# SOLVABILITY CONDITION FOR A NONSYMMETRIC RICCATI EQUATION APPEARING IN STACKELBERG GAMES

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## Abstract

Open-loop Stackelberg games are conceptual very interesting particularly for applications which contain a hierarchical structure. The existence of unique Stackelberg equilibria was shown to be tied to the existence of solutions to certain nonsymmetric Riccati equations, which are hard to solve. The paper reveals a connection between solutions of a standard algebraic Riccati equation and a nonsymmetric algebraic Riccati equation. As a consequence easy veri£able conditions are obtained to ensure a Stackelberg equilibrium, which makes these type of games accessible to applications.

## 1 Introduction

The algebraic Riccati equations considered in this paper appear in two player linear quadratic open-loop Stackelberg games on the in£nite time horizon $[0, \infty]$. It was shown in [6] that the nonsymmetric algebraic Riccati equation and its corresponding differential equation counterpart is of great relevance for Stackelberg games on the in£nite and £nite horizon, respectively. Therefore, explicit solvability conditions for these type of Riccati equations are highly desirable.

A linear quadratic Stackelberg game is de£ned by linear dynamics of the form

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = \xi, \tag{1}$$

where the matrices $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ $(i = 1, 2)$ are constant and $n, m_1, m_2 \in \mathbb{N}$. Furthermore, we consider quadratic type cost-functionals of the form

$$\begin{aligned} J_1(u_1, u_2) &= \int_0^\infty x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2 \; dt, \\ J_2(u_1, u_2) &= \int_0^\infty x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2 \; dt. \end{aligned} \tag{2}$$

It is assumed that the coef£cients in (2) are symmetric with $Q_i \in \mathbb{R}^{n\times n}$, $R_{ij} \in \mathbb{R}^{m_j\times m_j}$ for $1 \le i, j \le 2$. Furthermore, we only consider the open-loop information structure, i.e. the players are committed to follow a predetermined strategy. We agree that here player 2 is the leader, i.e. he is seeking a $J_2(u_1, u_2)$-minimizing strategy $u_2^*(t)$, a function of time only, that he announces before the game starts knowing how the follower reacts to any of his choices. The follower, i.e. player 1, will then minimize his cost-functional with a strategy $u_1^*(t)$ also as a function of time only. For the precise mathematical de£nition of a Stackelberg equilibrium we refer to [3].

In the second section we recall known results for games on the £nite time horizon $[0, t_f]$ which ensure the existence of a unique Stackelberg equilibrium; in this case the performance criteria are (for $i = 1, 2$)

$$\begin{aligned} J_i &= \frac{1}{2} x^T(t_f) K_{if} x(t_f) \\ &+ \frac{1}{2} \int_0^{t_f} (x^T(t) Q_i x(t) + \sum_{j=1}^{2} u_j^T(t) R_{ij} u_j(t)) \; dt. \end{aligned} \tag{3}$$

The third section contains the results for the in£nite time horizon showing that both situations are quite different. Section 4 brings the results for the in£nite time horizon together and indicates a connection to the £nite horizon case. In the last section the obtained results are discussed and further research directions for Stackelberg games are proposed.

We think that the Stackelberg concept has rich applications and to emphasize this we cite [5], where this gametheoretic concept was successfully applied to solve a tracking problem for ¤exible robots.

## 2 Games on the £nite time horizon

Let us brie¤y summarize some basic results that have been derived for the £nite time horizon. These have been formulated in the papers [8] and [4]. The £rst remarkable result is based on the convexity condition:

$$(C_1) \quad \begin{cases} R_{11} > 0, & R_{22} > 0 \\ R_{21} \ge 0 & \\ Q_1 \ge 0, & Q_2 \ge 0 \\ K_{1f} \ge 0, & K_{2f} \ge 0. \end{cases}$$

**Theorem 1 (Simaan, Cruz; 1973).** *Let the linear quadratic open-loop Stackelberg game ful£ll condition* $(C_1)$*. De£ne* $S_i := B_i R_{ii}^{-1} B_i^T$ *for* $i = 1, 2$ *and* $S_{21} := B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T$*. If the coupled system of matrix Riccati differential equations*

$$-\frac{d}{dt}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} A + \begin{pmatrix} -A & S_{21} & -S_1 \\ 0 & A^T & 0 \\ -Q_1 & 0 & A^T \end{pmatrix}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} - \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} 0 & S_1 & S_2 \end{pmatrix}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} \quad (4)$$

*has a unique solution* $\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix}$ *satisfying the boundary conditions* $P(0) = 0, K_1(t_f) = K_{1f}, K_2(t_f) = K_{2f} - K_{1f}P(t_f)$*, then*

$$u_1^* = -R_{11}^{-1} B_1^T K_1(t) x^*(t), \quad u_2^* = -R_{22}^{-1} B_2^T K_2(t) x^*(t)$$

*de£nes a Stackelberg equilibrium where* $x^*$ *is the solution of*

$$\dot{x} = (A - S_1 K_1 - S_2 K_2)x, \qquad x(0) = \xi.$$

The coupled system of Riccati equations (4) is also known as *Stackelberg Riccati (differential) equation*. The corresponding algebraic Riccati equation is called the *algebraic Stackelberg Riccati equation*. The next result is based on a value function approach and does not rely on the convexity condition.

**Theorem 2 (Freiling, Jank, Lee; 1999).** *For a linear quadratic open-loop Stackelberg game assume that* $R_{11} > 0$ *and* $R_{22} > 0$ *and let* $S_1, S_2$ *and* $S_{21}$ *be as above.*

1. *Let* $E_1(t) \in \mathbb{R}^{n \times n}$ *be the solution of the standard (i.e. here and below: symmetric) Riccati differential equation*

$$-\dot{E}_1 = E_1 A + A^T E_1 + Q_1 - E_1 S_1 E_1, \quad E_1(t_f) = K_{1f}. \quad (5)$$

2. *Let the solution* $E_2(t) \in \mathbb{R}^{2n \times 2n}$ *of the standard Riccati differential equation*

$$-\dot{E}_2 = \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix} + E_2 \begin{pmatrix} A & -S_1 \\ -Q_1 & -A^T \end{pmatrix} + \begin{pmatrix} A & -S_1 \\ -Q_1 & -A^T \end{pmatrix}^T E_2 - E_2 \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix} E_2 \quad (6)$$

*with terminal condition* $E_2(t_f) = \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix}$ *exist on* $[0, t_f]$.

3. *Let the coupled system*

$$-\frac{d}{dt}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} A + \begin{pmatrix} -A & S_{21} & -S_1 \\ 0 & A^T & 0 \\ -Q_1 & 0 & A^T \end{pmatrix}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} - \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} 0 & S_1 & S_2 \end{pmatrix}\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} \quad (7)$$

*with terminal values* $P(t_f) = 0, K_1(t_f) = K_{1f}$ *and* $K_2(t_f) = K_{2f}$ *have a solution on* $[0, t_f]$.

*Then there exists a unique open-loop Stackelberg equilibrium which is given by*

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) x^*(t),$$
$$u_2^*(t) = -R_{22}^{-1} B_2^T K_2(t) x^*(t),$$

*where* $x^*(t)$ *is a solution of the closed-loop equation*

$$\dot{x} = (A - S_1 K_1 - S_2 K_2)x, \qquad x(0) = \xi.$$

Suf£cient conditions for the existence of the solutions of (5) - (7) on $[0, t_f]$ can be found in Chapter 4 of [1]. Note also that these equations are numerically easier to treat than the nonlinear boundary value problem (4).

## 3 Games on the in£nite time horizon

One of the authors ([6]) considered open-loop Stackelberg games on the in£nite time horizon for the £rst time. Let us brie¤y cite the results from [6]. The next Theorem is based on the convexity condition and gives the equilibrium strategies in feedback form. The convexity condition reads now as follows:

$$(C_2) \qquad \begin{cases} R_{11} > 0 \quad R_{22} > 0 \\ R_{21} \geq 0 \\ Q_1 \geq 0 \quad Q_2 \geq 0. \end{cases}$$

**Theorem 3.** *If condition* $(C_2)$ *is ful£lled and if there exists a stabilizing solution* $X$ *to the nonsymmetric algebraic Riccati equation*

$$0 = X \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} X + \begin{pmatrix} Q_1 & 0 \\ Q_2 & -Q_1 \end{pmatrix} - XSX \quad (8)$$

*with*

$$S = \begin{pmatrix} B_1 & B_2 & 0 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ R_{21} & 0 & R_{11} \end{pmatrix}^{-1} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \\ 0 & B_1^T \end{pmatrix} =: \begin{pmatrix} S_1 & S_2 \\ S_{21} & -S_1 \end{pmatrix},$$

*i.e. the closed-loop matrix* $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} - SX$ *is stable, then the unique Stackelberg equilibrium exists and is given by*

$$\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = -\begin{pmatrix} R_{11}^{-1} & 0 & 0 \\ 0 & R_{22}^{-1} & 0 \end{pmatrix} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \\ 0 & B_1^T \end{pmatrix} Xz, \quad (9)$$

*where $z$ is the solution of the closed-loop equation*

$$\dot{z} = (\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} - SX)z, \quad z(0) = \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \quad (10)$$

It is interesting to mention that the algebraic Stackelberg Riccati equation associated to (7) does not show up here on the in£nite time horizon, but that a similar result as above can be obtained for the £nite time counterpart, which we take from [6].

**Corollary 1.** *Assume that condition $(C_1)$ is ful£lled. If the nonsymmetric Riccati differential equation*

$$-\dot{X} = X\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} X + \begin{pmatrix} Q_1 & 0 \\ Q_2 & -Q_1 \end{pmatrix} - XSX$$

*with* $X(t_f) = \begin{pmatrix} K_{1f} & 0 \\ K_{2f} & -K_{1f} \end{pmatrix}$ *admits a solution on* $[0, t_f]$, *then there exists a unique Stackelberg equilibrium.*

The second result is based on the value function type approach and on the Hilbert space approach presented by [8] and therefore avoids the convexity condition. Here, $L_+^{2,n}$ denotes the Hilbert space of square integrable functions (with respect to the Lebesgue measure) on the positive real line.

**Theorem 4.** *Assume that the coef£cients of the Stackelberg game are such that $A$ is stable and the matrices $R_{11}, R_{22}$ are positive de£nite. Let the stabilizing solution $E_1$ of the standard algebraic Riccati equation*

$$0 = \underbrace{X_1 A + A^T X_1 + Q_1 - X_1 S_1 X_1}_{=: \mathcal{R}_1(X_1)} \quad (11)$$

*exist and let also the stabilizing solution $E_2$ of the standard algebraic Riccati equation*

$$0 = \underbrace{X_2 H + H^T X_2 + Q - X_2 S X_2}_{=: \mathcal{R}_2(X_2)} \quad (12)$$

*exist, where*

$$H := \begin{pmatrix} A & -S_1 \\ -Q_1 & -A^T \end{pmatrix}; \quad B := \begin{pmatrix} B_2 \\ 0 \end{pmatrix}$$

$$Q := \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix}; \quad S := BR_{22}^{-1}B.$$

*Partition the solution* $E_2 = \begin{pmatrix} E_2^{11} & E_2^{12} \\ E_2^{21} & E_2^{22} \end{pmatrix}$, *where each block is of size $n \times n$. If $E_2^{22}$ is invertible, then there exists a unique Stackelberg equilibrium for any initial value $\xi \in \mathbb{R}^n$. The Stackelberg strategy for the leader is explicitly given by*

$$u_2^* = -\begin{pmatrix} R_{22}^{-1} B_2^T & 0 \end{pmatrix} E_2 z \quad (13)$$

*and $z$ is the solution of*

$$\dot{z} = (H - SE_2)z, \quad z_0 = \begin{pmatrix} I \\ -E_2^{-22} E_2^{21} \end{pmatrix} \xi. \quad (14)$$

*The Stackelberg strategy for the follower is attained by*

$$u_1^*(u_2^*) = -R_{11}^{-1} B_1^T (E_1 x^* + e_1^*) \quad (15)$$

*with $e_1^*$ as $L_+^{2,n}$-solution of*

$$\dot{e}_1^* = -(A - S_1 E_1)^T e_1^* + E_1 \begin{pmatrix} S_2 & 0 \end{pmatrix} E_2 z \quad (16)$$

*and $x^*$ as solution of*

$$\dot{x}^* = (A - S_1 E_1)x^* - S_1 e_1^* - \begin{pmatrix} S_2 & 0 \end{pmatrix} E_2 z; \quad x^*(0) = \xi \quad (17)$$

*and $z$ given by* (14).

## 4 Observations and main result

Before we study the connection between solutions of (12) and (8) we cite some well known results for concerning the standard (symmetric) algebraic Riccati equations appearing in the preceding Theorem. For proofs and further reading we refer to [7] or to Chapter 2 in [1].

**Lemma 1.** *Assume for the algebraic Riccati equation $\mathcal{R}_1(X_1) = 0$, that $R_{11} > 0$.*

1. *The algebraic Riccati equation has a semi-stabilizing solution $E_1^s$, if*
   (a) *$(A, B_1)$ is stabilizable;*
   (b) *there exists a solution $E = E^T$ to the inequality $\mathcal{R}_1(E) \geq 0$.*

   *The semi-stabilizing solution is moreover the maximal solution.*
2. *$E_1^s$ is stabilizing if there exists $E_0 = E_0^T$ such that $\mathcal{R}_1(E_0) > 0$.*
3. *$E_1^s$ is invertible and stabilizing, if there exists no $(Q_1, A)$ non-observable eigenvalue $\lambda$ with Re$\lambda \leq 0$.*

4. *If $(A, B_1)$ is stabilizable and $Q_1 \geq 0$, then $E_1^s \geq 0$.*

**Remark 1.** *If we replace in the preceding Lemma $R_{11}$ by $R_{22}$, $A$ by $H$, $Q_1$ by $Q$, $B_1$ by $\begin{pmatrix} B_2 \\ 0 \end{pmatrix}$ and $S_1$ by $S$ analogous results also apply for the (semi-) stabilizing solution $E_2^s$ of* (12).

From these consideration and Theorem 4 we are now able to obtain an existence result for the nonsymmetric algebraic Riccati equation (8).

**Theorem 5.** *Assume that the Stackelberg game ful£lls the following assumptions:*

1. *The matrix $A$ is stable, $R_{11} > 0$ and $R_{22} > 0$.*
2. *There exists $Z_0 = Z_0^T$ with $\mathcal{R}_1(Z_0) > 0$* **or** *the pair $(Q_1, A)$ has no non-observable eigenvalue in left complex half plane.*
3. *The pair $(H, S)$ is stabilizable.*
4. *There exists $Y_0 = Y_0^T$ with $\mathcal{R}_2(Y_0) > 0$* **or** *the pair $(Q, H)$ has no non-observable eigenvalues in the left complex half plane.*

*Then there exists a (unique) stabilizing solution $X_s$ to* (8), *which is explicitly given by*

$$X_s = \begin{pmatrix} -M_{22}^{-1}M_{21} & M_{22}^{-1} \\ M_{11} - M_{12}M_{22}^{-1}M_{21} & M_{12}M_{22}^{-1} \end{pmatrix}, \qquad (18)$$

*where the appearing matrices are given by the stabilizing solution*

$$E_2^s =: \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} > 0$$

*of* (12).

Before we prove the Theorem let us note that it also contains the case where the convexity condition $(C_2)$ holds and $A$ is stable. Consequently, Theorem 4 guarantees a unique Stackelberg equilibrium.

**Corollary 2.** *Assume that $A$ is stable and that the convexity condition $(C_2)$ holds for a Stackelberg game on the in£nite time horizon, then the semi-stabilizing solution $E_1^s$ to* (11) *exists. Furthermore, the semi-stabilizing solution $E_2^s \geq 0$ to* (12) *exists, if the pair $(H, S)$ is stabilizing. If moreover the pairs $(Q_1, A)$ and $(Q, H)$ have no non-observable eigenvalues in the left complex half plane, then $E_1^s$ and $E_2^s$ are both positive definite and stabilizing. The unique stabilizing solution $X_s$ to* (8) *exists and has the representation* (18).

*Proof of Theorem 5:* Consider the Hamiltonian matrix associated to (12), which is

$$H_{St} := \begin{pmatrix} A & -S_1 & -S_2 & 0 \\ -Q_1 & -A^T & 0 & 0 \\ -Q_2 & 0 & -A^T & Q_1 \\ 0 & -S_{21} & S_1 & A \end{pmatrix}. \qquad (19)$$

The corresponding matrix for the nonsymmetric equation (8)

$$\widetilde{H}_{St} := \begin{pmatrix} A & 0 & -S_1 & -S_2 \\ 0 & A & -S_{21} & S_1 \\ -Q_1 & 0 & -A^T & 0 \\ -Q_2 & Q_1 & 0 & -A^T \end{pmatrix} \qquad (20)$$

is orthogonal equivalent to $H_{St}$ by

$$H_{St} = T\widetilde{H}_{St}T^T, \qquad T := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{pmatrix}. \qquad (21)$$

Since $\begin{pmatrix} I \\ E_2^s \end{pmatrix}$ is an invariant subspace for $H_{St}$ we infer that $T^T \begin{pmatrix} I \\ E_2^s \end{pmatrix}$ is an invariant subspace for $\widetilde{H}_{St}$. It is easy to check that

$$T^T \begin{pmatrix} I \\ E_2^s \end{pmatrix} = \begin{pmatrix} I & 0 \\ M_{21} & M_{22} \\ 0 & I \\ M_{11} & M_{12} \end{pmatrix}.$$

Since $M_{22}$ is by our preceding observation invertible we get with

$$\begin{pmatrix} I & 0 \\ M_{21} & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & M_{22}^{-1} \end{pmatrix}$$

the representation formula (18) of the stabilizing solution $X_s$. The £rst part of the proof shows that $H_{St}$ and $\widetilde{H}_{St}$ have the same set of eigenvalues and therefore must $X_s$ be unique, if it exists. □

As pointed out in the last section the algebraic Riccati equation associated to (7) is not involved on the in£nite time horizon. However, it is worth noting that given the existence of the stabilizing solution $X_s$ of (8) at least one stabilizing solution $\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix}$ of

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} A + \begin{pmatrix} -A & S_{21} & -S_1 \\ 0 & A^T & 0 \\ -Q_1 & 0 & A^T \end{pmatrix} \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} + \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} \begin{pmatrix} 0 & S_1 & S_2 \end{pmatrix} \begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix} \qquad (22)$$

exists - this follows easily from the fact that Im $\begin{pmatrix} I \\ X_s \end{pmatrix}$ is $H_{St}$-invariant and contains therefore an $n$-dimensional $H_{St}$-invariant subspace of the form Im $\begin{pmatrix} I \\ P \\ K_1 \\ K_2 \end{pmatrix}$, which de£nes the desired solution of (22).

We mention the following interesting fact:

If a stabilizing solution $\begin{pmatrix} P \\ K_1 \\ K_2 \end{pmatrix}$ of (22) exists, it is unique under the following additional condition:

$A$ is stable and every eigenvalue of $A$ is $(Q_1, A)$ unobservable.

In this case one obtains from [2] that $H_{St}$ has $n$ eigenvectors of the form $\begin{pmatrix} 0 \\ 0 \\ 0 \\ p_j \end{pmatrix}$ corresponding to eigenvalues in the open left complex half plane, where $\{p_1, \ldots, p_n\}$ is a Jordan basis of $A$. Moreover, if this condition is ful£lled then, according to [1], Chapter 2, both (8) and (12) do not admit a stabilizing solution.

## 5 Discussion

In this paper we derived explicit conditions which ensure the existence of the stabilizing solution of nonsymmetric algebraic Riccati equations appearing in so called open-loop Stackelberg games on the in£nite time horizon. The result obtained is interesting from different points of view. First of all it brings together two approaches used in [6] to obtain existence results for Stackelberg equilibria. On the other hand it also shows that the inherent structure in the Stackelberg concept can be used to treat nonsymmetric Riccati equations with help of standard algebraic Riccati equations.

The next step is to £nd in a similar way existence results for the nonsymmetric Riccati differential equation corresponding to the algebraic equation. This would also help to understand Stackelberg games on the £nite time horizon a good deal better than we do nowadays.

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