CRHPC UNDER INPUT CONSTRAINTS; A BARRIER FUNCTION APPROACH

DAMIAN JERZY, ANDRZEJ KRÓLIKOWSKI‡

Poznań University of Technology, Institute of Control and Information Engineering,
Faculty of Electrical Engineering
60-965 Poznań, ul.Piotrowo 3A, POLAND.
‡Andrzej.Krolikowski@put.poznan.pl

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Abstract

Handling of rate and amplitude constraints in guaranteed stability model predictive control problem of linear discrete-time systems is considered. The method based on the barrier function is used to solve the constrained optimization problem. A simulation based analysis of closed-loop control system stability and performance is also given with respect to design parameters and constraints.

1 INTRODUCTION

Model predictive control is a popular control strategy because of its simplicity and successful industrial applications. One of the main advantages of the finite horizon predictive control is that it can handle constraints. Taking constraints into account in the design stage leads inherently to a solution of constrained optimisation problem. The application of quadratic programming (QP) techniques to solve the generalized predictive control (GPC) is widely used, see the comments given in [1] and [2]. For example, the constrained GPC (CGPC) has been discussed in [3] where the QP problem is transformed into the Linear Complementarity Problem (LCP) which in turn is solved using Lemke's algorithm. This reduces the amount of computation compared with the QP. Another attempt to reduce the computation burden is presented in [4], where the Lagrange multipliers method was used to handle separately with rate and amplitude constraints. Some other approaches can be found for example in [1], [2], [5], [6]. A general survey on constrained model predictive control for state space models is given in [7].

In this paper, a guaranteed stability predictive control named in [8] as a constrained receding-horizon predictive control (CRHPC) is first shortly described. Then this control approach is taken as a starting point to derive the predictive control algorithm under amplitude and rate constraints. As an optimization technique a method based on the logarithmic barrier function [13] is proposed. A simulation-based comparison of the performance with respect to design parameters and constraints is given. To this end, the second order unstable and non-minimum phase systems are taken for the simulation study. Additionally, the computational load of this constrained predictive control algorithm is also analyzed taking into account a potential real-time implementation.

2 THE GUARANTEED STABILITY PREDICTIVE CONTROL (CRHPC)

The input-output description of linear discrete-time system is given by

$$A(q^{-1})\Delta y_t = B(q^{-1})\Delta u_t \tag{1}$$

where y_t is the output, u_t is the control input and A, B, Δ are polynomials in the backward shift operator q^{-1}, i.e., $A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$, $B(q^{-1}) = b_1 q^{-1} + \cdots + b_n q^{-n}$, $\Delta(q^{-1}) = 1 - q^{-1}$.

The goal of the control is to cause the output y_t to follow a reference signal r_t taking into account the control effort. This can be expressed by the cost function of the form

$$J(N, q_y, q_u) = \sum_{i=1}^{N} q_{y,t+i}(y_{t+i} - r_{t+i})^2 + \sum_{i=0}^{N} q_{u,t+i}\Delta u_{t+i}^2 \tag{2}$$

where the weights $q_{y,t+i-1} \geq 0$, $q_{u,t+i-1} > 0$ and the prediction horizon N are basic design parameters of predictive controller. The CRHPC algorithm [2] minimizes the cost (2) subject to the set of m future terminal equality constraints

$$y_{t+N+j} = r_{t+N} \qquad j = 1, \ldots, m \tag{3}$$

$$\Delta u_{t+N+j} = 0 \qquad j = 1, \ldots, m \tag{4}$$

The cost function (2) and constraints (3),(4) can be given the form

$$J(N, q_y, q_u) = (y_1 - r_1)^T Q_y (y_1 - r_1) + \bar{u}^T Q_u \bar{u} \tag{5}$$

$$\bar{G}\bar{u} + f_2 = r_2 \tag{6}$$

where $\bar{u} = (\Delta u_t, \ldots, \Delta u_{t+N})^T$. The following relations also hold

$$y_1 = G\bar{u} + f_1 \tag{7}$$

$$y_2 = \bar{G}\bar{u} + f_2 \tag{8}$$

where the matrix G is composed of the step response coefficients $\{g_i\}$ of the control channel, i.e., the points of $\frac{B}{A\Delta}$

$$G = \begin{bmatrix} g_1 & 0 & \ldots & 0 & 0 \\ g_2 & g_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ g_N & g_{N-1} & \ldots & g_1 & 0 \end{bmatrix}$$

the matrix $\bar{G}$ is

$$\bar{G} = \begin{bmatrix} g_{N+1} & g_N & \cdots & g_1 \\ g_{N+2} & g_{N+1} & \cdots & g_2 \\ \cdots & \cdots & \cdots & \cdots \\ g_{N+m} & g_{N+m-1} & \cdots & g_m \end{bmatrix}$$

and

$$Q_y = diag(q_{y,t+1}, q_{y,t+2}, \ldots, q_{y,t+N})$$
$$Q_u = diag(q_{u,t}, q_{u,t+1}, \ldots, q_{u,t+N})$$
$$f_1 = (f_{t+1}, \ldots, f_{t+N})^T$$
$$r_1 = (r_{t+1}, \ldots, r_{t+N})^T$$
$$f_2 = (f_{t+N+1}, \ldots, f_{t+N+m})^T$$
$$r_2 = (r_{t+N+1}, \ldots, r_{t+N+m})^T$$
$$y_1 = (y_{t+1}, \ldots, y_{t+N})^T$$
$$y_2 = (y_{t+N+1}, \ldots, y_{t+N+m})^T$$

where f_{t+i} is a prediction of y_{t+i}, $i = 1, ..., N+m$.

The optimal control minimizing (5) subject to the equality constraint (6) is then [8]

$$\bar{u}^o = \tilde{G}[I - \bar{G}^T(\bar{G}\tilde{G}\bar{G}^T)^{-1}\bar{G}\tilde{G}]G^T Q_y e_1 + + \tilde{G}\bar{G}^T(\bar{G}\tilde{G}\bar{G}^T)^{-1}e_2 \quad (9)$$

where

$$\bar{u}^o = (\Delta u_t^o, \ldots, \Delta u_{t+N}^o)^T \quad (10)$$

and

$$\tilde{G} = (G^T Q_y G + Q_u)^{-1} \quad (11)$$
$$e_1 = r_1 - f_1 \quad (12)$$
$$e_2 = r_2 - f_2 \quad (13)$$

The first element of the sequence (10), i.e. Δu_t^o, is applied to the system. Then the optimization procedure starts again at the next time instant $(t+1)$ with the current data. This means that in the control law (9) only the vectors f_1, r_1, f_2, r_2 should be updated.

As shown in [8], the closed-loop system under CRHPC is asymptotically stable if

$$N \geq n+2, \;\; m = n+1$$

and

$$q_{u,t+i} = \bar{q}_u > 0, \;\; q_{y,t+i} = \bar{q}_y \geq 0 \;\; \forall i.$$

3 HANDLING OF INPUT CONSTRAINTS

The above described CRHPC is considered under the following control input constraints: the rate constraint

$$|\Delta u_t| \leq du \quad (14)$$

and the amplitude constraint

$$|u_t| \leq u \quad (15)$$

which formulates the problem of CRHPC under constraints to be solved below.

The input constraints can be represented as follows

$$R_1 \bar{u} \leq c_1 \quad (16)$$

where c_1 is a vector containing upper and lower constraints and R_1 is a block matrix of the form

$$R_1 = \begin{bmatrix} I \\ -I \\ T \\ -T \end{bmatrix} \qquad c_1 = \begin{bmatrix} I_1 du \\ I_1 du \\ I_1 u - I_1 u_{t-1} \\ I_1 u + I_1 u_{t-1} \end{bmatrix} \quad (17)$$

where $I_{1((N+1)\times 1)} = (1, \ldots, 1)^T$, $T_{((N+1)\times(N+1))}$ is a lower triangular matrix whose nonzero entries equal to 1 and $I_{((N+1)\times(N+1))}$ is the unit matrix.

Taking (7) into consideration the cost function (5) can be expressed in the form

$$J = \frac{1}{2}\bar{u}^T H \bar{u} + h_1^T \bar{u} + h_2 \quad (18)$$

and the equality constraint (6) takes a form

$$\bar{G}\bar{u} = e_2 \quad (19)$$

where

$$H = 2(G^T Q_y G + Q_u)$$
$$h_1^T = -2e_1^T Q_y G$$
$$h_2 = e_1^T Q_y e_1$$

Now, the problem consists in minimization of (18) under constraints (16), (19), however the equality constraint (19) can be relaxed as follows

$$e_2 - \epsilon_2 \leq \bar{G}\bar{u} \leq e_2 + \epsilon_2 \quad (20)$$

where ϵ_2 is a tolerance variable. The inequality constraints (16) and (20) can be put together in the form

$$R\bar{u} \leq d \quad (21)$$

where $R = (R_1^T, \bar{G}^T, -\bar{G}^T)^T$ and $d = [c_1^T, (e_2+\epsilon_2)^T, (-e_2+\epsilon_2)^T]^T$.

4 THE LOGARITHMIC BARRIER METHOD

Below the logarithmic barrier method is proposed to solve the constrained predictive control problem. Taking the cost function (18) and the constraint (21) into account the following logarithmic barrier function can be given [13], [14]

$$\phi_B(\bar{u}, \mu) = \frac{J(\bar{u})}{\mu} - \sum_{i=1}^{4N+4+2m} ln(d_i - \underline{r}_i^T \bar{u}) \quad (22)$$

where d_i is the i-th element of vector $d_{((4N+4+2m)\times 1)}$, r_i^T is the i-th row of matrix R, and $\mu > 0$ is the barrier parameter.

Because of the singularity of the logarithm at zero this barrier function will prevent the iterates from going outside the feasible region. Therefore the logarithmic barrier function method is called an interior point method. To implement this method the gradient and the Hessian of $\phi_B(\bar{u}, \mu)$ are needed

$$\underline{g}_1 \hat{=} \nabla \phi_B(\bar{u}, \mu) = \frac{1}{\mu}(H\bar{u} + h_1) - \sum_{i=1}^{4N+4+2m} \frac{\nabla f_i(\bar{u})}{f_i(\bar{u})} \quad (23)$$

$$H_1 \hat{=} \nabla^2 \phi_B(\bar{u}, \mu) = \frac{1}{\mu}H + \sum_{i=1}^{4N+4+2m} [\frac{\nabla^2 f_i(\bar{u})}{-f_i(\bar{u}} + \\ + \frac{\nabla f_i(\bar{u}) \nabla f_i(\bar{u})^T}{f_i^2(\bar{u})}] \quad (24)$$

where $f_i = -(d_i - r_i^T \bar{u})$, so $\nabla f_i(\bar{u}) = r_i$, $\nabla^2 f_i(\bar{u}) = 0$. Taking the Newton direction

$$\underline{p} = -H_1^{-1} \underline{g}_1 \quad (25)$$

the following criterion to terminate the approximate minimization of $\phi_B(\bar{u}, \mu)$

$$\| \underline{p} \|_{H_1} \leq \tau < 1 \quad (26)$$

where the norm is defined as

$$\| \underline{p} \|_{H_1} = \sqrt{\underline{p}^T H_1 \underline{p}} \quad (27)$$

and in the considered case depends on $\bar{u}$. Now the following algorithm for finding an ϵ-optimal solution can be proposed [13]:

Input:
ϵ - the accuracy parameter
τ - the proximity parameter
θ - the reduction parameter
μ_0 - the initial barrier value
$\bar{u}^0$ - a given interior feasible point such that
$\| \underline{p}(\bar{u}^0, \mu_0) \|_{H_1(\bar{u}^0, \mu_0)} \leq \tau$

begin
$\bar{u} := \bar{u}^0; \mu := \mu_0;$
while $\mu > \frac{\epsilon}{16N}$ **do**
begin (outer step)
$\mu := (1 - \theta)\mu;$
while $\| \underline{p} \|_{H_1} \geq \tau$ **do**
begin (inner step)
$\tilde{\alpha} := \arg\ \min_{\alpha > 0} \{\phi_B(\bar{u} + \alpha \underline{p}, \mu) : \bar{u} + \alpha \underline{p} \in F^0\}$
$\bar{u} := \bar{u} + \tilde{\alpha} \underline{p}$
end (inner step)
end (outer step)
end
where F^0 is the interior of the feasible region.

4.1 A simpler approach

It can be seen that in order to implement the barrier method the gradient and the Hessian of the cost function J are needed. From (9), the optimal CRHPC can be given the form

$$\bar{u}^o = \tilde{G}(G_1 e_1 + G_2 e_2) \quad (28)$$

Thus, the corresponding gradient and Hessian of the cost function J are

$$\nabla J(\bar{u}) = \tilde{G}^{-1}\bar{u} - G_1 e_1 - G_2 e_2 \quad (29)$$

$$\nabla^2 J(\bar{u}) = \tilde{G}^{-1} \quad (30)$$

and G_1, G_2 resulting from (9) have the following forms

$$G_1 = [I - \bar{G}^T (\bar{G}\tilde{G}\bar{G}^T)^{-1} \bar{G}\tilde{G}] G^T Q_y \quad (31)$$

$$G_2 = \bar{G}^T (\bar{G}\tilde{G}\bar{G}^T)^{-1} \quad (32)$$

The expressions (29),(30) can be used in (23), (24) to obtain the needed forms for $\underline{g}_1$ and H_1 and to implement the proposed algorithm. Then in the functions f_i only the constraint (16) is included.

5 SIMULATIONS

In order to illustrate the performance of the derived algorithm a few simulation runs are presented below. Since the main focus in this paper was put on solving constrained optimization problem by means of barrier method, most of the simulations were made with algorithm including equality constraint (6) in the relaxed form of inequality (20). The other method including this equality in the gradient presented in subsection 4.1 was used only for comparison purpose.

The following second-order systems were taken for simulation:

1. unstable with $a_1 = 1.8, a_2 = -0.9, b_0 = 1.0, b_1 = 0.5$
2. non-minimum phase with $a_1 = -1.5, a_2 = 0.7, b_1 = -1.0, b_2 = 2.0$

Simulation runs were performed for a square wave as a reference signal given by

$$r_{40T+t+5} = r(-1)^T \quad t = 0, 1, \ldots, 22, \qquad T = 0, 1, \ldots$$

If not given otherwise, the design controller parameters were set at $N = 4, m = 3, q_y = 1, q_u = 0,001$ and the relaxation parameter ϵ_2 set at 0.6 for both the systems 1 and 2. The design parameters for barrier method itself were as follows: $\epsilon = 1, \tau = 0.2, \theta = 0.4, \mu = 100$.

The signal values for both models where no constraints (unc) are present are given in the table 1 to facilitate assessing ability of the barrier method algorithm to suppress input rate and amplitude signals where $maxall$ denotes maximal signal values calculated in single run by the barrier method procedure which were taken from the whole control horizon. In turn index max corresponds to maximal signal value applied to the model, i.e. the first entry in the vector of controls.

In figs.1 and 2 the system 1 under rate constraint is simulated. It can be seen that even weak constraint prevents the output from overshooting, however tight constraints slow down considerably changes of the output. The simulation showed that at $du < 0.63$ the algorithm cannot find any appropriate solution

	$\Delta u_{t,maxunc}$	$u_{t,maxunc}$	$\Delta u_{t,maxallunc}$	$u_{t,maxallunc}$
1	2.403	2.630	3.815	2.938
2	1.192	0.726	1.239	0.726

Table 1: Maximal values of unconstrained input signals

$Model$	$Input\ rate$			$Input\ amplitude$		
	du	$\Delta u_{t,maxall}$	$\Delta u_{t,maxall}/\Delta u_{t,maxallunc}$	u	$u_{t,maxall}$	$u_{t,maxall}/u_{t,maxallunc}$
1	0.63	0.614	16.1%	1.6	1.599	54.4%
2	0.18	0.172	13.8%	0.23	0.222	30.5%

Table 2: Minimum feasible constraints of the barrier method

satisfying inequalities (20), therefore this value is considered to be minimal rate constraint which can be applied to the system. In the case of system 2 similar simulations are shown in fig.9, and here the corresponding value du is 0.18. Figs.3 and 10 present the amplitude-constrained run of the system 1 and 2, respectively. The amplitude can be safely suppressed to $u = 1.6$ for system 1 what means that the output still follows the reference signal, however quality of this control deteriorates at smaller constraint values. The simulation show that amplitude of signal can continuously be suppressed even further but the output is not able to follow properly the reference signal anymore then in consequence such control can not be accepted. At $u < 0.65$ the inequalities (20) are not fulfilled, it means the barrier method procedure can not find any feasible solution yet. For system 2 relevant constraint values are $u = 0.23$ at which the run is still acceptable and $u = 0.11$ which limits possibility to find solution of the optimization problem. Since inequalities (20) determine constraining capability of the algorithm it is worthwhile to see how the relaxation parameter ϵ_2 influences this. Fig.4 indicates that the relationship between ϵ_2 and minimal feasible rate constraint value du is almost linear: the more tight constraint given by ϵ_2 the poorer rate constraining capabilities the algorithm shows. An illustration to that is also fig.5 performed with $\epsilon_2 = 0.6$ which presents maximal error of future equality constraints given by (19). It is calculated as $max\ error = max(abs(\bar{k}_{j,i}))$, $j = 1, ..., m$, $i = 1, ..., 80$, where $\bar{k} = \bar{G}\bar{u} - e_2$. It is clear that at more tight rate constraint the error increases approaching the limit $\epsilon_2 = 0.6$ and at certain du no feasible solution can be found. With dotted line also the performance of the algorithm including equality constraints in gradient is presented in the same constraining conditions. As here no explicit control over fulfilling equation (19) is present, the relevant error is essentially bigger than in the previous method. On the other hand, lower number of constraint inequalities (see (16) and (17) vs. (21)) results in lower computational complexity compared to the previous method. This is illustrated in fig.6. It makes evident the fact that major computational capacity is required by the barrier method itself which starts constraining at $du = 3.81$. To improve accuracy of both methods the parameter $\epsilon = 0.1$ was taken and the simulations were performed as presented in figs.5, 6 and 7. A general feature of barrier methods related to inability to fully utilize available signal range is demonstrated in fig.7. Due to singularity of the barrier function at boundaries, the algorithm strives to move the solution away from the boundary so it never reaches it (this is value 100% in this figure). It becomes of course more and more difficult when the allowed range of the signal gets more tight. Similarly as described previously, $\Delta u_{t,max}$ denotes the maximal value in the whole run of the first entry in the control vector (10), and $\Delta u_{t,maxall}$ denotes the maximal value of the whole vector (10). Finally, fig.8 presents system 1 at simultaneous input rate and amplitude constraints put close to the minimal acceptable values. These conditions are of course even more severe for the system what results in slightly slower output changes compared to rate and amplitude constraints treated separately. To summarize capability of barrier method to constrain input signals the table 2 lists minimum feasible constraints which can be imposed for relevant models in assumed testing conditions. It can be seen that barrier method is able to considerably reduce the input signal value. However this is of course always a compromise between the input value, quality of following the reference and computational load.

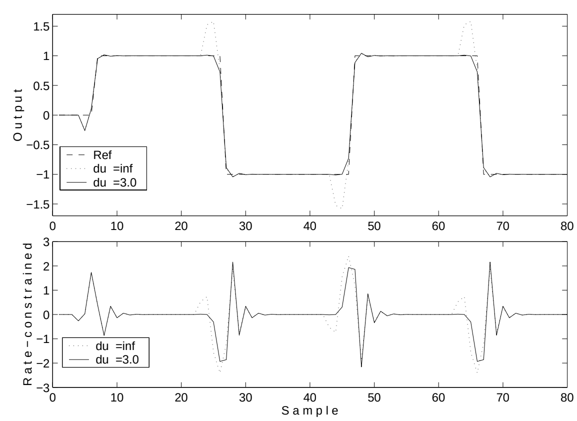

Fig. 1: Unstable system under rate constraint

6 CONCLUSIONS

The input-constrained CRHPC problem is presented and solved using logarithmic barrier algorithm. Second order unstable and non-minimum phase systems were taken for simulation of the control system operating with the proposed algorithm. The simulations showed that algorithm is able to considerably reduce rate and amplitude input constraints which can be handled simultaneously. However, the computational load of the method is significant and strongly dependent on degree of signal constraining. Therefore in the case of potential real-time application a compromise between signal suppressing, track-

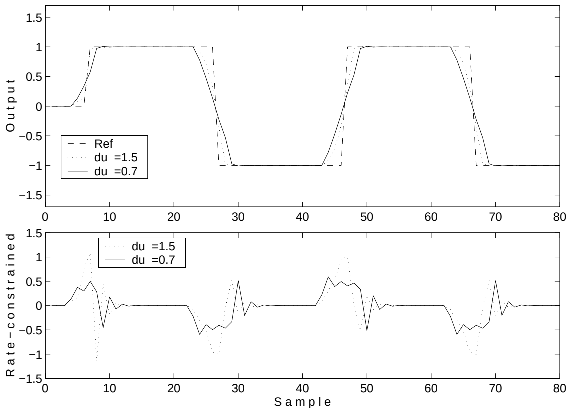

Fig. 2: Unstable system under rate constraint

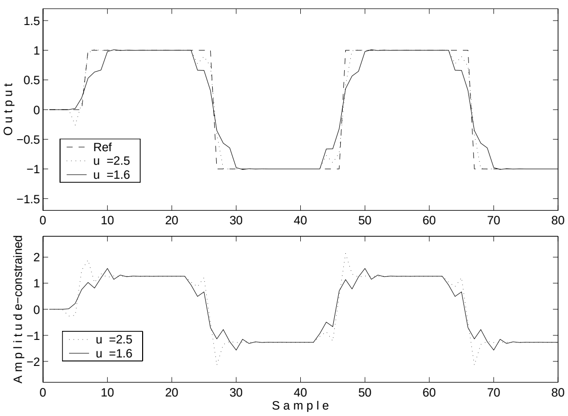

Fig. 3: Unstable system under amplitude constraint

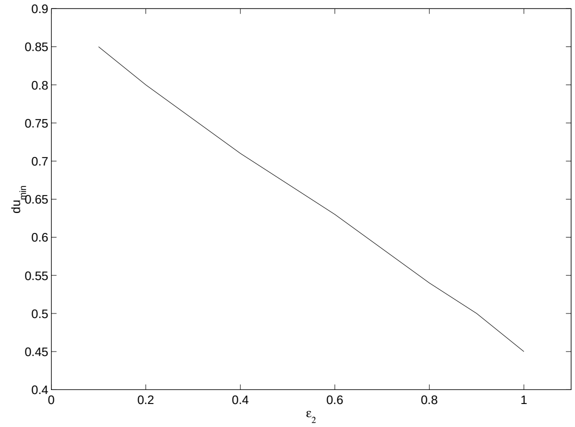

Fig. 4: Minimal feasible rate constraint vs. ϵ_2

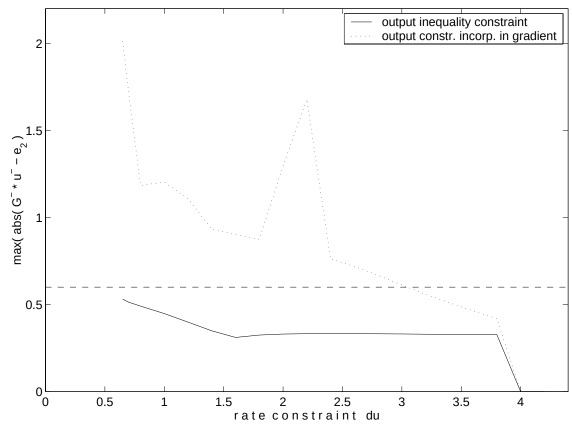

Fig. 5: $Max\ error$ of equality constraint vs. rate constraint

ing quality and computational load must be balanced.

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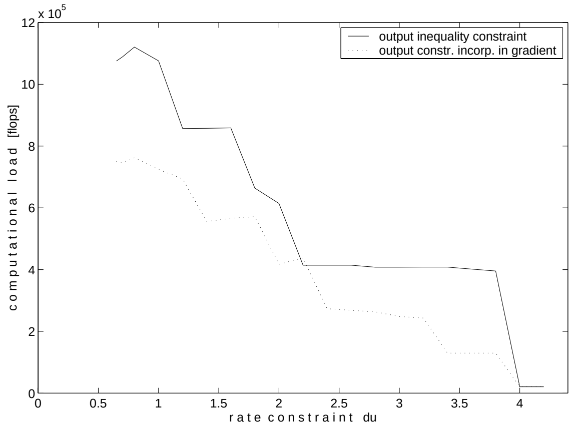

Fig. 6: Computational load vs. rate constraint

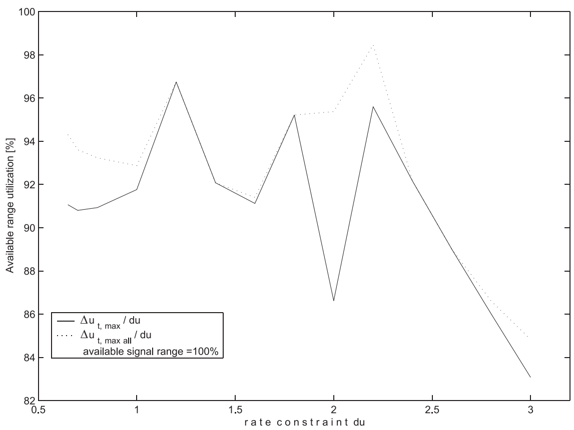

Fig. 7: Utilization of available range by control signal

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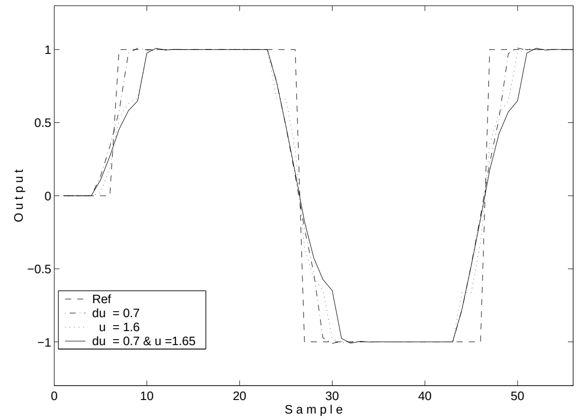

Fig. 8: Unstable system under rate and amplitude constraint

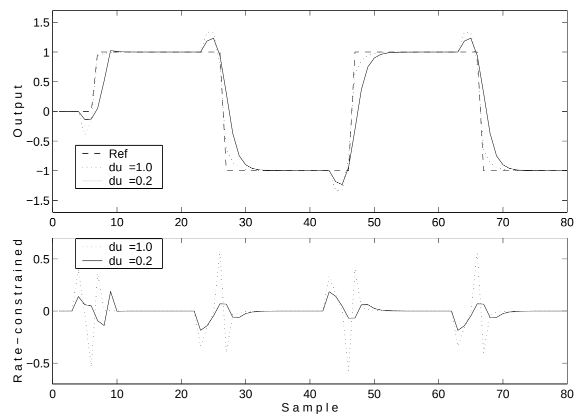

Fig. 9: Nmp system under rate constraint

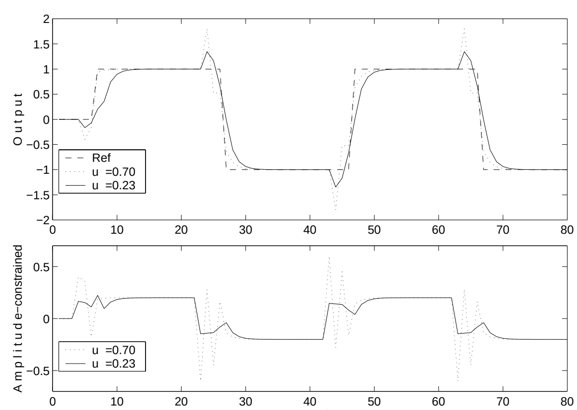

Fig. 10: Nmp system under amplitude constraint