

# DESIGN OF CROSS-DIRECTIONAL CONTROLLERS WITH OPTIMAL STEADY STATE PERFORMANCE

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## Abstract

Actuator constraint handling is necessary for many cross-directional controllers. We discuss how optimal steady state performance can be guaranteed by modifying an internal model control structure with a non-linear element. For the simple dynamics associated with most web processes this also gives good closed-loop dynamic behaviour. Thus unconstrained control design techniques may be applied directly to the constrained control problem.

## 1 Introduction

Cross-directional control design is required for a wide class of industrial web forming processes including paper making, plastic film extrusion, coating processes and steel rolling (see Fig 1). It has received considerable attention in the academic community—see for example [1] and references therein, as well as [2] and associated contributions.

There are two main schools of cross-directional control design (exemplified by Chapters 6 and 7 in [1]): firstly unconstrained control (perhaps with limited anti-windup) based on robust control methodologies and secondly constrained control achieved via MPC (model predictive control). Briefly the former guarantees robust dynamic behaviour while the latter offers improved steady state response (provided the model is sufficiently accurate [3, 4]). In this paper we discuss how the simple model structure common to most web forming processes allows optimal constraint handling to be incorporated as a modification to an IMC (internal model control) structure. It may be viewed either as an optimal anti-windup scheme for robust control design or as a methodology for MPC design according to taste.

We will make the standard assumption that the open loop behaviour of the output profile  $y(t)$  may be well approximated by the model

$$y(t) = z^{-k}h(z^{-1})Bu(t) + d(t) \quad (1)$$

Here  $y(t) \in \mathbb{R}^n$  represents the measured profile across the web and  $u(t) \in \mathbb{R}^m$  represents the array of actuators. Typically

$n > m$ . We assume the whole profile  $y(t)$  is available simultaneously; if raw measurements are obtained from a scanning sensor, then a periodic Kalman filter [5, 6] can be used to estimate  $y(t)$ . The dynamics are represented as a delay of  $k$  sample and a biproper transfer function  $h(z^{-1})$ . Usually  $h(z^{-1})$  is low order and often simply a first order response

$$h(z^{-1}) = (1 - \alpha)(1 - \alpha z^{-1})^{-1} \quad (2)$$

We have assumed, without loss of generality, that  $h(z^{-1})$  has unit gain. The  $(n \times m)$  interaction matrix  $B$  describes the steady state response of the actuators on the profile. Finally  $d(t) \in \mathbb{R}^n$  represents disturbances on the plant.

Let  $B$  be decomposed as  $B = \Phi \Sigma \Psi^T$  with  $\Phi$  and  $\Psi$  orthonormal (the description is rather general as we may allow either  $\Phi$  or  $\Psi$  to be the identity matrix [7]). One possibility is the singular value decomposition [8, 1], in which case  $\Sigma$  is diagonal (if  $n > m$  the upper block of  $\Sigma$  is diagonal, while the lower block is zero).

We can then write  $y(t)$  and  $u(t)$  in terms of basis functions which are the columns of  $\Phi$  and  $\Psi$  respectively. We will assume that these basis functions are “spectral” [9] in the sense that they are naturally ordered according to some smoothness criterion. With an abuse of terminology, we will classify them as low, medium and high frequency modes. The effect of the interaction matrix is assumed to attenuate for high frequencies. In the case of the singular value decomposition the modes are ordered according to the magnitude of the singular value.

Ultra-high frequency modes of  $\Phi$  are uncontrollable [8, 9]. With model mismatch the relative uncertainty is greater at high frequencies, and may result in closed-loop instability [1]. Even in the case of closed-loop stability, attempting to control uncertain modes may degrade steady state performance [3]. It is thus generally accepted that the controller should not act on high order modes. It may also be useful to restrict the dimension of the input space [10] — for example the actuators are usually constrained to sum to zero. We will assume the controller is designed to act only on  $r \leq \min(n, m)$  modes. It will be useful to define a reduced interaction matrix  $B_r \in \mathbb{R}^{(n,m)}$  which can be decomposed as  $B_r = \Phi_r \Sigma_r \Psi_r^T$  with  $\Phi_r \in \mathbb{R}^{(n,r)}$  representing the modes we wish to control,  $\Sigma_r \in \mathbb{R}^{(r,r)}$  and  $\Psi_r \in \mathbb{R}^{(m,r)}$ . We will write  $\eta(t) = \Phi_r^T y(t)$  and  $\mu(t) = \Psi_r^T u(t)$ .

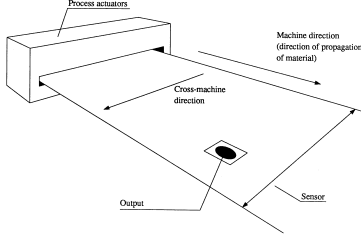


Figure 1: *Generic web forming process.*

It will be useful to define  $\Psi_r^\perp \in \mathbb{R}^{(m, m-r)}$  as an orthonormal matrix whose columns span the orthogonal complement to the space spanned by the columns of  $\Psi_r$ . The restriction on the input space may then be expressed as requiring  $(\Psi_r^\perp)^T u(t) = 0$ .

If sufficient modes are excluded it is possible to design robust controllers that do not violate actuator constraints [11, 10, 8, 1, 12]. It has been recommended [10] that  $r$  should be chosen to ensure no actuator touches the constraint boundary. The designs of [1] are based on IMC structures. The designs of [10, 12] are modifications of Dahlin controllers, themselves variants of IMC. Furthermore, *any* linear controller of the form

$$u(t) = -\Psi_r C(z^{-1}) \Phi_r^T y(t) \quad (3)$$

can be rearranged as an IMC

$$u(t) = -\Psi_r Q(z^{-1}) \Phi_r^T y(t) + \Psi_r Q(z^{-1}) \Phi_r z^{-k} h(z^{-1}) B_r u(t) \quad (4)$$

with

$$Q(z^{-1}) = C(z^{-1}) [I + z^{-k} h(z^{-1}) \Sigma_r C(z^{-1})]^{-1} \quad (5)$$

Limited anti-windup schemes have been proposed for such controllers [13, 10, 1, 14]. However there may be mid-frequency modes where the model mismatch is relatively small, but where unconstrained control action would require actuator constraint violation even in steady state. In such cases restricting  $r$  as above would result in economic disadvantage [3, 4]. Similarly anti-windup schemes that take no account of actuator directionality can lead to severe performance degradation [15, 16].

In such cases optimal steady state performance requires the solution of a constrained optimization. This was recognized for the cross-directional control problem in [17], when limitations in computing power were an impediment to implementation. Subsequently several MPC strategies with a quadratic program have been discussed in the literature, for example [9, 6]. Strategies for minimising  $l_1$  norms [18] and  $l_\infty$  norms [19] have also been discussed. A number of control designs that seek approximate solutions to the quadratic cost have also been discussed—for example [20, 21, 22, 23, 24, 1].

In this paper we discuss the design of cross-directional controllers that preserve the robust properties of unconstrained

methods, whilst ensuring, where possible, optimal steady state performance. We will also seek efficient computational implementation and good dynamic response. In particular we propose preserving the IMC structure for both control design and implementation. The constraints can then be satisfied by solving a deadbeat optimization problem. This is, of course, well known as a particular implementation of MPC [25] and may be considered [15] as a generalization of standard anti-windup schemes such as those in [26].

## 2 Constrained IMC

### 2.1 Control criteria

Suppose the disturbance  $d(t)$  in (1) is fixed. Then the ideal steady state performance criterion is to minimise  $\|\Phi_r^T y(t)\|$  for some norm. The projection term  $\Phi_r^T$  is included so that higher modes are not penalised. Model mismatch may cause steady state performance degradation [3]. If this degradation is significant, then the number of modes acted upon should be reduced. We will assume the number of modes is well-chosen.

It may not be possible to set  $\Phi_r^T y(t)$  to zero since the actuator movement is limited. Typically each actuator has a minimum and maximum value, with a further bending constraint placed on adjacent actuators. Thus for example we may have

$$\begin{aligned} u_{\min} &\leq u_i(t) \leq u_{\max} \\ \bar{u}_{\min} &\leq u_{i-1}(t) - 2u_i(t) + u_{i+1}(t) \leq \bar{u}_{\max} \end{aligned} \quad (6)$$

We will represent all such static constraints as requiring  $u(t) \in \mathbb{U}$ .

If a fixed (steady state) estimate  $\hat{d}_{ss}$  of  $d$  exists, then one method for attempting to minimise  $\|\Phi_r^T y(t)\|$  is to demand that in steady state  $u(t) = u_{ss}$  with

$$\begin{aligned} u_{ss} &= \arg \min_u \left\| \Phi_r^T (Bu + \hat{d}_{ss}) \right\| \\ \text{s.t. } u &\in \mathbb{U} \text{ and } (\Psi_r^\perp)^T u = 0 \end{aligned} \quad (7)$$

In some MPC schemes set points for inputs and outputs (and states) are computed at each sample using such a static optimisation criterion [6]. These set points are then used in the main dynamic optimisation, which is also solved at each sample.

Consider the basic IMC scheme depicted in Fig 2. A disturbance estimate is obtained as

$$\hat{d}(t) = y(t) - z^{-k} h(z^{-1}) B u(t) \quad (8)$$

In turn the disturbance estimate is passed through a linear filter (we will assume that the set point is zero) to give  $\hat{d}_f(t) = Q(z^{-1}) \hat{d}(t)$ . Finally the control action  $u(t)$  is generated via a static non-linear function  $u(t) = NL[\hat{d}_f(t)]$ . If we choose  $Q(1) = 1$  and the non-linearity to be

$$\begin{aligned} u(t) &= \arg \min_u \left\| \Phi_r^T (Bu + \hat{d}_f(t)) \right\| \\ \text{s.t. } u &\in \mathbb{U} \text{ and } (\Psi_r^\perp)^T u = 0 \end{aligned} \quad (9)$$

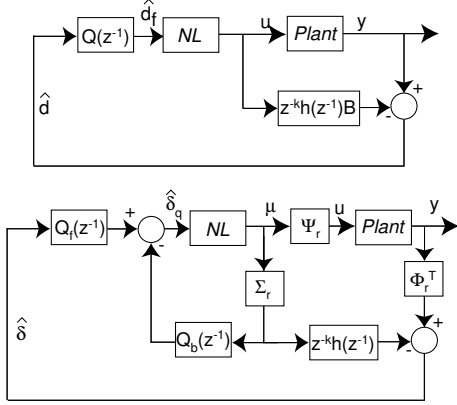


Figure 2: Basic IMC configuration with non-linear element for anti-windup, and in modal form with feedback around the non-linear element.

then we have a controller that satisfies (7), provided  $\hat{d}_f(t)$  converges to some  $\hat{d}_{ss}$ . Under such conditions, if  $y(t)$  also converges to a steady state value  $y_{ss}$  then  $\hat{d}_{ss} = y_{ss} - Bu_{ss}$ .

Furthermore, if no constraints are active, then (under the assumption that  $\Sigma_r$  has full rank),

$$\begin{aligned} u(t) &= -\Psi_r \Sigma_r^{-1} \Phi_r^T Q(z^{-1}) \hat{d}(t) \\ \hat{d}(t) &= y(t) - z^{-k} h(z^{-1}) B u(t) \end{aligned} \quad (10)$$

Thus we have a control scheme that behaves as a standard IMC when away from the constraints, but satisfies (when stable) the steady state optimality condition (7). This is achieved by including a static non-linearity in the control structure, and may be considered as an example of the anti-windup schemes recommended in [15]. The solution method for the non-linearity will depend on the choice of norm, as discussed below. In the remainder of this section, we discuss how the control structure may be tailored for the cross-directional control problem.

## 2.2 Implementation in modal form

Solving (9) is equivalent to finding  $u(t) = \Psi_r \mu(t)$  with

$$\mu(t) = \arg \min_{\mu} \left\| \Sigma_r \mu + \hat{\delta}_f(t) \right\| \quad \text{s.t. } \Psi_r \mu \in \mathbb{U} \quad (11)$$

where  $\hat{\delta}_f(t) = \Phi_r^T \hat{d}_f(t)$ . We may write  $\hat{\delta}_f(t) = Q_\delta(z^{-1}) \hat{\delta}(t)$  with  $\hat{\delta}(t) = \Phi_r^T \hat{d}(t)$  and  $Q_\delta(z^{-1}) = \Phi_r^T Q(z^{-1}) \Phi_r$ .

As stated this is merely a numerical modification. However it is more elegant to design the controller in the context of the modal decomposition [1, 9]. Thus we redefine  $Q_\delta(z^{-1})$  to be diagonal with  $Q_\delta(1) = I$ . Furthermore it may be seen that careful exploitation of the chosen basis function representation can lead to significant increase in computational efficiency. Nevertheless we stress that the use of basis functions is motivated by the requirement to ensure robust performance rather than computational efficiency.

It is insightful to consider both the unconstrained performance and the steady state performance. Without the constraint  $u(t) \in \mathbb{U}$  (and under the assumption  $\Sigma_r$  has full rank) this gives the solution

$$\begin{aligned} \mu(t) &= -\Sigma_r^{-1} Q_\delta(z^{-1}) \hat{\delta}(t) \\ u(t) &= \Psi_r \mu(t), \quad \hat{\delta}(t) = \Phi_r^T y(t) - z^{-k} h(z^{-1}) \Sigma_r \mu(t) \end{aligned} \quad (12)$$

Meanwhile in steady state (under the assumption of closed-loop stability)

$$\begin{aligned} \mu_{ss} &= \arg \min_{\mu} \left\| \Sigma_r \mu + \hat{\delta}_{ss} \right\| \quad \text{s.t. } \Psi_r \mu \in \mathbb{U} \\ u_{ss} &= \Psi_r \mu_{ss}, \quad \hat{\delta}_{ss} = \Phi_r^T y_{ss} - \Sigma_r \mu_{ss} \end{aligned} \quad (13)$$

## 2.3 Choice of $Q_\delta$

The most natural choice for  $Q_\delta(z^{-1})$  is a scalar transfer function  $q(z^{-1})$  times the identity matrix. With  $h(z^{-1})$  first order (2) a Dahlin controller  $q(z^{-1})$  would be chosen as

$$q(z^{-1}) = [(1 - \beta)(1 - \alpha z^{-1})][(1 - \alpha)(1 - \beta z^{-1})]^{-1} \quad (14)$$

with  $\beta$  a tuning parameter.

Our contention is that if the dynamics of the plant are simple, then designs for constrained systems should be based on design strategies for unconstrained systems. Thus we refer the reader to the modal designs in (for example) [13, 1, 12] for a more detailed discussion on choice of  $Q_\delta(z^{-1})$ . Where necessary, the transformation (5) should be exploited.

Note that if an IMC is designed for step output disturbances, it may give unacceptable response for slow output or input disturbances. Generally speaking good design requires a higher order filter  $Q_\delta(z^{-1})$  to ensure the appropriate sensitivities are small in closed loop. Most cross-directional control problems are regulator problems (i.e. the output set-point is zero), but if a servo response it required it may be better to incorporate such dynamics in the feedback path. See [27] for a discussion.

Often such controllers are designed mode by mode, with faster action on the lower order modes (where in general the model is better known), and slower action on the higher order modes. Provided integral action is incorporated, this will have no effect on the implicit steady state cost (7). However in [12] high gain proportional action is used in place of integral action.

## 2.4 Feedback round the nonlinear block

It is well-known that for anti-windup schemes where the non-linear element is saturation, better response can be obtained by incorporating a feedback term around the non-linearity [26]. The natural generalisation of this to our case is also depicted in Fig 2 with  $Q_b(z^{-1})$  strictly proper. The control law can be described as:

$$\begin{aligned} \hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k} h(z^{-1}) \Sigma_r \mu(t) \\ \hat{\delta}_q(t) &= Q_b(z^{-1}) \hat{\delta}(t) - Q_b(z^{-1}) \Sigma_r \mu(t) \\ \mu(t) &= \arg \min_{\mu} \left\| \Sigma_r \mu + \hat{\delta}_q(t) \right\| \quad \text{s.t. } \Psi_r \mu \in \mathbb{U} \\ u(t) &= \Psi_r \mu(t) \end{aligned} \quad (15)$$

Once again we consider two conditions. Without the constraint  $u(t) \in \mathbb{U}$  (and under the assumption  $\Sigma_r$  has full rank) this gives the solution

$$\begin{aligned}\mu(t) &= -\Sigma_r^{-1} (I - Q_b(z^{-1}))^{-1} Q_f(z^{-1}) \hat{\delta}(t) \\ u(t) &= \Psi_r \mu(t), \quad \hat{\delta}(t) = \Phi_r^T y(t) - z^{-k} h(z^{-1}) \Sigma_r \mu(t)\end{aligned}\quad (16)$$

Meanwhile in steady state

$$\begin{aligned}\mu_{ss} &= \arg \min_{\mu} \left\| (I - Q_b(1)) \Sigma_r \mu + Q_f(1) \hat{\delta}_{ss} \right\| \\ &\text{s.t. } \Psi_r \mu \in \mathbb{U} \\ u_{ss} &= \Psi_r \mu_{ss}, \quad \hat{\delta}_{ss} = \Phi_r^T y_{ss} - \Sigma_r \mu_{ss}\end{aligned}\quad (17)$$

In order to ensure equivalence with the previous case under these two conditions, it is sufficient that  $Q_f(1)$  is some scalar times the identity and

$$Q_b(z^{-1}) = I - Q_f(z^{-1}) [Q_\delta(z^{-1})]^{-1}\quad (18)$$

In particular if we choose

$$Q_f(z^{-1}) = \lambda Q_\delta(z^{-1}) + (1 - \lambda) Q_\delta(0)\quad (19)$$

for some scalar  $\lambda$  then  $Q_b(z^{-1})$  is guaranteed strictly proper. The choice of  $\lambda$  (in the context of a saturating non-linearity) is discussed in [26].

The relation of the resulting controller to one step horizon MPC is discussed in [28]. Here we note that minimum variance control gives one choice of  $\beta$  and  $\lambda$ . Suppose the plant is given by (1) and (2) with

$$d(t) = (1 - cz^{-1})(1 - z^{-1})^{-1} e(t)\quad (20)$$

where  $e(t)$  is some zero mean noise with  $\mathcal{E} \{e(t)e^T(t-k)\} = 0$  for  $k \geq 1$ . Suppose we want the minimum variance control

$$u(t) = \arg \min_u \mathcal{E} \left\{ \left\| \Phi_r^T \hat{y}(t+k) \right\|_2^2 \right\} \text{ s.t. } u(t) \in \mathbb{U}\quad (21)$$

In the unconstrained case the solution is the Dahlin controller [27]. With constraints this gives

$$\begin{aligned}Q_f(z^{-1}) &= (1 - \alpha)^{-1} (1 - c)(1 - cz^{-1})^{-1} I \\ Q_b(z^{-1}) &= -\alpha z^{-1} (1 - \alpha z^{-1})^{-1} I\end{aligned}\quad (22)$$

It can be observed that this corresponds to (14) with  $\beta = c$ . The choice of  $Q_f(z^{-1})$  and  $Q_b(z^{-1})$  corresponds to  $\lambda = \frac{\beta}{\beta - \alpha}$  in (19).

## 2.5 Choice of non-linear function

So far we have not specified the choice of norm on the non-linear function. The standard choice would be a 2-norm, which results in a quadratic program. But as we have separated the non-linearity from the dynamics it is straightforward to introduce any other choice of norm, without changing the dynamics away from the constraints. In particular both  $l_1$  norm [18] and  $l_\infty$  norm [19] criteria have been recommended for certain

cross-directional problems. Either may be substituted in this scheme, resulting in a linear program. Such choices should be motivated by the required steady state performance.

Similarly we may choose to weight each mode differently in the cost function. For example, it may be preferred to weight low order modes (where in general the model is better known) more heavily than high order modes.

In [29] the authors recommend adding a barrier to the steady state calculation in MPC. Such a barrier may also be included in the non-linear function. It has the effect of ensuring the inputs lie on the interior of the constraint set, while a limit is put on the associated performance degradation by the duality gap. Such optimization problems may be solved efficiently via modified interior point algorithms [30].

Finally in [31] a modified steady state criterion is proposed that is robust to model mismatch. It may also be incorporated into the non-linear function. The associated optimization is then usually a conic program.

## 3 Simulation example

Consider a cross-directional plant model in the form of (1) with  $m = 101$  and  $n = 501$ , the delay term  $k = 4$  and  $\alpha = 0.95$ . For the purposes of this simulation, we assume the true interaction matrix is given by  $B_p$  and an estimate of the interaction matrix is denoted by  $B$ . Their columns are both truncated sinc functions. The steady state responses for both  $B_p$  and  $B$  to a single actuator  $u_{51}$  are shown in Figure 3. It was found that for  $r > 27$ , either the closed-loop response was unstable or the steady-state variation became very large.

The disturbance term  $d(t)$  was constructed as follows. Let  $e(t) \in \mathbb{R}^n$  be a zero-mean coloured white noise sequence with covariance  $\mathcal{E} [e_i(t)e_j^T(t)] = 0.9^{|i-j|} 0.01/0.81$ . Then  $d(t)$  is given by  $d(t) = 0.01(1 - 0.99z^{-1})^{-1} e(t)$ . Note that once created, this disturbance was fixed and used for each simulation.

Constraints on the actuators are present in the form of (6) with  $u_{\min} = -1$ ,  $u_{\max} = 1$ ,  $\bar{u}_{\min} = -0.1$  and  $\bar{u}_{\max} = 0.1$ .

We introduce the measure  $\bar{y} = \sum_{i=N_1}^{N_2} \|y(i)\|_2^2$  to distinguish between different controller designs, with  $N_1 < N_2$  integers. In what follows we fix  $N_1 = 200$  and  $N_2 = 1000$ .

Consider the second controller design from Figure 2. We choose a Dahlin controller structure for  $Q_\delta(z^{-1})$  as given by (14) and consider two choices for  $\beta$  - namely  $\beta = 0.9$  and  $\beta = 0.3$ . The feedback term  $Q_b(z^{-1})$  is given by (18) with  $Q_f(z^{-1})$  given by (19) and  $\lambda = 0.5$ . An  $l_2$  norm was chosen for the optimization problem (15). The number of modes was chosen at  $r = 15$ . The measure  $\bar{y}$  for both choices of  $\beta$  are shown in Table 3. The results shown in Table 3 are ordered according to the value of  $\bar{y}$ . In this case the controller with faster nominal response performs better, as would be expected.

Consider a fixed value of  $\beta = 0.3$  and four choices for  $\lambda$  given by  $\lambda = 1, 0.5, 0, \beta/(\beta - \alpha)$ . The latter choice of  $\lambda$  corresponds to that for minimum variance control, albeit the value of  $\beta$  is

$\lambda$	$\beta$	$r$	$\nu$	Controller	$\bar{y}$
-	-	-	-	no	$2.4391e + 05$
0.5	0.3	9	0	yes	$1.9037e + 05$
0.5	0.9	15	0	yes	$1.7833e + 05$
1	0.3	15	0	yes	$1.7612e + 05$
$\beta/(\beta - \alpha)$	0.3	15	1	yes	$1.7372e + 05$
0.5	0.3	15	0	yes	$1.7339e + 05$
0	0.3	15	0	yes	$1.7326e + 05$
$\beta/(\beta - \alpha)$	0.3	15	0	yes	$1.7325e + 05$

Table 1: Set of simulation experiments with measure  $\bar{y}$  for different parameter values.

motivated differently here. The measure  $\bar{y}$  for all four cases is also shown in Table 3. Figure 4 shows the final profile across the strip at  $t = 845$  for both the open-loop response and the choice  $\lambda = \beta/(\beta - \alpha)$ . Once again the ordering is as expected, with the latter choice of  $\lambda$  performing best.

To illustrate the benefits of constrained control, consider a fixed value for  $\lambda = 0.5$ , a fixed value for  $\beta = 0.3$  and two choices for  $r$  - namely  $r = 9$  and  $r = 15$ . The smaller value of  $r = 9$  was selected as the largest  $r$  such that input constraints are not violated. Figure 5 shows the output variation at each time interval for both choices of  $r$ . The respective values of  $\bar{y}$  are shown in Table 3.

As a final comparison, we considered a related optimization problem to (15) which uses a logarithmic barrier function [29] with fixed weighting term  $\nu = 1$ , to represent the constraint set  $\mathcal{U}$ . The associated optimization problem is,

$$\mu(t) = \arg \min_{\mu} \|\Sigma_r \mu + \hat{\delta}_q(t)\|_2^2 - \nu \sum_{i=1}^M \ln(b_i - \{A \Psi_r \mu\}_i) \quad (23)$$

where  $\{A \Psi_r \mu\}_i$  is the  $i$ 'th element of  $A \Psi_r \mu$  and  $A$  and  $b$  represent the inequality constraints on the inputs  $Au \leq b$ . In the limit as  $\nu \rightarrow 0$ , the solution to (23) coincides with the solution to (15). We chose  $\beta = 0.3$ ,  $\lambda = \beta/(\beta - \alpha)$ ,  $r = 15$  and  $\nu = 1$ . The measure  $\bar{y}$  for this case is shown in Table 3.

The worst case completion time for the computation in all these results was 40 milliseconds on a Pentium IV, 1.8GHz machine. Computation issues are discussed in [28].

## 4 Conclusion

It has become standard to address the cross-directional control problem by decomposing the inputs and outputs into modes. For robust stability and performance it is necessary that the controller acts only on a reduced number of these modes.

If the controller acts on a sufficiently small number of modes the actuators will automatically lie within their constraints. But to maximise economic performance it may be necessary to include a greater number of modes, requiring actuator constraints to be taken explicitly into account. We have shown that in

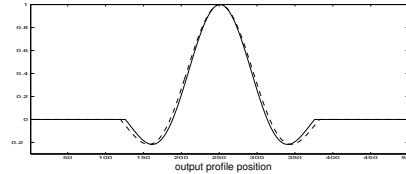


Figure 3: The cross-directional profiles for both  $B_p$  (solid) and  $B$  (dashed) with  $u_{51} = 1$ . Note that  $B$  is a stretched version of  $B_p$ .

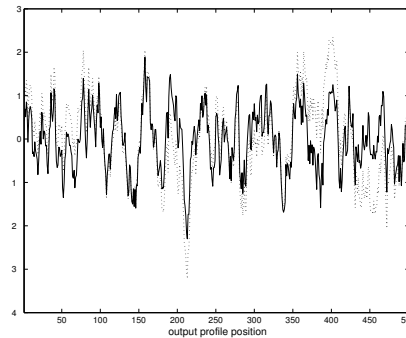


Figure 4: Cross-directional profile for  $t = 845$  for  $\beta = 0.3$ ,  $\lambda = \beta/(\beta - \alpha)$  (solid) and open loop (dotted).

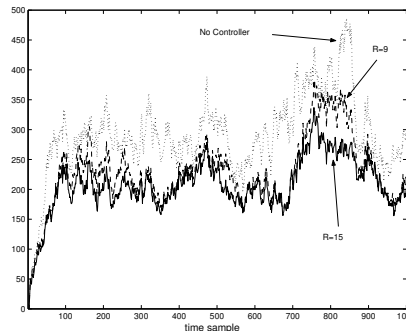


Figure 5: Output variation over the simulation period. Three cases are shown: no controller (dotted),  $R = 9$  (dashed) and  $R = 15$  (solid). The solid line corresponds to the same parameters as the closed-loop example shown in Figure 4

this case it is possible to guarantee optimal steady state performance with a modified IMC structure. For the simple dynamics associated with most web processes this also gives good closed-loop dynamic behaviour. Thus unconstrained control design techniques may be applied directly to the constrained control problem.

We have demonstrated such a control design with a simulation example. It illustrates that such a controller may be easily implemented in real time, even on a systems with fast sampling.

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