NESTED STRATEGIES FOR THE QUANTIZED FEEDBACK STABILIZATION

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Abstract

A state feedback with finitely many quantization levels yields only the so called practical stabilization, namely the convergence of any initial state belonging to a bigger bounded region into another smaller target region of the state space. The ratio between the measure of the starting region and the target region is called contraction of the closed loop system. In the performance analysis of a stabilization strategy based on a quantized state feedback, two parameters play a central role: the number of quantization levels used by the feedback and the convergence time of the closed loop system. In this paper we propose a general strategy yielding a family of stabilizing quantized feedbacks from a base one and we analyze the performance of three different applications of this method.

1 Introduction

In recent years a considerable interest has been devoted on control problems in which communication constraints are taken into consideration. Systems with communication constraints can be considered as instances of hybrid systems in which particular attention is devoted to the data flow. Control problems in this set up are very difficult to solve and a general theory seems still far to be developed. Some important contributions can be found in [3, 13, 2, 4, 1, 6, 9, 10, 12, 11, 8].

Discrete time systems with quantized feedback can be seen as particularly simple cases of dynamical systems in which the control requires a finite information flow. This class of systems can be analyzed in more detail even though they are nonlinear systems with wild behavior. In this set up the information flow has to be quantified in terms of the number of quantization levels of the feedback function. The problem in this context can be formulated as follows:

What is the minimal information flow required for fulfilling a certain control objective?

In control theory stabilization is considered the simplest control objective. In this case the previous question specializes as follows:

What is the minimal information flow required for stabilizing a discrete time unstable system?

In this paper we will show that this question makes sense only if we evaluate also the performance of the closed loop system. We will show that there are different stabilizing quantized feedback strategies requiring different information flows, but providing closed loop systems with different stability performances. Stability performance can be measured in different ways. In this contribution we choose to evaluate stability performance in terms of the convergence time. Other possible performance measures can be evaluated from the knowledge of the expected convergence time.

2 Problem statement

Consider the following discrete-time, one-dimensional linear model

$$x_{t+1} = ax_t + u_t, \tag{1}$$

where $a \in \mathbb{R}$. Most of the paper is devoted to the stabilization problem and so it is assumed that $|a| > 1$. Some results however holds true also for stable systems and so for $|a| \leq 1$.

Let $k : \mathbb{R} \to \mathbb{R}$ be a piecewise constant function with only finitely many discontinuities. If we use k as a static feedback in the system (1), namely we let $u_t = k(x_t)$, we obtain the closed loop system

$$x_{t+1} = \Gamma(x_t), \tag{2}$$

where $\Gamma(x) := ax + k(x)$ is a piecewise affine map with a fixed slope a.

Remark: In fact, the definition we gave is not precise if we do not define what happens at the boundary points of the intervals. We assume there is a finite family of disjoint open intervals I_h such that $D := \cup_h I_h$ is dense in $\mathbb{R}$ and such that $k(x) = u_h$ for every $x \in I_h$. In this way the associated closed loop map is defined as a map

$$\begin{aligned} &\Gamma : D \to \mathbb{R} \\ &\Gamma(x) = ax + u_h \quad \text{if } x \in I_h\,. \end{aligned} \tag{3}$$

In order to consider iterations of Γ we need to restrict the domain by considering

$$\Omega = \bigcap_{n=0}^{\infty} \Gamma^{-n}(D)\ . \tag{4}$$

It is clear that $\Gamma(\Omega) \subseteq \Omega$. Notice that $\mathbb{R} \setminus \Omega$ is a countable subset of $\mathbb{R}$ and since most of the questions considered in this paper are related to mean properties, it will be sufficient to consider Γ as a map defined on Ω, disregarding all the orbits which will eventually get to a discontinuity point. However, in those situations in which it is necessary to introduce a more abstract definition of state evolution (see [7]).

It is obvious that, by using quantized feedback controllers only a "practical stability" can be obtained as detailed in the following definitions.

Definition: Invariance and almost invariance. Given a closed interval I, we say that I is Γ-invariant if every orbit (x_t) of Γ with $x_0 \in I$ is such that $x_t \in I$ for every t. It is almost Γ-invariant if the assertion above is true for almost every initial condition x_0 with respect to the Lebesgue measure. When an interval I is invariant or almost invariant we will use in any case the notation $\Gamma : I \to I$.

Definition: Stability and almost stability. Given two closed intervals $J \subseteq I$, we say that Γ is (I, J)-*stable* if I and J are invariant by Γ and if for every orbit (x_t) of Γ with $x_0 \in I$, there exists an integer $t \geq 0$ such that $x_t \in J$. We say that Γ is *almost* (I, J)-*stable* if I and J are almost invariant and the convergence to J as defined above occurs for almost all initial condition in the orbit $x_0 \in I$, with respect to the Lebesgue measure. A quantized feedback map $k : \mathbb{R} \to \mathbb{R}$ is said to be *(almost)* (I, J)-*stabilizing* if the corresponding closed loop map Γ is (almost) (I, J)-stable.

Assume that Γ is almost (I, J)-stable. The first entrance time function

$$T_{(I,J)} : I \cap \Omega \to \mathbb{N} \cup \{+\infty\}$$

is defined by

$$T_{(I,J)}(x) = \inf\{n \in \mathbb{N} \mid \Gamma^n x \in J\} = \sum_{n=1}^{\infty} \mathbf{1}_{I \setminus J}(\Gamma^n x), \quad (5)$$

where $\mathbf{1}_{I\setminus J}(\cdot)$ denotes the indicator function of the set $I \setminus J$. We put $T_{(I,J)}(x) := +\infty$ if $\Gamma^t x \notin J$ for all t. Notice that the map $T_{(I,J)}$ is always finite exactly when we have stability, while it is almost surely finite when we have almost stability.

Remark: Notice that, if we want to extend the function $T_{(I,J)}$ to the all I, we can not use definition (5). Indeed, there is a possible ambiguity for orbits touching discontinuity points since, given $x \in I$, there can be infinitely many orbits having x as initial condition and therefore $\Gamma^n x$ is not uniquely defined. In this case definition (5) should be replaced as follows: we say that $T_{(I,J)}(x) = n$ if every orbit $(x_t) \in X_\Gamma$ such that $x_0 = x$ is such that $x_t \in J$ for any $t \geq n$ and if there exists an orbit $(x_t) \in X_\Gamma$ such that $x_0 = x$ and such that $x_{n-1} \notin J$.

For any choice of a probability density f in I, denote by $\mathbb{P}_f$ the probability measure induced by f and by $\mathbb{E}_f$ the expected value with respect to such a probability. Then, the expected value of the entrance time is given by

$$\mathbb{E}_f(T_{(I,J)}) = \int_I T_{(I,J)}(x) f(x) dx.$$

It is clear that

$$\begin{aligned} \mathbb{E}_f(T_{(I,J)}) &= \int_I \left[\sum_{n=1}^{\infty} \mathbf{1}_{I\setminus J}(\Gamma^n x) f(x) \right] dx = \\ &= \sum_{n=1}^{\infty} n \mathbb{P}_f[T_{(I,J)} = n] = \sum_{n=0}^{\infty} \mathbb{P}_f[T_{(I,J)} > n]. \end{aligned}$$

The most natural density to assume is the uniform density on I and for this reason probability measure and the expected value with respect to this density with be simply denoted by the symbols $\mathbb{P}$ and $\mathbb{E}$, respectively.

In the sequel, for any given (almost) (I, J)-stabilizing quantized feedback k yielding an (almost) (I, J)-stable piecewise affine closed loop map Γ, we will denote by $\mathbf{T}(k)$ or $\mathbf{T}(\Gamma)$ the relative expected entrance time. Notice that this quantity depends only on the restriction of Γ to $I \setminus J$ and so we can assume that Γ is defined only on $I \setminus J$. For this reason the right parameter measuring the information flow will be the number of quantization intervals in $I \setminus J$ which will be denoted by symbols $\mathbf{N}(k)$ or $\mathbf{N}(\Gamma)$. Finally the ratio between the length of I and the length of J will be called contraction rate and will be denoted by $C(k)$ or $C(\Gamma)$.

The performance analysis of the quantized stabilization consists in determining, for a given $C > 1$, $N \in \mathbb{N}$ and $T > 0$, whether there exists or not a (almost) stabilizing quantized feedback k such that $C(k) = C$, $\mathbf{N}(k) = N$ and $\mathbf{T}(k) = T$, or, in other words, in estimating the set

$$\mathcal{A} := \{(C, N, T) \,:\, C(k) = C, \mathbf{N}(k) = N, \mathbf{T}(k) = T, \,\exists\, k(\cdot)\}$$

Remark: The analysis proposed in this paper can be extended to a family of more general performance measures. Let

$$V : I \to \mathbb{R}$$

be such that $0 \leq V(x) \leq 1$ for every $x \in I$ and $V(x) = 0$ for every $x \in J$. Another measure of the transient properties of the closed loop system is the following number

$$\mathbb{E}\left(\sum_{n=0}^{\infty} V(\Gamma^n x) \right).$$

It is clear that, if $V(x) = \mathbf{1}_{I\setminus J}(x)$, then the previous cost coincides with the expected entrance time in J. If $V(x)$ is a general continuous function, then, for any $\alpha \in [0, 1]$ we have that

$$\alpha \mathbf{1}_{I\setminus J(\alpha)}(x) \leq V(x) \leq \mathbf{1}_{I\setminus J}(x),$$

where $J(\alpha) := \{x \in I \,:\, V(x) \leq \alpha\}$. This fact implies that

$$\alpha \mathbb{E}(T_{J(\alpha)}) \leq \mathbb{E}\left(\sum_{n=0}^{\infty} V(\Gamma^n x) \right) \leq \mathbb{E}(T_{(I,J)}).$$

This shows that the dependence of this generalized performance index and of the expected entrance time on the parameters $C(\Gamma)$ and $\mathbf{N}(\Gamma)$ will be similar.

3 Nested quantized feedback strategies

Consider the linear discrete time system (1), where $|a| > 1$, and consider two intervals $J \subseteq I$. We want to stabilize it through a quantized state feedback, i.e. we want to find a quantized feedback map k such that the closed loop system (2) drives (almost) any initial state $x_0 \in I$ into a state evolution which, after a transient, enters the interval J. Several solutions to this problem can be proposed. In fact we will show that, starting from a base quantized feedback, it is possible to construct a family of quantized feedbacks by iterating the base one.

Assume that $k(x)$ is a (almost) (I, J)-stabilizing quantized feedback with contraction rate $C(k)$, $\mathbf{N}(k)$ quantization intervals and expected entrance time $\mathbf{T}(k)$. Let $F(x)$ be an affine map such that $J = F(I)$. It is clear that the quantized feedback

$$F \circ k \circ F^{-1} \;:\; F(I) \to F(I)$$

is (almost) $(F(I), F^2(I))$-stabilizing. Observe that the corresponding closed loop map is $F \circ \Gamma \circ F^{-1}$. The same construction can be iterated, obtaining for every $i = 0, 1, \ldots, \tau - 1$ the quantized feedback $F^i \circ k \circ F^{-i}$ which is (almost) $(F^i(I), F^{i+1}(I))$-stabilizing. Consider now the quantized feedback defined as follows

$$k^{(\tau)}(x) := F^i \circ k \circ F^{-i}(x) \quad \text{if} \quad x \in F^i(I) \setminus F^{i+1}(I).$$

This quantized feedback is called nested.

It is clear that $k^{(\tau)}$ will be will be $(I, F^\tau(I))$-stabilizing if $k(x)$ is (I, J)-stabilizing. Less obvious is to show that $k^{(\tau)}$ will be almost $(I, F^\tau(I))$-stabilizing if $k(x)$ is almost (I, J)-stabilizing (see [7]). It is clear moreover that $C(k^{(\tau)}) = C(k)^\tau$ and $\mathbf{N}(k^{(\tau)}) = \tau\mathbf{N}(k)$. As far as the expected entrance time $\mathbf{T}(k^{(\tau)})$ is concerned, it is difficult in general to estimate its dependence on the number τ of nestings.

Consider the map

$$\Psi : I \to I : x \mapsto F^{-1} \circ \Gamma^{T_{(I,J)}(x)}(x), \tag{6}$$

where $T_{(I,J)}(x)$ is the first entrance time function for k. It is clear that, if the uniform density on I is invariant with respect to the transformation Ψ, then $\mathbf{T}(k^{(\tau)}) = \tau\mathbf{T}(k)$. In this case from a triple $(C, N, T) \in \mathcal{A}$ we can obtain a sequence of triples $(C^n, \tau N, \tau T)) \in \mathcal{A}$, for all $\tau \in \mathbb{N}$. This method will be used in the following subsections to obtain three specific quantized feedback strategies.

In general we can not guarantee that Ψ will possess invariant probability densities. It can be shown that this is the case if $T(x)$ is bounded ([7]. In this case we have the following result.

Proposition 1 *Assume that Γ is (I, J)-stable. There exists a probability density $\overline{f}$ and a bounded sequence $\{a_\tau\}$ such that*

$$\mathbf{T}(k^{(\tau)}) = \tau\mathbb{E}_{\overline{f}}(T) + a_\tau. \tag{7}$$

This has the following consequence. If the triple (C, N, T) is in $\mathcal{A}$ and corresponds to a situation in which the entrance time function is bounded, then we can obtain a sequence of triples $(C^\tau, \tau N, \tau\overline{T} + a_\tau) \in \mathcal{A}$, for all $\tau \in \mathbb{N}$, where $\overline{T}$ is the expected entrance time with respect to a suitable probability density and $\{a_\tau\}$ is a bounded sequence.

4 Three stabilizing quantized feedback strategies

The method presented in the previous section will be used in the following subsections to obtain three specific quantized feedback strategies. In the sequel we assume for simplicity that $I = [-1, 1]$ and $J = [\epsilon, \epsilon]$, with $\epsilon \leq 1$ and so we have that $C = 1/\epsilon$. In this section we will simply write $C, \mathbf{N}, \mathbf{T}$ dropping the explicit dependence from k.

4.1 Deadbeat quantized feedback strategy

The first strategy, which has been analyzed in a certain detail by Delchamps in [3], consists in approximating the 1-step deadbeat controller $k(x) := -ax$ by its quantized version, i.e., by a uniform quantized function $k(x)$ such that $-ax - \epsilon \leq k(x) \leq -ax + \epsilon$. One possibility is to take

$$k(x) := -(2h+1)\epsilon \quad \text{for} \quad h\frac{2\epsilon}{a} < x \leq (h+1)\frac{2\epsilon}{a}. \tag{8}$$

This controller drives any state belonging to I into J in one step. In this case we have that

$$\mathbf{N} = 2\left\lceil |a|\frac{C-1}{2} \right\rceil \simeq |a|C.$$

and that

$$\mathbf{T} = \sum_{n=1}^{\infty} \mathbb{P}[T_J \geq n] = \mathbb{P}[T_J \geq 1] = 1 - \mathbb{P}[J] = 1 - 1/C.$$

Using the nesting strategy presented above we can construct a τ steps deadbeat quantized feedback simply iterating the 1 step deadbeat quantized feedback. We only need to pay attention to the fact that the uniform density in I is invariant with respect to the map Ψ defined in (6). This happens if $|a|(C-1)/2$ is an integer. Assume that this is the case and denote it by n. We obtain a triple contraction rate, quantization intervals, expected entrance time equal to

$$\left(\frac{2n+|a|}{|a|}, 2n, \frac{2n}{2n+|a|}\right) \in \mathcal{A}\,.$$

Using the strategy presented above, we can iterate the construction τ times, obtaining in this way a sequence of triples

$$\left(\left(\frac{2n+|a|}{|a|}\right)^\tau, 2\tau n, \tau\frac{2n}{2n+|a|}\right) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

which provides a family of quantized feedbacks parametrized by the two integers τ, n. We are mainly interested in understanding what asymptotic behavior can be obtained of $\mathbf{N}$ and

$\mathbf{T}$ as $C \to \infty$. To this aim observe that

$$\frac{\mathbf{N}/|a|}{\mathbf{T}C^{1/\mathbf{T}}} = \left(\frac{2n+|a|}{|a|}\right)^{-\frac{|a|}{2n}} \in [1/e, 1].$$

Making the change of variable

$$C = \left(\frac{2n+|a|}{|a|}\right)^{\tau}, \qquad n = \frac{|a|}{2}(C^{\frac{1}{\tau}} - 1) \tag{9}$$

we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \tau(C^{\frac{1}{\tau}} - 1) \\ \mathbf{T} &= \tau(1 - C^{-\frac{1}{\tau}}) \end{aligned}$$

where τ is any function of C that, by (9), can be chosen arbitrarily subject to the fact that $\tau(C)/\log C$ is bounded from above. If in particular τ is fixed, we obtain

$$\begin{aligned} \mathbf{N}/|a| &\sim \tau C^{\frac{1}{\tau}} \\ \mathbf{T} &\sim \tau \end{aligned}$$

the symbol $\sim$ meaning that the ratio of the two functions tends to 1 as $C \to \infty$. If instead we think of τ as a possible function of C, we can distinguish two different behaviors: the case when $\tau(C)/\log C \to 0$ and the case when $\tau(C) \sim K \log C$. In the first case we have that

$$\mathbf{N}/|a| \sim \mathbf{T}C^{1/\mathbf{T}}.$$

and moreover $\mathbf{N}/\log C \to \infty$, namely we have a super-logarithmic growth of the number of quantization intervals, while the expected entrance time have a sublogarithmic growth $\mathbf{T}/\log C \to 0$. In the second situation when $\tau(C) \sim K \log C$ we have that both $\mathbf{N}$ and $\mathbf{T}$ grow logarithmically in C. More precisely, we have that

$$\begin{aligned} \mathbf{N}/|a| &\sim K(e^{1/K} - 1)\log C \\ \mathbf{T} &\sim K(1 - e^{-1/K})\log C \end{aligned}$$

4.2 Logarithmic quantized feedback strategy

The second strategy is based on the quantized feedback (we assume $a > 0$, the case $a < 0$ being completely analogous)

$$k(x) = \begin{cases} -a+1 & \text{if } \epsilon \le x \le 1 \\ +a-1 & \text{if } -1 \le x \le -\epsilon \end{cases}$$

where

$$\epsilon = \frac{a-1}{a+1}.$$

In this way we obtain an almost (I, J)-stabilizing quantized feedback where $I = [-1, 1]$ and $J = [-\epsilon, \epsilon]$.

In this case we have a contraction rate $1/\epsilon$ and 2 quantization intervals. The expected entrance time can be found by noticing that

$$\Gamma^{-n}(I \setminus J) = [-1, -\epsilon_n] \cup [\epsilon_n, 1],$$

where $\epsilon_n = 1 - 2/(a+1)a^n$, which implies that the expected entrance time is

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}[T_{(I,J)} > n] &= \sum_{n=0}^{\infty} \mathbb{P}[\Gamma^{-n}(I \setminus J)] = \\ &= \frac{2}{a+1}\sum_{n=0}^{\infty} a^{-n} = \frac{2a}{a^2-1} \end{aligned}$$

In general, when we do not restrict to positive a, we obtain a triple contraction rate, quantization intervals, expected entrance time equal to

$$\left(\frac{|a|-1}{|a|+1}, 2, \frac{2|a|}{|a|^2-1}\right) \in \mathcal{A}.$$

Using the strategy presented above, we can iterate the construction τ times. In this case it is less obvious to show that the Lebesgue measure is invariant with respect to the map Ψ defined from Γ as in (6). To show this observe preliminarly that, if we assume that $\Gamma(x) = x$ for all $x \in J$, then

$$\lim_{n\to\infty} \Gamma^n(x) = \Gamma^{T_{(I,J)}(x)}(x), \qquad \text{for almost all } x \in I$$

which implies that $\Gamma^n(x)$ converges to $\Gamma^{T_{(I,J)}(x)}(x)$ in distribution. Observe moreover that, if the density function f_n of the random variable $\Gamma^n(x)$ is of the form

$$f_n(a) = \begin{cases} \alpha_n & \text{if } a \in J \\ \beta_n & \text{if } a \in I \setminus J, \end{cases}$$

then also f_{n+1} has the same structure with $\alpha_{n+1} = 2\beta_n/|a| + \alpha_n$ and $\beta_{n+1} = \beta_n/|a|$. This implies that

$$\lim_{n\to\infty} f_n(a) = \begin{cases} 1/\epsilon & \text{if } a \in I_1 \\ 0 & \text{if } a \in I_0 \setminus I_1 \end{cases}$$

from which we can argue that the Lebegues measure is invariant with respect to the map Ψ.

These facts allow us to obtain a sequence of triples

$$\left(\left(\frac{|a|+1}{|a|-1}\right)^{\tau}, 2\tau, \frac{2|a|}{|a|^2-1}\tau\right) \in \mathcal{A}, \qquad \tau \in \mathbb{N}.$$

Making the change of variable

$$C = \left(\frac{|a|+1}{|a|-1}\right)^{\tau}, \qquad \tau = \frac{\log C}{\log(|a|+1) - \log(|a|-1)}$$

we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \frac{2}{|a|}\,\frac{\log C}{\log(|a|+1) - \log(|a|-1)} \\ \mathbf{T} &= \frac{2|a|}{|a|^2-1}\,\frac{\log C}{\log(|a|+1) - \log(|a|-1)} \end{aligned}$$

These expressions motivate the name logarithmic quantizer which is commonly given to this quantized feedback. The strategy obtained in this way coincides with the one proposed in [4, 6] which yields a Lyapunov stability.

4.3 Chaotic quantized feedback strategy

In [6] another possible quantized feedback yielding almost stability has been proposed. This control strategy exploits the chaotic behavior of the state evolution inside $I = [-1, 1]$ produced by the feedback map

$$k_0(x) := -(2h+1) \quad \text{for} \quad \frac{2}{a}h < x \leq \frac{2}{a}(h+1), \tag{10}$$

when we have that $|a| \geq 2$. In this way we have that, for almost every initial condition x_0, the state evolution x_t is mantained inside the interval I and is dense in this interval. For this reason x_t will visit the interval $J = [-\epsilon, \epsilon]$. Therefore, if we modify this feedback map in J as follows

$$k(x) = \begin{cases} k_0(x) & \text{if } x \in I \setminus J \\ k_1(x) & \text{if } x \in J \end{cases} \tag{11}$$

where $k_1(x)$ is any quantized feedback making J invariant (take for istance $k_1(x) = \epsilon k_0(x/\epsilon)$) we obtain that the state will move chaotically inside I till it will enter the interval J and there it will be entrapped. In this way we obtain a feedback map requiring

$$\mathbf{N} = \lceil |a| \rceil$$

quantization intervals. In this case the evaluation of the expected entrance time can be done using Markov chain techniques. Assume that $\epsilon = 2^{-n}$. It is clear that, for evaluating the expected entrance time, we can refere to the system with feedback $k_0(x)$. Define the sets $I_i := [-i2^{-n}, -(i-1)2^{-n}] \cup [(i-1)2^{-n}, i2^{-n}]$, $i = 1, \ldots, 2^n$. In this way we have that $J = I_1$. Assuming that that initial state x_0 is uniformly distributed in I, we can argue that

$$\mathbb{P}[x_0 \in I_i] = 2^{-n}.$$

Assuming that the iterated state x_t is uniformly distributed in each quantization interval I_i, then the structure of the closed loop map $\Gamma_0(x) = ax + k_0(x)$ ensures that also the updated state $x_{t+1} = \Gamma_0(x_t)$ will have the same property. Moreover we have that

$$\mathbb{P}[x_{t+1} \in I_j | x_t \in I_i] = \Pi_{ij}$$

where Π_{ij} is the i, j-element of the matrix

$$\Pi = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2^n \times 2^n}$$

If we introduce the column vector

$$\pi := 2^{-n} [1 \quad 1 \quad \cdots \quad 1 \quad 1] \in \mathbb{R}^{1 \times 2^n},$$

then (see [**?**, page ???]) the expected first entrance time in the state 1 is given by the formula

$$\mathbb{E}(T_{(I,J)}) = \frac{d}{dz} w(z)_{|z=1},$$

where

$$w(z) := \frac{\pi \Pi(z) e_1}{e_1^T \Pi(z) e_1}$$

and where $\Pi(z) := \sum_{n \geq 0} \Pi^n z^n$ and $e_1 := [1\ 0\ \cdots\ 0]^T$. Since $\pi \Pi = \pi$, then

$$\pi \Pi(z) := \frac{1}{1-z} \pi.$$

On the other hand it can be seen that

$$e_1^T \Pi(z) e_1 = 1 + 2^{-n} \frac{z^n}{1-z},$$

obtaining in this way

$$w(z) = \frac{1}{z^n + (1-z)2^n}$$

and

$$\mathbf{T} = \frac{d}{dz} w(z)_{|z=1} = 2^n - n.$$

In this way we obtained the triple

$$(2^n, 2, 2^n - n) \in \mathcal{A}.$$

Using the strategy presented above we can iterate this construction τ times. It can be shown that also in this case the Lebesgue measure is invariant with respect to the closed map Ψ defined from Γ as in (6). To show this we use the same kind of reasoning used in the previous subsection. Again, by defining Γ in such a way that $\Gamma(x) = x$ for all $x \in J$, we have that the random variable $\Gamma^n(x)$ converges to $\Gamma^{T_{(I,J)}(x)}(x)$ in distribution. Observe moreover that, if the density function f_n of the random variable $\Gamma^n(x)$ is constant in each quantization interval I_i, then it can be shown that also f_{n+1} has the same property. This implies that also the limit density will be a function which is constant in each set I_i and in particular in J. From this we can argue that the Lebesgue measure is invariant with respect to the map Ψ. These facts allow us to obtain a sequence of triples

$$(2^{\tau n}, \tau 2, \tau 2^n - \tau n) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

The previous reasoning can be extended to any situation in which $|a|$ is an integer. In this case it can be obtained sequence of triples

$$(|a|^{\tau n}, \tau |a|, \tau |a|^n - \tau n) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

which provides a family of quantized feedbacks parametrized by the two integers τ, n. We are mainly interested in understanding what asymptotic behavior can be obtained for $\mathbf{N}$ and $\mathbf{T}$ as $C \to \infty$. To this aim observe that

$$\frac{\mathbf{T}}{\frac{\mathbf{N}}{|a|} C^{\frac{|a|}{\mathbf{N}}}} = 1 - \frac{n}{|a|^n} \in [1 - \frac{1}{e \ln |a|}, 1].$$

Making the change of variable

$$C = |a|^{\tau n}, \qquad n = \frac{\log C}{\tau \log |a|} \tag{12}$$

we obtain that

$$\begin{aligned} \mathbf{N}/|a| &= \tau \\ \mathbf{T} &= \tau C^{\frac{1}{\tau}} - \frac{\log C}{\log |a|} \end{aligned}$$

where τ is any function of C that, by (12), can be chosen arbitrarily subject to the fact that $\tau(C)/\log C$ is bounded from above. If in particular τ is fixed, we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \tau \\ \mathbf{T} &\sim \tau C^{\frac{1}{\tau}}. \end{aligned}$$

If instead we think of τ as a possible function of C, we can distinguish the case when $\tau(C)/\log C \to 0$ and the case when $\tau(C) \sim K \log C$. In the first case we have that

$$\mathbf{T} \sim \frac{\mathbf{N}}{|a|} C^{\frac{|a|}{\mathbf{N}}}$$

and moreover $\mathbf{N}/\log C \to 0$, namely a sublogarithmic growth of the number of quantization intervals, while the expected entrance time have a superlogarithmic growth $\mathbf{T}/\log C \to \infty$. In the second situation when $\tau(C) \sim K \log C$ we have that both $\mathbf{N}$ and $\mathbf{T}$ grow logarithmically in C. More precisely, we have that

$$\begin{aligned} \mathbf{N}/|a| &= K \log C \\ \mathbf{T} &= \left(K e^{1/K} - \frac{1}{\log |a|} \right) \log C \end{aligned}$$

Chaotic stabilizers can also be considered for non integers slopes a. Some preliminary results on this case have been obtained in [6]. In the forecoming paper [5] it is proved the following more refined result.

Theorem 1 *Let a be such that $|a| > 2$, $I = [-1, 1]$ and $J = [-\epsilon, \epsilon]$. for $0 < \epsilon < 1$. There exists an almost (I, J)-stabilizing quantized feedback $k : I \to \mathbb{R}$ such that*

$$\begin{aligned} \mathbf{N} &= \lceil |a| \rceil + 1 \\ \mathbf{T} &\leq KC, \end{aligned}$$

where K is a positive constant only depending on a.

Remark: Observe that these three stabilization methods suggest that looking for a stabilizing quantized feedback with minimal quantization intervals is rather naive. In fact the last strategy would be clearly the optimal one. This is not true since the different strategies requires different information flow, but they provides closed loop systems with different stability performances. The following table summarizes the properties of the different quantized feedback strategies.

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