ON OPTIMALITY AND CERTAINTY EQUIVALENCE IN OUTPUT FEEDBACK CONTROL OF CONSTRAINED UNCERTAIN LINEAR SYSTEMS

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Abstract

We address the problem of finite-horizon optimal control of uncertain discrete-time SISO linear systems with input constraints. The uncertainty for the problem analyzed is related to incomplete state information (output feedback) and stochastic disturbances. We analyze the optimal solutions for short optimization horizons when the disturbances and the initial state are Gaussian random vectors. We also consider two suboptimal strategies, one of which is known as Certainty Equivalent Control.

1 Introduction

Control problems where constraints are imposed on certain variables and where performance is measured by means of an additive cost function are usually encountered in practice. One widely-known technique that addresses such problems is Model Predictive Control (MPC) [14].

In this kind of problems, an optimal control strategy is understood as one that minimizes a cost function while insuring that all constraints will be satisfied. Finding the optimal control strategy involves the solution of an optimization problem. Depending on the equations that describe the system and the information that can be obtained from the system (measurements), different kinds of optimization problems arise. Two main forms of system descriptions can be distinguished: deterministic and stochastic.

The problem of constrained control of uncertain systems with a stochastic description is still not fully resolved within the framework of MPC [14]. In [11], output feedback predictive control of non-linear systems with uncertain parameters is addressed. In this work, the authors consider no constraints associated with the control. Due to the difficulties of the optimization for a general non-linear system, only suboptimal solutions are presented. In reference [12], state feedback control of unconstrained linear systems with uncertain parameters is addressed. In reference [4], the case of state feedback, input constraints and scalar disturbances, is considered. A randomized algorithm (Monte Carlo sampling) is used to approximate the optimal solution. Examples for an optimization horizon of only one time instant are presented. In [5], the same authors analyze a similar case, but where state constraints are also considered. They utilize a short optimization horizon (5 time instants) and implement the solution over a longer horizon in a receding horizon fashion.

When there is incomplete state information, an observer-based strategy that seems natural for control in the presence of stochastic disturbances is the one that uses the so-called *Certainty Equivalence* (CE) principle. Specifically, CE consists of estimating the state and then using these estimates as if they were the true state in the control law that results when the problem is formulated as a deterministic problem (no uncertainty). This strategy is motivated by the unconstrained control problem for linear systems with quadratic cost, for which the obtained strategy is indeed optimal [3, 9]. The use of CE in MPC leads to CE-MPC, and due to its simplicity, this strategy has been advocated in the literature [15] and reported in a number of applications [2], [13] and [16]. Notwithstanding the widespread use of CE-MPC in applications, it must be stressed that CE-MPC, generally, results in a suboptimal control strategy. Two factors can be highlighted that render CE-MPC suboptimal: (1) the state estimate is assumed to be the true current state, and (2) the stochastic behavior of the system is neglected over the prediction horizon.

The main aim of this work is to focus on the case where the disturbances and the initial state are Gaussian random vectors and to provide an example where the CE solution differs from the optimal one. This work follows a line of research started in [17].

2 Problem Statement

We consider the following time-invariant discrete-time linear system

$$\begin{aligned} \mathbf{x}_{k+1} &= A\,\mathbf{x}_k + Bu_k + \mathbf{w}_k, && (1)\\ y_k &= C\,\mathbf{x}_k + v_k, && (2) \end{aligned}$$

where $\mathbf{x}_k, \mathbf{w}_k \in \mathbb{R}^n$ and $u_k, y_k, v_k \in \mathbb{R}$. The disturbances $\mathbf{w}_k$ and v_k are assumed to be sequences of independent and identically distributed Gaussian random vectors with zero mean and covariance matrices $R_{\mathbf{w}}$ and R_v, respectively. The scalar control, u_k, is constrained to take values in the following set

$$\mathcal{U} = \{u : -\Delta \le u \le \Delta\} \subset \mathbb{R}, \tag{3}$$

for a given constant value $\Delta > 0$. The initial state, $\mathbf{x}_0$, is also a Gaussian random vector, having mean $\mathbf{x}_{00}$ and covariance matrix P_{00}. We assume that the pair (A, B) is reachable and that the pair (A, C) is observable.

We further assume that, at the event k, the value of the state $\mathbf{x}_k$ is not available to the controller. Instead, all the past inputs and past and current outputs (measurements) are available to the controller at time instant k, and contained in the *information vector*, denoted by I^k, where

$$I^k = \{y_0, y_1, \dots, y_k, u_0, u_1, \dots u_{k-1}\} \in \mathbb{R}^{2k+1}, \tag{4}$$

and we see that $I^{k-1} \subset I^k$.

At time instant k, the controller must calculate the control action u_k based on the available information I^k in order that the constraints (3) be satisfied and a cost function minimized. Before posing the optimal control problem, we need the following definition.

Definition 1 (Admissible control policies) *A policy Π_N is a finite sequence of functions $\pi_k(\cdot) : \mathbb{R}^{2k+1} \to \mathbb{R}$ for $k = 0, 1, \dots, N-1$,* i.e.,

$$\Pi_N = \{\pi_0(\cdot), \pi_1(\cdot), \cdots, \pi_{N-1}(\cdot)\}.$$

*A policy Π_N is called an "**admissible control policy**" if and only if*

$$\pi_k(I^k) \in \mathcal{U} \quad \forall I^k \in \mathbb{R}^{2k+1}, \quad \text{for} \quad k = 0, \dots, N-1.$$

Further, the class of all admissible control policies will be denoted by

$$\overline{\Pi}_N = \{\Pi_N : \Pi_N \text{ admissible}\}.$$

The optimal control problem we address can now be stated as follows.

Definition 2 (Finite-horizon Optimal Control Problem) *Given the probability density functions (pdf's) $p_{\mathbf{x}_0}(\cdot)$, $p_{\mathbf{w}}(\cdot)$ and $p_v(\cdot)$ of the initial state, $\mathbf{x}_0$, and the disturbances $\mathbf{w}_k$ and v_k, respectively, the problem considered is that of finding the control policy Π_N^{OPT}, called the **optimal control policy**, belonging to the class of all admissible control policies $\overline{\Pi}_N$, that minimizes the cost*

$$V_N(\Pi_N) = \underset{\substack{\mathbf{x}_0, \mathbf{w}_k, v_k \\ k=0,\dots,N-1}}{\mathbf{E}} \left\{ g_T(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \pi_k(I^k)) \right\}, \tag{5}$$

subject to

$$\mathbf{x}_{k+1} = A\,\mathbf{x}_k + B\,\pi_k(I^k) + \mathbf{w}_k, \tag{6}$$

$$y_k = C\,\mathbf{x}_k + v_k, \tag{7}$$

$$I^{k+1} = \{I^k, y_{k+1}, u_k\}, \tag{8}$$

for $k = 0, \dots, N-1$, where the terminal cost $g_T(\cdot)$ and the cost per stage $g(\cdot,\cdot)$ are given by

$$\begin{aligned} g_T(\mathbf{x}_N) &= \mathbf{x}_N^{\mathrm{T}}\, S\, \mathbf{x}_N\,, \\ g(\mathbf{x}_k, \pi_k(I^k)) &= \mathbf{x}_k^{\mathrm{T}}\, Q\, \mathbf{x}_k + R\pi_k^2(I^k)\,, \end{aligned} \tag{9}$$

*for $S, R > 0$ and $Q \ge 0$. The integer N is called the **optimization horizon**.*

We can thus express the optimal control policy as

$$\Pi_N^{\text{OPT}} = \arg \inf_{\Pi_N \in \overline{\Pi}_N} V_N(\Pi_N), \tag{10}$$

*with the following resulting **optimal cost***

$$V_N^{\text{OPT}} = \inf_{\Pi_N \in \overline{\Pi}_N} V_N(\Pi_N). \tag{11}$$

Remark 2.1 *We would like to emphasize that the optimization problem thus stated takes into account the fact that new information will be available to the controller at future time instants. This is called* closed-loop *optimization and can be distinguished from* open-loop *optimization where the control values $\{u_0, u_1, \dots, u_{N-1}\}$ are selected all at once, at stage 0 [9]. For deterministic systems, in which there is no uncertainty, it is not necessary to make this distinction because minimizing the cost over all sequences of controls or over all control policies yields the same result.*

Throughout this work, the matrix S in Equation (9) will be adopted as a solution of the following Algebraic Riccati Equation [1]

$$S = A^{\mathrm{T}} S A + Q - K^{\mathrm{T}} \bar{R} K\,, \tag{12}$$

where

$$K \triangleq \bar{R}^{-1} B^{\mathrm{T}} S A\,, \qquad \bar{R} \triangleq R + B^{\mathrm{T}} S B\,. \tag{13}$$

3 Optimal Solutions

The problem described in the previous section can be tackled through the use of Dynamic Programming (DP) [9, 8]. DP is an algorithm based on Bellman's Principle of Optimality [6] that proceeds sequentially backwards in time by solving optimization subproblems. In the following subsections, the form of the optimal solutions for optimization horizons $N = 1$ and $N = 2$ are stated for future reference. These expressions are taken from the authors' parallel work [18].

3.1 Optimal Solution for $N = 1$

By applying the DP algorithm, the optimal solution for the case where the prediction horizon is equal to only one time instant can be obtained.

Proposition 1 *For $N = 1$, the solution of the optimal control problem stated in Definition 2 is of the form $\Pi_1^{\text{OPT}} = \{\pi_0^{\text{OPT}}(\cdot)\}$, with*

$$\pi_0^{\text{OPT}}(I^0) = -\mathrm{sat}_\Delta(K\,\mathbf{E}\{\mathbf{x}_0\,|\,I^0\}), \quad \forall I^0 \in \mathbb{R}\,, \tag{14}$$

where K was defined in (13) *and* $\mathrm{sat}_\Delta : \mathbb{R} \to \mathbb{R}$ *is defined as*

$$\mathrm{sat}_\Delta(z) = \begin{cases} \Delta & \textit{if } z > \Delta\,, \\ z & \textit{if } |z| \le \Delta\,, \\ -\Delta & \textit{if } z < -\Delta\,. \end{cases} \tag{15}$$

The proof of this proposition is given in [18].

Remark 3.1 *It is worth noting that when $N = 1$ the optimal control law π_0^{OPT} depends on the information I^0 only through the conditional expectation $\mathbf{E}\{\mathbf{x}_0\,|I^0\}$. Therefore, this conditional expectation is a sufficient statistic for the problem considered,* i.e., *it provides all the necessary information to implement the control.*

3.2 Optimal Solution for $N = 2$

We now consider the case where the optimization horizon is increased to two instants, *i.e.*, $N = 2$.

Proposition 2 *For $N = 2$, the solution of the optimal control problem stated in Definition 2 is of the form $\Pi_2^{\mathrm{OPT}} = \{\pi_0^{\mathrm{OPT}}(\cdot), \pi_1^{\mathrm{OPT}}(\cdot)\}$, with*

$$\pi_1^{\mathrm{OPT}}(I^1) = -\mathrm{sat}_\Delta(K\,\mathbf{E}\{\mathbf{x}_1\,|I^1\}), \quad \forall I^1 \in \mathbb{R}^3\,, \tag{16}$$

$$\pi_0^{\mathrm{OPT}}(I^0) = \arg\inf_{u_0 \in \mathcal{U}} \Big[\, (u_0 + K\,\mathbf{E}\{\mathbf{x}_0\,|I^0\})^2 + \overline{R}\,\mathbf{E}\{\Phi_\Delta(K\,\mathbf{E}\{\mathbf{x}_1\,|I^1\})|I^0, u_0\} \,\Big] \;, \forall I^0 \in \mathbb{R}, \tag{17}$$

where $\Phi_\Delta : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi_\Delta(z) = [z - \mathrm{sat}_\Delta(z)]^2. \tag{18}$$

Again, the proof of this proposition is given in [18].

4 Optimal Solutions in the Gaussian Case

In this section, we utilize the expressions (14), (16) and (17) to explicitly express the optimal solution (as a function of the available information) when the disturbances $\mathbf{w}_k$ and v_k, and the initial state, $\mathbf{x}_0$, are Gaussian random vectors. To achieve this aim, we will need to use the Kalman Filter, which calculates the parameters of the conditional distribution of the state given the information, at every time instant.

4.1 The Kalman Filter

We assume that the disturbances $\mathbf{w}_k$ and v_k are Gaussian random vectors with zero mean and covariance matrices R_w and R_v, respectively. The initial state, $\mathbf{x}_0$, is also assumed to be a Gaussian random vector, with mean $\mathbf{x}_{00}$ and covariance matrix P_{00}. When these assumptions hold, the conditional pdf's of the state at any time instant given the information at any (not necessarily the same) time instant are also Gaussian. The conditional mean and covariance of the state satisfy the following recursion, known as the Kalman Filter algorithm, which is usually separated into two sets of equations: prediction and measurement update [8].

Prediction:

$$\begin{aligned} \mathbf{E}\{\mathbf{x}_k\,|I^{k-1}, u_{k-1}\} &= A\,\mathbf{E}\{\mathbf{x}_{k-1}\,|I^{k-1}\} + Bu_{k-1} \\ \mathbf{cov}\{\mathbf{x}_k\,|I^{k-1}\} &= A\,\mathbf{cov}\{\mathbf{x}_{k-1}\,|I^{k-1}\}A^{\mathrm{T}} + R_{\mathbf{w}} \end{aligned} \tag{19}$$

Measurement update:

$$L_k = \mathbf{cov}\{\mathbf{x}_k\,|I^{k-1}\}C^{\mathrm{T}}(C\,\mathbf{cov}\{\mathbf{x}_k\,|I^{k-1}\}C^{\mathrm{T}} + R_v)^{-1},$$

$$\mathbf{E}\{\mathbf{x}_k\,|I^k\} = \mathbf{E}\{\mathbf{x}_k\,|I^{k-1}, u_{k-1}\} + L_k\left(y_k - C\,\mathbf{E}\{\mathbf{x}_k\,|I^{k-1}, u_{k-1}\}\right),$$

$$\mathbf{cov}\{\mathbf{x}_k\,|I^k\} = (\mathbf{I}_n - L_kC)\,\mathbf{cov}\{\mathbf{x}_k\,|I^{k-1}\}, \tag{20}$$

where $\mathbf{I}_n$ denotes the identity matrix of order n.

Remark 4.1 *From the above equations, it can be noted that the covariance matrices are not affected by the value taken by the information at any time instant. However, the covariance matrices depend on the number of times the algorithm has been iterated,* i.e., *they depend on the time instant k.*

4.2 Explicit Optimal Solution for $N = 1$

In Equation (14), the expectation $\mathbf{E}\{\mathbf{x}_0\,|I^0\}$ needs to be evaluated in order to express the optimal solution as a function of the information $I^0 = y_0$. Using Equations (20) for $k = 0$, we have

$$\mathbf{E}\{\mathbf{x}_0\,|y_0\} = (\mathbf{I}_n - L_0C)\,\mathbf{x}_{00} + L_0y_0, \tag{21}$$

where $L_0 = P_{00}C^{\mathrm{T}}(CP_{00}C^{\mathrm{T}} + R_v)^{-1}$, which is an affine function of its arguments and can be written as

$$\mathbf{E}\{\mathbf{x}_0\,|y_0\} = M^0_{\mathbf{x}_0}\,\mathbf{x}_{00} + M^0_{y_0}y_0. \tag{22}$$

Substituting Equation (21) into (14) yields

$$\pi_0^{\mathrm{OPT}}(y_0) = -\mathrm{sat}_\Delta\left[K(\mathbf{I}_n - L_0C)\,\mathbf{x}_{00} + KL_0y_0\right], \tag{23}$$

which constitutes the optimal control policy for $N = 1$ in the Gaussian case.

4.3 Explicit Optimal Solution for $N = 2$

In order to express the optimal solution given in Equation (16), the expectation $\mathbf{E}\{\mathbf{x}_1\,|I^1\}$ has to be evaluated. Using Equations (19) and (20), we can express $\mathbf{E}\{\mathbf{x}_1\,|I^1\} = \mathbf{E}\{\mathbf{x}_1\,|y_0, y_1, u_0\}$ as an affine function of its arguments, of the form

$$\mathbf{E}\{\mathbf{x}_1\,|y_0, y_1, u_0\} = f(y_0, y_1, u_0) = M^1_{\mathbf{x}_0}\,\mathbf{x}_{00} + M^1_{y_0}y_0 + M^1_{y_1}y_1 + M^1_{u_0}u_0. \tag{24}$$

Substituting this expression for $\mathbf{E}\{\mathbf{x}_1\,|I^1\}$ in Equation (16) yields

$$\pi_1^{\text{OPT}}(I^1) = \\ -\operatorname{sat}_\Delta\left[K(M^1_{\mathbf{x}_0}\,\mathbf{x}_{00} + M^1_{y_0}y_0 + M^1_{y_1}y_1 + M^1_{u_0}u_0)\right]. \quad (25)$$

Performing the same substitution into Equation (17) and using Equation (22) gives

$$\pi_0^{\text{OPT}}(I^0) = \arg\inf_{u_0\in\mathcal{U}}\Bigg[\ [u_0 + K(M^0_{\mathbf{x}_0}\,\mathbf{x}_{00} + M^0_{y_0}y_0)]^2 + \\ \overline{R}\,\mathbf{E}\left\{\Phi_\Delta[Kf(y_0,y_1,u_0)]|I^0,u_0\right\}\Bigg]\ , \quad (26)$$

Before the minimization can be performed, the remaining expected value in Equation (26) has to be calculated. Defining $z = Kf(y_0,y_1,u_0)$, the necessary calculation amounts to solving the following integral

$$\mathbf{E}\{\Phi_\Delta(z)|I^0,u_0\} = \int_{-\infty}^{\infty}\Phi_\Delta(z)p_z(z|I^0,u_0)dz, \quad (27)$$

where

$$p_z(z|I^0,u_0) \sim N(\eta,\sigma^2). \quad (28)$$

The variables η and σ^2 can be easily determined from the statistics provided by the Kalman Filter

$$\eta = \eta(I^0,u_0) = K(A\,\mathbf{E}\{\mathbf{x}_0\,|I^0\} + Bu_0), \quad (29)$$

$$\sigma^2 = (KL_0)^2(CA\,\mathbf{cov}\{\mathbf{x}_0\,|I^0\}A^{\text{T}}C^{\text{T}} + CR_wC^{\text{T}} + R_v), \quad (30)$$

with $\mathbf{E}\{\mathbf{x}_0\,|I^0\}$ as in (21) and $\mathbf{cov}\{\mathbf{x}_0\,|I^0\} = (\mathbf{I}_n - L_0C)P_{00}$.

After integration, Equation (26) can be expressed as

$$\pi_0^{\text{OPT}}(I^0) = \arg\inf_{u_0\in\mathcal{U}}\Bigg[\ \left[u_0 + K(M^0_{\mathbf{x}_0}\,\mathbf{x}_{00} + M^0_{y_0}y_0)\right]^2 \\ + \overline{R}\mathbf{W}[\eta(y_0,u_0),\sigma]\Bigg]\ , \text{ where} \quad (31)$$

$$\begin{aligned}\mathbf{W}(\eta,\sigma) &= [\sigma^2 + (\eta+\Delta)^2]\,\mathbb{G}(\eta,\sigma,-\Delta)\\ &+ [\sigma^2 + (\eta-\Delta)^2][1-\mathbb{G}(\eta,\sigma,\Delta)]\\ &- \frac{(\eta+\Delta)\sigma}{\sqrt{2\pi}}\exp\Big(\frac{-(\Delta+\eta)^2}{2\sigma^2}\Big)\\ &- \frac{(\Delta-\eta)\sigma}{\sqrt{2\pi}}\exp\Big(\frac{-(\Delta-\eta)^2}{2\sigma^2}\Big), \text{ with}\end{aligned}$$

$$\mathbb{G}(\eta,\sigma,\alpha) = \int_{-\infty}^{\alpha}\frac{1}{\sigma\sqrt{2\pi}}\ \exp\left\{\frac{-(z-\eta)^2}{2\sigma^2}\right\}dz. \quad (32)$$

Due to the fact that the solution to this integral cannot be expressed as an explicit function of its arguments (and parameters), we are not able to proceed further. In the next section, we consider two suboptimal strategies.

5 Suboptimal Strategies for the Gaussian Case

5.1 Certainty Equivalent Control

As mentioned in the introduction, Certainty Equivalent Control (CEC) is a control strategy in which, at each stage, the control applied is the optimal control of an associated *deterministic* problem derived from the original problem by removing all uncertainty. Specifically, the associated problem is derived by setting the disturbance $\mathbf{w}_k$ to a fixed typical value, *e.g.* $\mathbf{E}\{\mathbf{w}_k\}$, and by also assuming perfect state information. After finding the solution to the associated deterministic problem, the control to apply is implemented using some estimate of the state $\hat{\mathbf{x}}_k(I^k)$ in place of the true state (which was assumed to be known for solving the associated problem). That is, we obtain the optimal policy for the deterministic problem ($^{\text{DET}}$)

$$\Pi_N^{\text{DET}} = \{\pi_0^{\text{DET}}(\cdot),\ldots,\pi_{N-1}^{\text{DET}}(\cdot)\}, \quad (33)$$

where $\pi_k^{\text{DET}}:\mathbb{R}^n\to\mathbb{R}$ for $k=0,1,\ldots,N-1$. Then, the CEC evaluates the deterministic laws at the estimate of the state, *i.e.*,

$$u_k^{\text{CE}} = \pi_k^{\text{DET}}\left(\hat{\mathbf{x}}_k(I^k)\right). \quad (34)$$

The associated deterministic problem for linear systems with a quadratic cost is an example of a case where the control policy can be explicitly obtained for any finite optimization horizon [7]. The following example illustrates this for an optimization horizon $N=2$; for a procedure to obtain the pre-computable laws for larger optimization horizons see [7] and [19].

5.1.1 Closed-loop CEC for $N=2$

For $N=2$, the deterministic policy $\Pi_2^{\text{DET}} = \{\pi_0^{\text{DET}}(\cdot),\pi_1^{\text{DET}}(\cdot)\}$ is given by [10]:

$$\pi_1^{\text{DET}}(x) = -\operatorname{sat}_\Delta(Kx) \quad \forall\, x\in\mathbb{R}^n \quad (35)$$

$$\pi_0^{\text{DET}}(x) = \begin{cases}-\operatorname{sat}_\Delta(Gx+H) & \text{if } x\in\mathcal{X}_1\\ -\operatorname{sat}_\Delta(Kx) & \text{if } x\in\mathcal{X}_2\\ -\operatorname{sat}_\Delta(Gx-H) & \text{if } x\in\mathcal{X}_3\,.\end{cases} \quad (36)$$

K is given by (13) and

$$G = \frac{K+KBKA}{1+(KB)^2}\,, \qquad H = \frac{KB}{1+(KB)^2}\Delta\,.$$

The sets $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ form a partition of $\mathbb{R}^n$, and are given by

$$\begin{aligned}\mathcal{X}_1 &= \{x : KA_Kx < -\Delta\}\ ,\\ \mathcal{X}_2 &= \{x : |KA_Kx| \le \Delta\}\ ,\\ \mathcal{X}_3 &= \{x : KA_Kx > \Delta\}\ ,\end{aligned}$$

with

$$A_K = A - BK.$$

Therefore, a closed-loop CEC consists in using the controls

$$\begin{aligned}u_0^{\text{CE}} &= \pi_0^{\text{DET}}\left(\hat{\mathbf{x}}_0(I^0)\right)\\ u_1^{\text{CE}} &= \pi_1^{\text{DET}}\left(\hat{\mathbf{x}}_1(I^1)\right).\end{aligned} \quad (37)$$

5.2 Partially Stochastic CEC

This variant of CEC treats the problem as one of perfect state information but takes stochastic disturbances into account. To solve the optimization problem, it is assumed that the state is, and will be, known to the controller. In reality, and as with CEC, the value of the state is provided by an estimator that receives the available information as input and generates $\hat{\mathbf{x}}_k(I^k)$. A Partially Stochastic CEC (PS-CEC) admissible policy

$$\Lambda_N = \{\lambda_0(\cdot), \ldots, \lambda_{N-1}(\cdot)\} \tag{38}$$

is a sequence of admissible control laws $\lambda_k(\cdot) : \mathbb{R}^n \to \mathcal{U}$ that map the (estimates of the) states into admissible control actions. The PS-CEC solves the following perfect state information problem.

Definition 3 (PS-CEC Optimal Control Problem) *Assuming the state $\hat{\mathbf{x}}_k$ will be available to the controller at time instant k to calculate the control and given the pdf $p_{\mathbf{w}}(\cdot)$ of the disturbances $\mathbf{w}_k$, find the admissible control policy Λ_N^{OPT} that minimizes the cost*

$$\hat{V}_N(\Lambda_N) = \underset{\substack{\mathbf{w}_k \\ k=0,\ldots,N-1}}{\mathbf{E}} \left\{ g_T(\hat{\mathbf{x}}_N) + \sum_{k=0}^{N-1} g(\hat{\mathbf{x}}_k, \lambda_k(\hat{\mathbf{x}}_k)) \right\} \tag{39}$$

subject to $\hat{\mathbf{x}}_{k+1} = A\hat{\mathbf{x}}_k + Bu_k + \mathbf{w}_k$, for $k = 0, 1, \ldots, N-1$.

The optimal control policy for perfect state information thus found will be used, as in CEC, to calculate the control action based on the estimate $\hat{\mathbf{x}}_k$ provided by the estimator, *i.e.*,

$$\hat{u}_k^{\text{OPT}} = \lambda_k^{\text{OPT}}(\hat{\mathbf{x}}_k(I^k)). \tag{40}$$

The notation $\hat{u}_k^{\text{OPT}}$ is used to show that the PS-CEC solution can be regarded as an approximation to the optimal solution. Next, we apply this suboptimal strategy to the problem of interest.

5.2.1 PS-CEC for $N = 1$

In [18] it was shown that if $\hat{\mathbf{x}}_0(I^0) = \mathbf{E}\{\mathbf{x}_0 \,|I^0\}$, then the PS-CEC solution coincides with the optimal one, which for the Gaussian case is given in Equation (23).

5.2.2 PS-CEC for $N = 2$

The PS-CEC solution when $N = 2$ is given by [18]

$$\hat{u}_0^{\text{OPT}} = \arg \inf_{u_0 \in \mathcal{U}} \Bigg[(u_0 + K\hat{\mathbf{x}}_0)^2 + \overline{R}\,\mathbf{E}\left\{ \Phi_\Delta[K(A\hat{\mathbf{x}}_0 + Bu_0 + \mathbf{w}_0)] | \hat{\mathbf{x}}_0, u_0 \right\} \Bigg] . \tag{41}$$

As in Section 4.3, the expected value in the previous expression has to be evaluated in order to proceed with the minimization. Defining $z\prime = K(A\hat{\mathbf{x}}_0 + Bu_0 + \mathbf{w}_0)$ we see that

$$p_{z\prime}(z\prime|\hat{\mathbf{x}}_0, u_0) \sim N(\eta\prime, \sigma\prime^2), \tag{42}$$

where

$$\begin{aligned} \eta\prime &= KA\hat{\mathbf{x}}_0 + KBu_0, \\ \sigma\prime^2 &= KR_{\mathbf{w}}K^{\mathrm{T}}. \end{aligned}$$

Using the results in Section 4.3, we can find

$$\hat{u}_0^{\text{OPT}} = \arg \inf_{u_0 \in \mathcal{U}} \left[(u_0 + K\hat{\mathbf{x}}_0)^2 + \overline{R}\mathbf{W}(\eta\prime, \sigma\prime) \right] . \tag{43}$$

6 Example for $N = 2$

A simple numerical example can be used to show that the optimal control, $\pi_0^{\text{OPT}}(I^0)$ (see Section 4.3), depends not only on $\mathbf{E}\{x_0|I^0\}$ but also on $\mathbf{cov}\{x_0|I^0\}$. Indeed, Figure 1 shows an example of the dependence of the optimal control as a function of σ for the conditional expectation $\mathbf{E}\{x_0|I^0\}$ fixed at the value $[3.636 \ -16.092]^{\mathrm{T}}$, for the following system and parameters:

$$A = \begin{bmatrix} 0.5764 & -0.0014 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0.368 & 0.057 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 0.2 \end{bmatrix} \quad \begin{matrix} R = 50 \\ \Delta = 1 \end{matrix} \quad R_w = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \quad R_v = 0.01\,.$$

We see in Figure 1, that the optimal control policy is a function

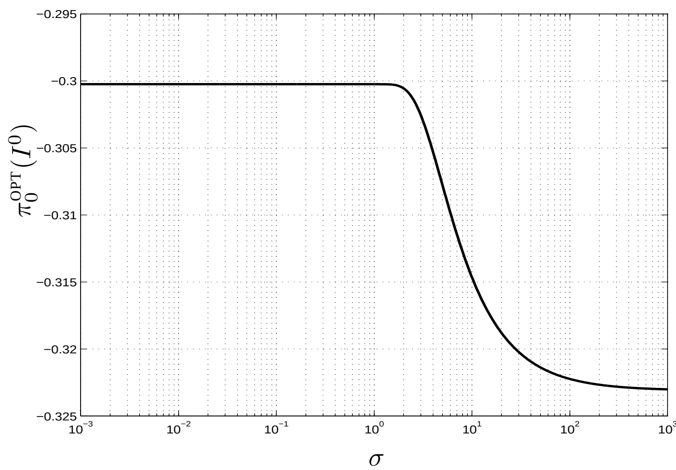

Figure 1: Optimal control with fixed $\mathbf{E}\{x_0|I^0\}$ as a function of σ.

of the standard deviation σ, which depends on $\mathbf{cov}\{\mathbf{x}_0\,|I^0\}$. Thus, since the Certainty Equivalent scheme consists in finding $\mathbf{E}\{x_0|I^0\}$ and calculating the control action based only on this value, we have shown that CEC definitely does not yield the optimal solution for this particular case.

It is worth noting some interesting features depicted in Figure 1. It is easy (though tedious) to show that the function $\mathbf{W}[\eta(y_0, u_0), \sigma]$ is a convex function of u_0 and has a unique minimum at $u_0^{\mathbf{W}} = -KA\,\mathbf{E}\{x_0|I^0\}/(KB)$. Thus, the *unconstrained* optimal control will be between the values $u_0^q = -K\,\mathbf{E}\{x_0|I^0\}$, which is the minimizer of the quadratic term of Equation (31) and $u_0^{\mathbf{W}}$. From the convexity of $\mathbf{W}[\eta(y_0, u_0), \sigma]$ and the quadratic term, and the fact that the sum of convex functions is also a convex function, it follows that the constrained minimum will be a clipped version of the unconstrained minimum.

When the distribution of the noise degenerates at its mean, *i.e.*, in the limit when R_w and R_v tend to zero, $\mathbf{E}\{\Phi_\Delta(z)|I^0, u_0\} \to \Phi_\Delta(\mathbf{E}\{z|I^0, u_0\})$. In this case, the optimal solution reverts to the Certainty Equivalent Control solution [10], given by Equation (36):

$$\pi_0^{\text{OPT}}(I^0) \to \pi_0^{\text{DET}}(\mathbf{E}\{x_0|I^0\})$$

On the other hand, when the noise becomes large, *i.e.*, $\sigma \to \infty$, it can be shown that

$$\mathbf{W}(\eta, \sigma) \to \eta^2 + g(\sigma),$$

and the optimal control for this case becomes

$$\pi_0^{\text{OPT}}(I^0) \to -\text{sat}_\Delta\left(G\,\mathbf{E}\{x_0|I^0\}\right)$$

These results are shown in Figure 1 for a particular value of $\mathbf{E}\{x_0|I^0\}$. An interesting feature of this example is that there exists a range of values of σ for which the Certainty Equivalent Control solution approximates the optimal solution very well. This motivates further research aimed at quantifying the size of this range in terms of the parameters of the system, the cost and the noise.

7 Conclusions

We have analyzed the optimal control problems for SISO linear systems with input constraints where the disturbances and initial state are Gaussian random vectors, for prediction horizons $N = 1$ and $N = 2$. Because of the Gaussian nature of the variables involved and the linearity of the system, the Kalman Filter provides all of the necessary statistics to solve the problem. We have also analyzed two suboptimal strategies.

Specifically, we have given an example where the optimal control depends not only on the state estimate provided by the Kalman Filter, but also on its covariance. This establishes that CEC does not yield the optimal solution in this case. We have also studied limiting cases when the variance of the noise tends to zero and infinity. As expected, for the former case the optimal solution reverts to CEC. Moreover, the example shows that there is a range of values of the variance (not only for the limiting case) for which Certainty Equivalence approximates the optimal solution with remarkable accuracy.

Further research is needed to quantify the conditions under which Certainty Equivalence yields a suitable approximation to the optimal solution.

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