0}^{i} g_{n-j}[n-i]\frac{d^j y[i]}{dt^j} = v = u + \phi_1(y) + \xi_1, \quad 1 \le i \le n+r-1 \quad (31)$$

which substituted in (30) gives ($1 \le i \le n+r-1$)

$$\sum_{j=0}^{i} g_{n-j}[n+r-i]\frac{d^j y[i]}{dt^j} = \frac{dy}{dt} - b_2 y - G_1^T w - \bar{\eta}_1 + \xi_1 \, . \quad (32)$$

Since $y(t)$, $\bar{\eta}_1(t)$, $\xi_1(t)$ and $w(t)$ (by virtue of assumption 4)) are bounded, and the polynomials

$$s^{n+r-i} + g_{n+r-i+1}[n+r-i]s^{n+r-i+1} + \cdots + g_{n+r}[n+r-i], \quad 1 \le i \le n+r-1$$

are Hurwitz, it follows that $y[i](t)$, $1 \le i \le n+r-1$, are bounded and, from (23), $v(t)$ and $u(t)$ are also bounded. Therefore, from (15), $\mu[i](t)$ are bounded, $1 \le i \le n+r-1$. By virtue of (24), $\dot{\tilde{y}}$ and $\dot{\tilde{\eta}}$ are bounded so that $(\tilde{y}^2 + \tilde{\eta}^T\tilde{\eta})$ is uniformly continuous. From (27), we can write

$$\int_0^t \left\| \begin{bmatrix} \tilde{y}(\tau) \\ \tilde{\eta}(\tau) \end{bmatrix} \right\|^2 d\tau \le -\frac{1}{c}\int_0^t \dot{V}(\tau)d\tau = \frac{1}{c}[V(0) - V(t)]$$

which, applying Barbalat's Lemma [**?**], implies

$$\lim_{t\to\infty} \tilde{y}(t) = 0 \, .$$

If the relative degree is $1 < \rho \le n$, we set

$$v = v^* + \tilde{v} \quad (33)$$

and assume $v^*$ as the fictitious control input. The subsequent steps closely follow the control design procedure used in [**?**] with the modifications suggested in [**?**] to avoid overparametrization. □

## 3 Example

In order to illustrate the control design presented in the previous section, we consider the following system

$$\begin{aligned} \dot{x}_1 &= x_2 + y^2 + w_1 \\ \dot{x}_2 &= u \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\theta w_1 \\ y &= x_1 \, . \end{aligned} \quad (34)$$

In this case we have $\rho = 2$ and we assume that the frequency $\theta$ of the sinusoidal disturbance affecting system (34) is unknown. The matrix $G = [1, 0]^T$ is assumed to be known so that some simplifications in the control design, with respect to the general case, are allowed. By means of the input-filtered transformation

$$\begin{aligned} \dot{\bar{\mu}} &= -\bar{\lambda}\bar{\mu} + u \\ \bar{x}_1 &= x_1, \quad \bar{x}_2 = x_2 - \bar{\mu} \end{aligned} \quad (35)$$

we obtain

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 + \bar{\mu} + y^2 + w_1 \\ \dot{\bar{x}}_2 &= \bar{\lambda}\bar{\mu} \\ y &= \bar{x}_1 \, . \end{aligned} \quad (36)$$

The use of the output-filtered transformation

$$\begin{aligned} \zeta_1 &= \bar{x}_1, \zeta_2 = \bar{x}_2 - \xi \\ \dot{\xi} &= -\bar{\lambda}\xi - \bar{\lambda}y^2 \end{aligned} \quad (37)$$

and the definition of a new variable

$$v = \bar{\mu} + y^2 + \xi \quad (38)$$

allow us to obtain the system

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + v + w_1 \\ \dot{\zeta}_2 &= \bar{\lambda}v \\ y &= \zeta_1 \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\theta w_1 \, . \end{aligned} \quad (39)$$

We now perform the linear change of coordinates

$$\begin{aligned} z_1 &= \zeta_1 \\ z_2 &= \zeta_2 + w_1 \\ z_3 &= w_2 + \theta\zeta_1 \\ z_4 &= \theta\zeta_2 \end{aligned} \quad (40)$$

yielding

$$\begin{aligned} \dot{z} &= A_c z + \begin{bmatrix} 1 \\ \bar{\lambda} \\ \theta \\ \theta\bar{\lambda} \end{bmatrix} v + \begin{bmatrix} 0 \\ -\theta \\ 0 \\ 0 \end{bmatrix} y \\ y &= C_c z \, . \end{aligned} \quad (41)$$

The input-filtered transformation

$$\begin{aligned} \dot{\mu}[1] &= -\lambda_1\mu[1] + v \\ \dot{\mu}_1[2] &= -\lambda_1\mu_1[2] + \mu_2[2] \\ \dot{\mu}_2[2] &= -\lambda_2\mu_2[2] + v \\ \dot{\mu}_1[3] &= -\lambda_1\mu_1[3] + \mu_2[3] \\ \dot{\mu}_2[3] &= -\lambda_2\mu_2[3] + \mu_3[3] \\ \dot{\mu}_3[3] &= -\lambda_3\mu_3[3] + v \\ y[1] &= \mu[1] \\ y[2] &= \mu_1[2] \\ y[3] &= \mu_1[3] \\ \bar{z} &= z - \sum_{i=2}^{4}\delta_i\sum_{j=2}^{i} g[j]\mu_{j-1}[i-1] \end{aligned} \quad (42)$$

maps (41) into

$$\begin{aligned}\dot{\bar{z}} &= A_c\bar{z} + \begin{bmatrix} 0 \\ -\theta \\ 0 \\ 0 \end{bmatrix} y + g(v + \sum_{i=2}^{4} \delta_i y[i-1]) \\ y &= C_c \bar{z} \,. \end{aligned} \quad (43)$$

By means of the output-filtered transformation

$$\begin{aligned}\chi_1 &= \bar{z}_1, \chi_j = \bar{z}_j - \xi_{j-1}[2][i](-\theta) \\ \dot{\xi}[2] &= \begin{bmatrix} -g_2 & 1 & 0 \\ -g_3 & 0 & 1 \\ -g_4 & 0 & 0 \end{bmatrix} \xi[2] + \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (44)$$

we obtain

$$\begin{aligned}\dot{\chi} &= A_c\chi + g(v + \sum_{i=2}^{4} \delta_i y[i-1] - \theta\xi_1[2]) \\ y &= C_c\chi \end{aligned} \quad (45)$$

which by the linear change of coordinates

$$\begin{aligned} y &= \chi_1 \\ \eta_i &= \chi_{i+1} - g_{i+1}\chi_1, \quad 1 \le i \le 3 \end{aligned} \quad (46)$$

is further transformed into

$$\begin{aligned}\dot{y} &= \eta_1 + g_2 y + v + \sum_{i=2}^{4} \delta_i y[i-1] - \theta\xi_1[2] \\ \dot{\eta} &= \Gamma\eta + \bar{g}y \end{aligned} \quad (47)$$

with

$$\Gamma = \begin{bmatrix} -g_2 & 1 & 0 \\ -g_3 & 0 & 1 \\ -g_4 & 0 & 0 \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} g_3 - g_2^2 \\ g_4 - g_2 g_3 \\ -g_2 g_4 \end{bmatrix} .$$

We define $v = \tilde{v} + v^*$ and consider $v^*$ as the control input, choosing

$$\begin{aligned} v^* &= -k\tilde{y} + \dot{y}_r - \hat{\eta}_1 - g_2 y - \sum_{i=2}^{4} \hat{\delta}_i y[i-1] + \hat{\theta}\xi_1[2] \\ \dot{\hat{\eta}} &= \Gamma\hat{\eta} + \bar{g}y \,. \end{aligned} \quad (48)$$

The overall error dynamics are

$$\begin{aligned} \dot{\tilde{y}} &= -k\tilde{y}\tilde{\eta}_1 + \tilde{v} + \sum_{i=2}^{4} \tilde{\delta}_i y[i-1] - \tilde{\theta}\xi_1[2] \\ \dot{\tilde{\eta}} &= \Gamma\tilde{\eta} \\ \dot{\tilde{v}} &= \dot{v} - \dot{v}^* = -\bar{\lambda}\bar{\mu} + u + 2y\dot{y} + \dot{\xi} - \dot{v}^* \\ &= -\bar{\lambda}\bar{\mu} + u + 2y\dot{y} + \dot{\xi} + k\dot{y} - k\dot{y}_r - \ddot{y}_r + \dot{\hat{\eta}}_1 + g_2\dot{y} \\ &\quad + \sum_{i=2}^{4} \dot{\hat{\delta}}_i y[i-1] + \sum_{i=2}^{4} \hat{\delta}_i \dot{y}[i-1] - \dot{\hat{\theta}}\xi_1[2] - \hat{\theta}\dot{\xi}_1[2] \\ &= u - \bar{\lambda}\bar{\mu} + \dot{\xi} - k\dot{y}_r - \ddot{y}_r + \dot{\hat{\eta}}_1 + \sum_{i=2}^{4} \dot{\hat{\delta}}_i y[i-1] \\ &\quad + \sum_{i=2}^{4} \hat{\delta}_i \dot{y}[i-1] - \dot{\hat{\theta}}\xi_1[2] - \hat{\theta}\dot{\xi}_1[2] + (\eta_1 + g_2 y \\ &\quad + v + \sum_{i=2}^{4} \delta_i y[i-1] - \theta\xi_1[2])(2y + k + g_2) \end{aligned} \quad (49)$$

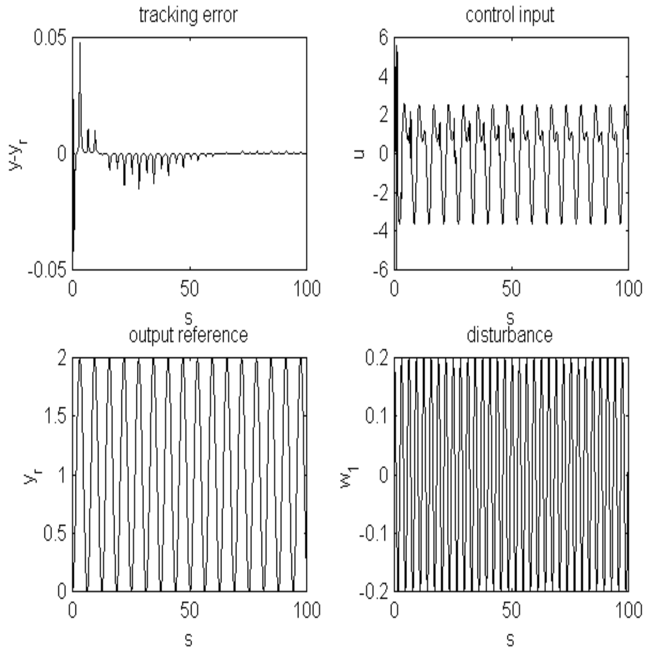

Figure 1: Simulation results.

The control $u$ and the parameter estimates are thus defined as

$$\begin{aligned} u &= \bar{\lambda}\bar{\mu} - \dot{\xi} + k\dot{y}_r + \ddot{y}_r - \dot{\hat{\eta}}_1 - \sum_{i=2}^{4} \dot{\hat{\delta}}_i y[i-1] \\ &\quad - \sum_{i=2}^{4} \hat{\delta}_i \dot{y}[i-1] + \dot{\hat{\theta}}\xi_1[2] + \hat{\theta}\dot{\xi}_1[2] \\ &\quad -(\hat{\eta}_1 + g_2 y + v + \sum_{i=2}^{4} \hat{\delta}_i y[i-1] \\ &\quad - \hat{\theta}\xi_1[2])(2y + k + g_2) - k_1\tilde{v} - k_2 4y^2\tilde{v} \\ \dot{\hat{\theta}} &= -c_1\xi_1[2]\tilde{y} - (2y + k + g_2)\xi_1[2]\tilde{v} \\ \dot{\hat{\delta}}_i &= c_i\Big(y[i-1]\tilde{y} + (2y + k + g_2)y[i-1]\tilde{v}\Big), \ 2 \le i \le 4 \end{aligned}$$

Some numerical simulations have been carried out with reference to the following parameters: $k = 3$, $k_1 = 5$, $k_2 = 5$, $\bar{\lambda} = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $c_i = 400$, $1 \le i \le 4$, $\theta = 4$, $y_r(t) = 1 - \cos t$. The initial conditions were assumed to be zero except for the disturbance $w_1(0) = 0.2, w_2(0) = 0$. The results of the simulations are illustrated in fig. 1, where the time histories of the tracking error $\tilde{y}$, the control input $u$, the output reference $y_r$ and the disturbance $w_1$ are reported.

## 4 Conclusions

The problem of tracking any smooth bounded output reference signal for minimum phase nonlinear systems with output dependent nonlinearities and additive unknown sinusoidal disturbances has been addressed and solved globally by output feedback, i.e. for any initial condition. While the system is supposed to be known, only the order of the linear exosystem which generates the unknown disturbances is required to be known. A second order nonlinear example illustrates the design techniques and the achievable performance.

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