FINITE-TIME CONTROL WITH POLE PLACEMENT

F. Amato*, **M. Ariola**†, **C. Cosentino**†

* Dipartimento di Medicina Sperimentale e Clinica, Università degli Studi Magna Græcia di Catanzaro, Via T. Campanella 115, 88100, Catanzaro, Italy, e-mail: amato@unicz.it

† Dipartimento di Informatica e Sistemistica, Università degli Studi di Napoli Federico II, Via Claudio 21, 80125, Napoli, Italy, e-mail: {ariola,carcosen}@unina.it

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Abstract

In this paper we deal with the problem of designing an output feedback controller which guarantees, at the same time, that the closed-loop poles are in specified regions of the complex plane and that the system under control is finite time bounded. This is accomplished by means of a dynamic compensator in the controller-observer form. The design procedure is divided in two steps: first, supposing that the state is available, a state feedback controller which gives the desired closed-loop properties is designed; then a state observer which tries to retain the properties guaranteed by the state feedback controller is synthesized. All the conditions are expressed in terms of Linear Matrix Inequalities and therefore the problem can be solved by efficient numerical optimization algorithms.

1 Introduction

The problem we consider in this paper can be summarized as follows: given a linear system subject to disturbances, design a controller which places the poles of the closed loop system in a specified region of the complex plane and, at the same time, guarantees finite time boundedness of the state variables.

Pole placement is often required in control applications to ensure a desired transient behaviour of the closed loop system. A classical example is represented by the flying qualities requirements in the field of aircraft control; such requirements specify, for example, an upper bound for the damping related to the phugoid mode [7]. Sufficient conditions guaranteeing the existence of a state and output feedback controller placing the system poles in a desired region have been provided in the literature (see [4] and the bibliography therein) in terms of feasibility problems involving Linear Matrix Inequalities (LMIs) [2].

On the other hand the concept of finite time control dates back to the Sixties, when the idea of finite time stability (FTS) was introduced in the control literature [8], [5]. A system is said to be finite time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval.

It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and viceversa. While LAS deals with the behaviour of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept which is useful to study the behaviour of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturations or the exit from the linear regime) during the transients.

FTS in the presence of exogenous inputs leads to the concept of finite time boundedness (FTB). In other words a system is said to be FTB if, given a bound on the initial condition and a characterization of the set of admissible inputs, the state variables remain below the prescribed limit for all inputs in the chosen set. It is clear that FTB implies FTS but the converse is not true.

FTS and FTB are open loop concepts. The finite time stabilization problem concerns the design of a linear controller which ensures the FTS or the FTB of the closed loop system. Sufficient conditions for finite time stabilization in the presence of *constant* disturbances and zero reference input have been provided in [1] for the state feedback case.

To understand the importance of considering at the same time FTS and pole placement, we refer again to the aircraft example. In that context, given a certain flight condition and the corresponding linearized model, a typical requirement for the designer is that of synthesizing a controller which "augments the stability" of the linear model (Stability Augmentation System) guaranteeing a desired damping; however, at the same time, it is important to ensure that the state variables do not exceed some bounds to avoid the exit from the linear regime and/or the attainment of nonphysical values.

This paper provides novel contributions in several ways. The first result is a sufficient condition guaranteeing the existence of a state feedback controller placing the poles of the closed loop system in a desired region of the complex plane and ensuring finite time boundedness in presence of both nonzero reference inputs and exogenous disturbances, which are assumed to belong to a larger class than the one considered in [1].

When the state is not fully available, we consider finite time stabilization via output feedback. The design, in this case, is divided in two steps. First we design a state feedback controller guaranteeing the desired pole placement and FTB of the system supposing that the state were available; then a state observer which tries to retain the finite-time properties guaranteed by the state feedback controller is synthesized.

In both the state and output feedback cases, we shall provide conditions for the existence of the controller in terms of LMIs feasibility problems.

The paper is organized as follows. In Section 2 some preliminary definitions are given and the problems we deal with are precisely stated. In Section 3 sufficient conditions for the existence of a state feedback controller guaranteeing pole placement and finite time stabilization are provided. In Section 4 the output feedback problem is considered; finally the conclusions are drawn in Section 5.

2 Problem Statement and Preliminaries

The following definitions deal with various finite time control problems.

Definition 1 (Finite time stability (FTS)). Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, and a positive definite symmetric matrix R, the linear system

$$\dot{x}(t) = Ax(t)\,, \quad x(0) = x_0 \tag{1}$$

is said to be finite time stable (FTS) with respect to (c_1, c_2, T, R), if

$$x_0^T R x_0 \leq c_1 \Rightarrow x^T(t) R x(t) < c_2 \quad \forall t \in [0, T]\,.$$

◇

Remark 1. Lyapunov Asymptotic Stability (LAS) and FTS are independent concepts: a system which is FTS may be not LAS; conversely a LAS system could be not FTS if, during the transients, its state exceeds the prescribed bounds. △

Definition 2 (Finite time boundedness (FTB)). Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix R and a class of signals $\mathcal{W}$, the linear system

$$\dot{x}(t) = Ax(t) + Gw(t)\,, \quad x(0) = x_0 \tag{2}$$

is said to be finite time bounded with respect to $(c_1, c_2, \mathcal{W}, T, R)$ if

$$x_0^T R x_0 \leq c_1 \Rightarrow x^T(t) R x(t) < c_2 \quad \forall t \in [0, T]\,, \tag{3}$$

for all $w(\cdot) \in \mathcal{W}$. ◇

Remark 2. An important difference between LAS and FTB relies in the fact that, for linear systems, LAS is a structural property of the system which is not affected by the inputs, while FTB clearly depends on the kind and amplitude of the inputs acting on the system. For instance, referring to aircraft control, it is important that during the execution of a certain task the state variables do not exceed some threshold under all admissible pilot (reference) inputs and/or in the presence of wind gusts (disturbance). △

Note that FTS and FTB refer to open loop systems. The next definition puts together FTS and FTB in the design context.

Definition 3 (Finite time stabilization via state feedback (FTSSF)). Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix R and two classes of signals $\mathcal{W}_r$ and $\mathcal{W}_d$, the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)\,, \quad x(0) = x_0 \tag{4}$$

is said to be finite time stabilizable via state feedback with respect to $(c_1, c_2, \mathcal{W}_r \times \mathcal{W}_d, T, R)$ if there exists a control law in the form

$$u(t) = Kx(t) + r(t) \tag{5}$$

such that the closed loop system obtained by the connection of (4) and (5), namely

$$\dot{x}(t) = (A + BK)\,x(t) + \begin{pmatrix} B & E \end{pmatrix} \begin{pmatrix} r(t) \\ d(t) \end{pmatrix}\,, \quad x(0) = x_0\,,$$

is FTB with respect to $(c_1, c_2, \mathcal{W}_r \times \mathcal{W}_d, T, R)$. ◇

Remark 3. In [1] a sufficient condition for FTSSF has been provided when $r(t) = 0$ and $\mathcal{W}_d$ is the class of constant, norm bounded disturbances. In this paper we consider a more general situation, since both $r(t) \neq 0$ and, as we shall precise later, the class $\mathcal{W}_d$ we consider in this work is larger than the one considered in [1]. △

Next, we consider dynamic output feedback control. Note that, when a dynamical controller is used, the state of the closed loop system is given by the composition of the system state and the controller state. Since we are only interested in the finite time stabilization of the state of the original system (5), we give the following definition.

Definition 4 (Finite Time Stabilization via output feedback (FTSOF)). Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix R and two classes of signals $\mathcal{W}_r$ and $\mathcal{W}_d$, the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)\,, \quad x(0) = x_0 \tag{6a}$$
$$y(t) = Cx(t) \tag{6b}$$

is said to be finite time stabilizable via output feedback with respect to $(c_1, c_2, \mathcal{W}_r \times \mathcal{W}_d, T, R)$ if there exists a dynamic controller in the form

$$\dot{\xi}(t) = A_K \xi(t) + B_K y(t) + E_K r(t)\,, \quad \xi(0) = 0 \tag{7a}$$
$$u(t) = C_K \xi(t) + D_K r(t) \tag{7b}$$

such that (6a) subject to (7b), namely

$$\dot{x}(t) = Ax(t) + \begin{pmatrix} BC_K & BD_K & E \end{pmatrix} \begin{pmatrix} \xi(t) \\ r(t) \\ d(t) \end{pmatrix}\,, \quad x(0) = x_0\,,$$

is FTB with respect to $(c_1, c_2, \mathcal{W}_r \times \mathcal{W}_d \times \mathcal{W}_\xi, T, R)$, where $\mathcal{W}_\xi$ is the set of all solutions of (7a). ◇

In the next assumption we characterize the classes of signals $\mathcal{W}_r$ and $\mathcal{W}_d$ considered in this paper.

Assumption 1. The class $\mathcal{W}_r$ and $\mathcal{W}_d$ are defined as follows

$$\mathcal{W}_r := \Big\{ r(\cdot) \,:\, \dot{r}(t) = A_r r(t)\,,\, r(0) = r_0\,, \\ r_0^T \Gamma_r r_0 \leq \rho\,,\, \Gamma_r > 0 \Big\}$$
$$\mathcal{W}_d := \Big\{ d(\cdot) \,:\, \dot{d}(t) = A_d d(t)\,,\, d(0) = d_0\,, \\ d_0^T \Gamma_d d_0 \leq \delta\,,\, \Gamma_d > 0 \Big\}\,.$$

$\diamond$

Note that the sets $\mathcal{W}_r$ and $\mathcal{W}_d$ capture, among others, the classes of canonical polynomial and sinusoidal inputs. Constant disturbances (considered in [1]) are a particular sub-class of the set $\mathcal{W}_d$.

Now let us consider the pole placement problem. According to [3] we refer to the so-called LMI regions of the complex plane.

Definition 5 (LMI Region [3]). An LMI region is any subset $\mathcal{D}$ of the complex plane defined as

$$\mathcal{D} := \big\{ z \in \mathbb{C} \,:\, \Lambda + z\Theta + z^*\Theta^T < 0 \big\} \tag{8}$$

where Λ and Θ are real matrices and Λ is symmetric. $\diamond$

LMI regions include, among others, the strip, the disk and the cone with apex at the origin. For example the half plane $\Re(z) < -\alpha$ is characterized by the inequality

$$z + z^* + 2\alpha < 0\,,$$

while the conic sector with apex at the origin and inner angle 2θ is described by

$$\begin{pmatrix} \sin\left(\theta(z+z^*)\right) & \cos\left(\theta(z-z^*)\right) \\ \cos\left(\theta(z-z^*)\right) & \sin\left(\theta(z+z^*)\right) \end{pmatrix} < 0\,.$$

Definition 6 ($\mathcal{D}$-Stability). System (1) is said to be $\mathcal{D}$-stable if all the eigenvalues of A are in the region $\mathcal{D}$. $\diamond$

A generalization of the Lyapunov Theorem for linear systems leads to the following necessary and sufficient condition for $\mathcal{D}$-stability; in the sequel, given two matrices F and G, the symbol $F \otimes G$ denotes the Kronecker product of F and G [6].

Theorem 1 ($\mathcal{D}$- Stability [3]). System (1) is $\mathcal{D}$-stable if and only if there exists a positive definite matrix P such that

$$\Lambda \otimes P + \Theta \otimes (AP) + \Theta^T \otimes (PA^T) < 0\,. \tag{9}$$

$\square$

Note that (9) is an LMI in the variable P.

Now we are ready to state the main problems we will consider in this paper.

Problem 1 (FTSSF and Pole Placement).
Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix R, the classes of signals $\mathcal{W}_r$ and $\mathcal{W}_d$ considered in Assumption 1, *find* a control law in the form (5) which guarantees FTSSF of the linear system (4) and places the closed loop poles in the region $\mathcal{D}$. $\diamond$

Next we consider the output feedback case. To this end note that the pole placement problem is greatly simplified if we consider a dynamical compensator in the controller-observer form. Such compensator can be obtained from (7) by letting

$$A_K = A + BK + LC$$
$$B_K = -L\,,\, E_K = B\,,\, C_K = K\,,\, D_K = I\,.$$

Indeed in this case the well known Separation Principle ensures that the poles of the closed loop system are that ones of $A+BK$ and $A + LC$. Following this consideration, in the sequel we shall refer to output feedback dynamical controllers in the form

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + L(C\xi(t) - y(t))\,, \quad \xi(0) = 0 \tag{10a}$$
$$u(t) = K\xi(t) + r(t)\,. \tag{10b}$$

Problem 2 (FTSOF and Pole Placement).
Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix R, the classes of signals $\mathcal{W}_r$, $\mathcal{W}_d$ considered in Assumption 1, *find* a dynamical controller in the form (10) which attains FTSOF of the linear system (6) and places the poles of $A + BK$ in the region $\mathcal{D}$. $\diamond$

Remark 4. In order to be consistent with the formulation of the FTSOF problem, in Problem 2 our only concern is the placement of the poles of the original system (6). $\triangle$

3 A Sufficient Condition for FTSSF with Pole Placement

In this section we shall provide a sufficient condition for the solvability of Problem 1.

First consider the following result.

Lemma 1. Consider the region $\mathcal{D}$ defined in (8); then system (2) is $\mathcal{D}$-stable and FTB wrt $(c_1, c_2, \mathcal{W}, T, R)$, where

$$\mathcal{W} := \Big\{ w(\cdot) \,:\, \dot{w}(t) = A_w w(t)\,,\, w(0) = w_0\,, \\ w_0^T \Gamma w_0 \leq \mu\,,\, \Gamma > 0 \Big\}\,, \tag{11}$$

if, letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, $\tilde{Q}_2 = \Gamma^{1/2} Q_2 \Gamma^{1/2}$, there exist three symmetric positive definite matrices Q_1, Q_2, Q_3, a nonnegative scalar α and two positive scalars λ_1, λ_2, such that

$$\begin{pmatrix} \Psi_a & G \\ G^T & \Psi_b \end{pmatrix} < 0 \tag{12a}$$
$$\lambda_1 I < Q_1 < I \tag{12b}$$
$$Q_2 < \lambda_2 I \tag{12c}$$
$$\begin{pmatrix} \lambda_2\mu - c_2 e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & -\lambda_1 \end{pmatrix} < 0 \tag{12d}$$
$$\Lambda \otimes Q_3 + \Theta \otimes (AQ_3) + \Theta^T \otimes (Q_3 A^T) < 0\,, \tag{12e}$$

where

$$\begin{aligned} \Psi_a &:= A\tilde{Q}_1 + \tilde{Q}_1 A^T - \alpha \tilde{Q}_1 \\ \Psi_b &:= A_w^T \tilde{Q}_2 + \tilde{Q}_2 A_w - \alpha \tilde{Q}_2\,. \end{aligned}$$

Proof. Assume there exist positive definite matrices Q_1, Q_2, a number $\alpha \geq 0$ and positive scalars λ_i, $i = 1, 2$, satisfying conditions (12b)-(12d). We have

$$\begin{aligned} \frac{c_1}{\lambda_{\min}(Q_1)} + \lambda_{\max}(Q_2)\mu &< \frac{c_1}{\lambda_1} + \lambda_2 \mu \\ &< c_2 e^{-\alpha T} \\ &< \frac{c_2 e^{-\alpha T}}{\lambda_{\max}(Q_1)}\,. \end{aligned} \tag{13}$$

Now define $P_1 = Q_1^{-1}$, $P_2 = Q_2$; in this way inequality (13) can be rewritten

$$\lambda_{\max}(P_1)c_1 + \lambda_{\max}(P_2)\mu < \lambda_{\min}(P_1)c_2 e^{-\alpha T} \tag{14}$$

Let

$$\begin{aligned} V(x,w) &= x^T R^{1/2} P_1 R^{1/2} x + w^T \Gamma^{1/2} P_2 \Gamma^{1/2} w \\ &:= x^T \tilde{P}_1 x + w^T \tilde{P}_2 w \end{aligned} \tag{15}$$

and denote, as usual, by $\dot{V}$ the derivative of V along the solutions of system (2). Suppose that the condition

$$\dot{V}(x(t), w(t)) < \alpha V(x(t), w(t)) \tag{16}$$

holds for all $t \in [0, T]$ and all $w(\cdot) \in \mathcal{W}$. We will first demonstrate that conditions (16) and (14) imply that system (2) is FTB with respect to $(c_1, c_2, \mathcal{W}, T, R)$. Then we will show that condition (16) is equivalent to (12a).

Our first claim is that conditions (16) and (14) imply the Finite-Time Boundedness of system (2) with respect to $(c_1, c_2, \mathcal{W}, T, R)$. Introducing the matrix

$$P = \begin{pmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{pmatrix}$$

and the vector

$$z = \begin{pmatrix} x \\ w \end{pmatrix}$$

it is easy to show that from (16) it follows that

$$z^T(t)Pz(t) < z^T(0)Pz(0)e^{\alpha t} \tag{17}$$

and

$$\begin{aligned} z^T(t)Pz(t) &\geq \lambda_{\min}(P_1)x^T(t)Rx(t) \\ &\quad + \lambda_{\min}(P_2)w^T(t)\Gamma w(t) \\ &\geq \lambda_{\min}(P_1)x^T(t)Rx(t) \end{aligned} \tag{18a}$$
$$\begin{aligned} z^T(0)Pz(0)e^{\alpha t} &\leq \big(\lambda_{\max}(P_1)x^T(0)Rx(0) \\ &\quad + \lambda_{\max}(P_2)w^T(0)\Gamma w(0)\big)e^{\alpha t} \\ &\leq \left(\lambda_{\max}(P_1)c_1 + \lambda_{\max}(P_2)\mu\right)e^{\alpha T}\,. \end{aligned} \tag{18b}$$

Putting together (17) and (18) we have

$$x^T(t)Rx(t) < \frac{\lambda_{\max}(P_1)c_1 + \lambda_{\max}(P_2)\mu}{\lambda_{\min}(P_1)} e^{\alpha T}\,. \tag{19}$$

From (19) and (14) it readily follows that $x^T(t)Rx(t) < c_2$ for all $t \in [0, T]$.

Now we shall prove that condition (12a) is equivalent to (16). It is simple to recognize that inequality (16) is equivalent to the following (we omit the time argument for brevity)

$$\begin{aligned} &x^T A^T \tilde{P}_1 x + w^T G^T \tilde{P}_1 x + x^T \tilde{P}_1 A x + x^T \tilde{P}_1 G w \\ &+ w^T A_w^T \tilde{P}_2 w + w^T \tilde{P}_2 A_w w - \alpha x^T \tilde{P}_1 x - \alpha w^T \tilde{P}_2 w < 0 \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\begin{pmatrix} x^T & w^T \end{pmatrix} \cdot \\ &\begin{pmatrix} A^T \tilde{P}_1 + \tilde{P}_1 A - \alpha \tilde{P}_1 & \tilde{P}_1 G \\ G^T \tilde{P}_1 & A_w^T \tilde{P}_2 + \tilde{P}_2 A_w - \alpha \tilde{P}_2 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} < 0 \end{aligned} \tag{20}$$

From (20), pre and post-multiplying by

$$\begin{pmatrix} \tilde{Q}_1 & 0 \\ 0 & I \end{pmatrix}$$

the proof follows.

For what concerns the $\mathcal{D}$-Stability, it is ensured guaranteeing that condition (12e) holds, as stated in Theorem 1. □

Based on Lemma 1 we can state the following result.

Theorem 2. Problem 1 admits a feasible solution if letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, $\tilde{Q}_2 = \Gamma_r^{1/2} Q_2 \Gamma_r^{1/2}$, $\tilde{Q}_3 = \Gamma_d^{1/2} Q_3 \Gamma_d^{1/2}$, there exist three symmetric positive definite matrices Q_1, Q_2, Q_3, a nonnegative scalar α, three positive scalars

λ_i, $i = 1, 2, 3$, and a matrix N such that

$$\begin{pmatrix} \Psi_1 & B & E \\ B^T & \Psi_2 & 0 \\ E^T & 0 & \Psi_3 \end{pmatrix} < 0$$
$$\lambda_1 I < Q_1 < I$$
$$Q_2 < \lambda_2 I$$
$$Q_3 < \lambda_3 I$$
$$\begin{pmatrix} \lambda_2 \rho + \lambda_3 \delta - c_2 e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & -\lambda_1 \end{pmatrix} < 0$$
$$\Lambda \otimes \tilde{Q}_1 + \Theta \otimes \left(A\tilde{Q}_1 \right) + \Theta \otimes (BN) + \Theta^T \otimes \left(\tilde{Q}_1 A^T \right) + \Theta^T \otimes \left(N^T B^T \right) < 0 \,.$$

where

$$\Psi_1 := A\tilde{Q}_1 + \tilde{Q}_1 A^T + BN + N^T B^T - \alpha \tilde{Q}_1$$
$$\Psi_2 := A_r^T \tilde{Q}_2 + \tilde{Q}_2 A_r - \alpha \tilde{Q}_2$$
$$\Psi_3 := A_d^T \tilde{Q}_3 + \tilde{Q}_3 A_d - \alpha \tilde{Q}_3$$

If the above set of LMI is feasible, then a controller in the form (5) solving Problem 1 can be found by letting $K = N\tilde{Q}_1^{-1}$.

Proof. It is sufficient to apply Lemma 1 with the following substitutions

$$A \leftarrow A + BK \,, \quad G \leftarrow \begin{pmatrix} B & E \end{pmatrix} \,, \quad w(t) \leftarrow \begin{pmatrix} r(t) \\ d(t) \end{pmatrix} \,,$$
$$\mathcal{W} \leftarrow \mathcal{W}_r \times \mathcal{W}_d \,, \quad A_w \leftarrow \begin{pmatrix} A_r & 0 \\ 0 & A_d \end{pmatrix} \,,$$
$$\Gamma \leftarrow \begin{pmatrix} \Gamma_r & 0 \\ 0 & \Gamma_d \end{pmatrix} \,, \quad \tilde{Q}_2 \leftarrow \begin{pmatrix} \tilde{Q}_2 & 0 \\ 0 & \tilde{Q}_3 \end{pmatrix} \,,$$

and to define a new optimization variable $N := K\tilde{Q}_1$. □

Remark 5. Note that we used the same matrix variable $\tilde{Q}_1$ in the first and in the last condition of Theorem 1. This is necessary in order to obtain a condition for FTSSF with pole placement entirely expressed in terms of LMIs. △

4 The Output Feedback Case

When the state of the system under consideration is not fully available, it can be estimated via a classical Luenberger observer. However the observer may destroy the finite time stabilization obtained via the state feedback, due to the inaccuracy of the state estimate during the transients.

To better investigate this point let us consider the closed loop system obtained connecting the controller (10) to system (6)

$$\dot{x}(t) = Ax(t) + BK\xi(t) + Br(t) + Ed(t) \,, \quad x(0) = x_0$$
$$\dot{\xi}(t) = -LCx(t) + (A + BK + LC)\,\xi(t) + Br(t) \,, \quad \xi(0) = 0 \,.$$

It is well known that, using the state transformation which brings the system into the state-estimation error base

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \,, \tag{21}$$

the system dynamics can be rewritten as

$$\dot{x}(t) = (A + BK)\,x(t) + Br(t) + Ed(t) - BKe(t) \,, \quad x(0) = x_0 \tag{22}$$

with

$$\dot{e}(t) = (A + LC)\,e(t) + Ed(t) \,, \quad e(0) = x_0 \,. \tag{23}$$

Therefore the system state evolution is influenced by the behaviour of the exogenous input $e(t)$. Since we assume that in the previous phase a controller K solving Problem 1 has beed designed according to Theorem 2, for $e(t) = 0$ system (22) is obviously FTB and its poles are located in the region $\mathcal{D}$; however the presence of a nonzero $e(t)$, while not affecting the locations of the poles of $A + BK$, could bring the norm of the state $x(t)$ outside the prespecified bounds.

On the basis of the above discussion, we will develop a methodology to find an observer gain L in (10) such that the FTB property of the system

$$\dot{x} = (A + BK)\,x + Br + Ed$$

is not lost in presence of the estimation error.

Theorem 3. Assume that K is a static feedback controller which solves Problem 1 for the system (6a). Let $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, $\tilde{Q}_2 = R^{1/2} Q_2 R^{1/2}$, $\tilde{Q}_3 = \Gamma_r^{1/2} Q_2 \Gamma_r^{1/2}$, $\tilde{Q}_4 = \Gamma_d^{1/2} Q_3 \Gamma_d^{1/2}$. If there exist four symmetric positive definite matrices Q_1, Q_2, Q_3, Q_4, a nonnegative scalar α, four positive scalars λ_i, $i = 1, \dots, 4$, and a matrix M such that

$$\begin{pmatrix} \Phi_1 & -BK & B & E \\ -K^T B^T & \Phi_2 & 0 & E \\ B^T & 0 & \Phi_r & 0 \\ E^T & E^T & 0 & \Phi_d \end{pmatrix} < 0 \tag{24a}$$
$$\lambda_1 I < Q_1 < I \tag{24b}$$
$$Q_2 < \lambda_2 I \tag{24c}$$
$$Q_3 < \lambda_3 I \tag{24d}$$
$$Q_4 < \lambda_4 I \tag{24e}$$
$$\begin{pmatrix} \lambda_2 c_1 + \lambda_3 \rho + \lambda_4 \delta - c_2 e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & -\lambda_1 \end{pmatrix} < 0 \tag{24f}$$

where

$$\Phi_1 := A\tilde{Q}_1 + \tilde{Q}_1 A^T + BK\tilde{Q}_1 + \tilde{Q}_1 K^T B^T - \alpha \tilde{Q}_1$$
$$\Phi_2 := A^T \tilde{Q}_2 + \tilde{Q}_2 A + MC + C^T M^T - \alpha \tilde{Q}_2$$
$$\Phi_r := A_r^T \tilde{Q}_3 + \tilde{Q}_3 A_r - \alpha \tilde{Q}_3$$
$$\Phi_d := A_d^T \tilde{Q}_4 + \tilde{Q}_4 A_d - \alpha \tilde{Q}_4$$

then Problem 2 admits a solution; a possible solution is given by the pair K, L, where $L = \tilde{Q}_2^{-1} M$.

Proof. System (22) can be put in the form (2) by making the following substitutions

$$A \leftarrow A + BK\,, \quad G \leftarrow \begin{pmatrix} -BK & B & E \end{pmatrix}\,,$$

$$w(t) \leftarrow \begin{pmatrix} e(t) \\ r(t) \\ d(t) \end{pmatrix}\,, \quad \mathcal{W} \leftarrow \mathcal{W}_e \times \mathcal{W}_r \times \mathcal{W}_d\,,$$

$$A_w \leftarrow \begin{pmatrix} A + LC & 0 & E \\ 0 & A_r & 0 \\ 0 & 0 & A_d \end{pmatrix}\,, \quad \Gamma \leftarrow \begin{pmatrix} R & 0 & 0 \\ 0 & \Gamma_r & 0 \\ 0 & 0 & \Gamma_d \end{pmatrix}\,,$$

$$\tilde{Q}_2 \leftarrow \begin{pmatrix} \tilde{Q}_2 & 0 & 0 \\ 0 & \tilde{Q}_3 & 0 \\ 0 & 0 & \tilde{Q}_4 \end{pmatrix}\,,$$

where

$$\mathcal{W}_e := \Big\{ e(\cdot) \,:\, \dot{e}(t) = (A + LC)e(t) + Ed(t)\,, \\ e(0) = x(0)\,,\, e_0^T R e_0 \leq c_1\,,\, R > 0 \Big\}\,,$$

and $M = \tilde{Q}_2 L$; then the conditions (24a)-(24f) are easily derived from Lemma 1. □

Remark 6. In Theorem 3, because of the separation principle, we are not concerned with the regional pole placement any more. Therefore conditions (24a)–(24f) are only aimed at retaining the FTB properties of the closed-loop system (22). △

5 Conclusions

In this paper, given a linear system, we have considered the problem of guaranteeing at the same time, via output feedback, the finite-time boundedness of the system state during the transients and the placement of the closed-loop poles in some specified regions. The proposed controller design is divided in two phases. First a state feedback controller which gives the desired propeties in terms of pole location and finite-time boundedness is designed; then a state observer is built in such a way that the finite-time performances obtained at the first step are retained. The sufficient conditions for the existence of the controller are given in terms of Linear Matrix Inequalities; this allows to solve our problem by means of efficient numerical optimization algorithms.

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