HIGHER ORDER SLIDING MODE CONTROL BASED ON OPTIMAL LINEAR QUADRATIC CONTROL

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Abstract

In this paper, a new robust higher order sliding mode controller for uncertain minimum-phase nonlinear systems is designed. The problem is solved in three steps: a) the higher order sliding mode problem is formulated in input-output term; b) the problem is viewed in uncertain linear context by considering uncertain nonlinear functions as bounded non structured parametric uncertainties; c) following the optimal sliding-mode design for linear systems, a time varying manifold is designed through the minimization of a quadratic cost function over a finite time interval with a fixed final state. The control law which engenders the sliding on the time varying surface allows the establishment of an r^{th} order sliding mode. The designed controller is well-adapted to practical implementation and all the features of linear quadratic control can be used to synthesize the controller's gain.

1 Introduction

It is well known that the standard sliding mode features are high accuracy and robustness with respect to various internal and external disturbances. The basic idea is to force the state via discontinuous feedback to move on a prescribed manifold (called the *sliding manifold*). Specific problem involved by this technique is the chattering effect, *i.e.* dangerous high-frequency vibrations of the controlled system. In [12], the author relates the chattering behaviour to the discontinuity of the "sign" function which appears in the control law on the sliding manifold. To overcome this problem, one can replace the "sign" function in a small vicinity of the surface by a smooth approximation; that implies deterioration of accuracy and robustness. Note also that there exist other approaches to reduce the effects of the chattering, by using observers [12], or generalized sliding mode controllers [10].

Recently, a new approach called "higher order sliding mode" has been proposed [1], [3], [6]. Instead of influencing the first sliding manifold time derivative, the "sign" function is acting on its higher time derivative. Keeping the main advantages of the standard sliding mode control, the chattering effect is eliminated and higher order precision is provided. In the case of second order sliding mode control ($r = 2$), many works have given solutions. Several second order sliding mode algorithms are proposed in [1], [3], [6]. Arbitrary-order sliding controller for single-input single-output systems (SISO) with finite time convergence has been proposed in [7]. At our best knowledge, these works are the most complete published on the r^{th} order sliding mode approach. The algorithm proposed in [7] is inspired by the so-called "terminal sliding modes control" [14]. By tuning only one "gain" parameter and from the knowledge of the relative degree of the output [5], the controller allows the tracking of smooth signals. The aim of this paper is to present a new arbitrary-order sliding mode controller for uncertain SISO minimum-phase nonlinear systems. The main objective of this new approach is to propose a controller for which the implementation is simple, the convergence time is finite and the robustness is ensured. The controller design combines standard sliding mode control with linear quadratic (LQ) one over a finite time interval with a fixed final state [8]. The infinite-horizon linear quadratic control has been used by [13], [12] to synthesize sliding mode surfaces for multi-input linear systems. Actually, the problem of the higher order sliding mode control of SISO minimum-phase uncertain systems can be formulated in input-output terms only through the differentiation of the sliding variable [7], and is equivalent to the finite time stabilization of integrators chain with nonlinear uncertainties. These latter are considered as bounded non structured parametric uncertainties: in this case, the system can be viewed as an uncertain linear system. Then, following the optimal sliding mode formulation for linear systems [12], and considering the uncertain linear system, an optimal time varying switching manifold is determined by minimizing a quadratic cost function over a finite time interval $[t_0, t_f]$ with a fixed final state. The standard sliding mode over this manifold (which depends on the sliding variable σ and its $(r-1)$ first time derivatives) leads to the establishment of r^{th} sliding mode in finite time with respect to σ.

The algorithm needs the relative degree ρ [5] of the system with respect to the sliding variable σ and the bounds of uncertainties. This algorithm has several advantages: first, the convergence time is fixed *a priori* via the parameter t_f and the control law can be adjusted via t_f and two weighting matrices P_f and Q. Furthermore, this strategy can be applied for all value of sliding mode order (greater or equal to the relative degree). Finally, the structure of the controller is well-adapted to practical implementations.

The paper is organized as follows. The linear quadratic optimal control problem over a finite time interval with a fixed final state is briefly recalled in Section 2. The problem of higher order sliding mode control is stated in Section 3, in which it is shown that the problem is equivalent to stabilize to zero an uncertain linear system in finite time. The control strategy allowing the establishment of r^{th} order sliding mode in finite time is described in Section 4. Section 5 is devoted to illustrate the features of the controller through application to the control of a kinematic car model [7].

2 Background

2.1 Linear quadratic control over a finite time interval with a fixed final state

Consider the controllable linear system

$$\dot{x} \quad = \quad Ax + Bu \tag{1}$$

where $u \in I\!R^p$ is the control input and $x \in I\!R^n$ is the state vector. The objective of the LQ control over a finite time interval $[t_0, t_f]$ with a fixed final state is to find a control law $u \in L_2[0 \quad \infty)$ which minimizes the quadratic cost functional

$$J \quad = \quad \frac{1}{2}\{x^T(t_f)P_f x(t_f) + \int_{t_0}^{t_f}(x^TQx + u^TRu)\,\mathrm{d}t\} \tag{2}$$

for every initial state x_0 and $0 \leq t_0 < t_f < +\infty$ under the final state constraint

$$x(t_f) = x_d(t_f) \tag{3}$$

where $x_d(t_f)$ is the desired final state. P_f, Q and R denote the so-called *weighting matrices*. P_f and Q are supposed to be symmetrical and positive semidefinite, and R a symmetrical positive definite matrix. Since the matrix $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$ is symmetrical and non-negative definite, there exist unique full rank matrix C with elements in $I\!R$ so that

$$Q = C^T\, C. \tag{4}$$

The solution to the previous problem is summarized in the following theorem.

Theorem 1 ([8]) *Consider the linear system (1) with the pair (A,B) controllable and the pair (C,A) observable. Then, over the interval $[t_0, t_f]$, the optimal control u that stabilizes (1) to $x_d(t_f)$ in finite time for every initial state value x_0 and minimizes the quadratic cost function (2) with respect to the linear system (1) is given by*

$$\begin{array}{rcl} u(t) & = & -(R^{-1}B^TP(t) - R^{-1}B^TVH^{-1}V^T)x \\ & & -R^{-1}B^TVH^{-1}x_d(t_f) \end{array} \tag{5}$$

where $P(t) \in I\!R^{(n-p)\times(n-p)}$ is the unique non-negative definite solution of the differential matrix Riccati equation (with $P(t_f) = P_f$)

$$-\dot{P} \quad = \quad A^TP + PA + Q - PBR^{-1}B^TP, \tag{6}$$

$V \in I\!R^{(n-p)\times(n-p)}$ and $H \in I\!R^{(n-p)\times(n-p)}$ are the solutions of

$$-\dot{V} = (A - BR^{-1}B^TP)^TV, \quad t \leq t_f,\ V(t_f) = I \tag{7}$$

and

$$\dot{H} = V^TBR^{-1}B^TV, \quad t \leq t_f, \quad H(t_f) = 0 \tag{8}$$

with I the unit matrix. ■

3 Problem formulation

Consider the nonlinear SISO system

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & \sigma(x,t) \end{array} \tag{9}$$

where $x \in I\!R^n$ is the state variable, $u \in I\!R$ is the input control and $\sigma(x,t) \in I\!R$ is the output function (sliding variable). $f(x)$, $g(x)$ and $\sigma(x,t)$ are smooth functions. In this section, it is shown that the problem of higher order sliding mode control with respect to $\sigma(x,t)$ of nonlinear system (9) can be expressed in terms of stabilization of a linear system with uncertainties. Consider the nonlinear system (9), and assume that

H1. the relative degree ρ of system (9) with respect to σ is known and the associated zero dynamics are stable.

H2. $u \in \mathcal{U} = \{u : |u| < u_M\}$ where u_M is a real constant; furthermore, the solution of (9) is well defined $\forall\, t \geq 0$.

The r^{th} order sliding mode is defined through the following definition

Definition 1 [1] *Given the sliding variable σ, and $r \in I\!N$ with $r \geq 1$. The"r^{th} order sliding set" of σ, denoted $\mathcal{S}^r$, is defined as*

$$\mathcal{S}^r \quad = \quad \{x \in \mathcal{X} \mid \sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0\} \tag{10}$$

with $\mathcal{X} \subset I\!R^n$. The integer r is called "sliding mode order". ■

Definition 2 [6] *Consider the not-empty r^{th} order sliding set $\mathcal{S}^r$, and assume that it is locally an integral set in the Filippov sense, i.e. it consists of Filippov's trajectories of the discontinuous dynamics system [4]. The behaviour of (9) satisfying (10) is called "r^{th} order sliding mode" with respect to the sliding variable σ.* ■

Definition 2 means that system (9) satisfies an r^{th} order sliding mode with respect to σ if its trajectories lie on the intersection of the $r-1$ manifolds $\sigma = 0$, $\dot{\sigma} = 0$, $\cdots$, $\sigma^{(r-2)} = 0$ and $\sigma^{(r-1)} = 0$ in $\mathcal{X}$.

Our control goal is to fulfill the constraint $\sigma(x,t) = 0$ in finite time. The r^{th} order sliding mode control approach allows the finite time stabilization to zero of the sliding variable σ and its $r-1$ first time derivatives by defining a suitable discontinuous control function which is either the actual control if $\rho = r$, or its $(r-\rho)^{th}$ time derivative if $r > \rho$.

- **Case 1 :** $r = \rho$. Introduce new local coordinates $z = (z_1, \cdots, z_r, \cdots, z_n)$ where $[z_1, z_2, \cdots, z_r]^T = [\sigma, \dot{\sigma}, \cdots, \sigma^{(r-1)}]^T$ and $[z_{r+1}, \cdots, z_n]$ are chosen so that z is a local state coordinates transformation. Then, the problem of higher order sliding mode control with respect to σ is equivalent to the finite time stabilization of the following system

$$\left\{ \begin{array}{lcll} \dot{z}_i & = & z_{i+1}, & i = 1, \cdots, r-1 \\ \dot{z}_r & = & a(z,t) + b(z,t)u & \\ y & = & z_1 & \end{array} \right. \tag{11}$$

with $b = \dfrac{\partial}{\partial u}\left[\sigma^{(r)}\right]$ and $a = \sigma^{(r)} - bu$.

- **Case 2 :** $r > \rho$. This case is more general: in fact, it is necessary to increase the dimension of the system by addition of the actual control input u and its $r - \rho - 1$ first time derivatives as state variables. Then, one gets a new "extended" system

$$\begin{array}{rcl} \dot{\bar{x}} & = & \bar{f}(\bar{x}) + \bar{g}(\bar{x})u^{(r-\rho)} \\ & = & \begin{bmatrix} f(x) + g(x)\bar{x}_{n+1} \\ \bar{x}_{n+2} \\ \vdots \\ \bar{x}_{n+r-\rho} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \cdot u^{(r-\rho)} \end{array} \tag{12}$$

with $\bar{x} = [\bar{x}_1 \ \cdots \ \bar{x}_n \ \bar{x}_{n+1} \ \cdots \ \bar{x}_{n+r-\rho}]^T = [x^T \ u \ \dot{u} \ \cdots \ u^{(r-\rho-1)}]^T$. Note that the relative degree of (12) with respect to σ versus the "new" input $v = u^{(r-\rho)}$ equals r. Assume that the extended system has stable zero dynamics with respect to σ. Then, the r^{th} order sliding mode with respect to σ is equivalent to the finite time stabilization of system

$$\left\{ \begin{array}{lcll} \dot{z}_i & = & z_{i+1} & i = 1, \cdots, r-1 \\ \dot{z}_r & = & \varphi(z,t) + \gamma(z,t)v(t) & \end{array} \right. \tag{13}$$

where $z = [z_1 \cdots z_r \ \cdots z_{n+r-\rho}]^T \in \mathcal{Z} \subset I\!R^{n+r-\rho}$ is a new coordinates transformation such that $z_1 = \sigma$, $z_2 = \dot{\sigma}$, $\cdots$, $z_r = \sigma^{(r-1)}$, $\gamma = \dfrac{\partial}{\partial v}\left[\sigma^{(r)}\right]$ and $\varphi = \sigma^{(r)} - \gamma u$.

Consider the following assumption

H3. Functions $\varphi(z,t)$ and $\gamma(z,t)$ are bounded uncertain functions and, in the sequel of the paper, without loss of generality, $\gamma(z,t)$ is supposed to be positive : there exist $K_m \in I\!R$, $K_M \in I\!R$, $C_0 \in I\!R$ such that

$$\begin{array}{l} 0 < K_m < \gamma(z,t) < K_M \\ |\varphi(z,t)| \leq C_0. \end{array} \tag{14}$$

Under Assumption H3, the system (13) can be viewed as a chain of integrators with uncertain bounded terms. Then, the problem is stated as the finite time stabilization of (13) in a linear uncertain context, while considering the nonlinear functions γ and φ as bounded non structured parametric uncertainties. One can summarize the problem statement of higher sliding mode control in the following way:

Consider the nonlinear system (12) with a relative degree r with respect to σ. The r^{th} order sliding mode control with respect to σ is equivalent to the finite time stabilization to zero of the uncertain linear system

$$\begin{array}{rcl} \dot{Z}_1 & = & A_{11}Z_1 + A_{12}Z_2 \\ \dot{Z}_2 & = & \varphi + \gamma v \end{array} \tag{15}$$

where $Z_1 = [\sigma \cdots \sigma^{(r-2)}]^T$, $Z_2 = \sigma^{(r-1)}$, $0 < K_m < \gamma < K_M$, $|\varphi| \leq C_0$ and A_{11}, A_{12} defined by

$$A_{11} = \begin{bmatrix} 0 & 1 & \ldots & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ldots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As previously mentioned , if $r = \rho$, then $v = u$; if $r > \rho$, then $v = u^{(r-\rho)}$.

4 A solution to the r^{th} order sliding mode control

4.1 Optimal switching manifold design

We suggest to stabilize the perturbed linear system (15) in finite time while minimizing the following quadratic cost over a finite time interval $[t_0, \ t_f]$ ($t_0 \geq 0$ and $t_f < +\infty$)

$$J \ = \ \frac{1}{2}Z(t_f)^T P_f Z(t_f) + \frac{1}{2}\int_{t_0}^{t_f} Z^T Q Z \mathrm{d}t, \tag{16}$$

under the following fixed final states constraint

$$Z(t_f) = Z_d(t_f) = 0 \tag{17}$$

with $Z = [Z_1^T \ \ Z_2^T]^T$. The positive symmetrical matrix Q is defined as

$$Q \ = \ \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{18}$$

where Q_{11}, Q_{12} and Q_{22} are $((r-1)\times(r-1))$-, $((r-1)\times(1))$- and (1×1)-dimensional matrices respectively. Criterion (16) becomes

$$J \ = \ \frac{1}{2}\int_{t_0}^{t_f} Z_1^T Q_{11} Z_1 + 2Z_1^T Q_{12} Z_2 + Z_2^T Q_{22} Z_2 \, \mathrm{d}t. \tag{19}$$

Let ω defined as

$$\omega \ = \ Z_2 + Q_{22}^{-1} Q_{12}^T Z_1. \tag{20}$$

From (20), dynamics of Z_1 (15) and criterion (19) can be written as

$$\dot{Z}_1 \ = \ (A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T)Z_1 + A_{12}\omega \tag{21}$$

and

$$J = \frac{1}{2}\int_{t_0}^{t_f} Z_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T)Z_1 + \omega^T Q_{22}\omega \, \mathrm{d}t. \tag{22}$$

In (21), consider Z_1 as the state variable, and ω as the control input; the problem leads back to the resolution of the *LQ* problem (22) for the linear time invariant system (21) formulated in section 2.2. By analogy with Theorem 1, one gets

Theorem 2 *Consider the system (21) with $Q_{22} > 0$, pair $(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T, A_{12})$ controllable and pair $(C, A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T)$ observable with*

$$C^T C = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T \tag{23}$$

Then, over the finite time interval $[t_0, t_f]$, a control ω stabilizing (21) to $Z(t_f) = 0$ in finite time and minimizing the quadratic cost function (22), with respect to the linear invariant system (21) for every initial value $Z(t_0)$, is given by

$$\omega = -(Q_{22}^{-1}A_{12}^T P(t) - Q_{22}^{-1}A_{12}^T V(t)H(t)^{-1}V(t)^T)Z_1 \tag{24}$$

where $P(t) \in I\!R^{(r-1)\times(r-1)}$ is the unique solution to the differential Riccati equation

$$\begin{aligned} -\dot{P} &= P(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T) + (A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T)^T P \\ &\quad -PA_{12}Q_{22}^{-1}A_{12}^T P + (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T) \end{aligned} \tag{25}$$

with a given $P(t_f) = P_f$. $V \in I\!R^{(r-1)\times(r-1)}$ and $H \in I\!R^{(r-1)\times(r-1)}$ are the solutions to two linear differential equations ($t \le t_f$, $V(t_f) = I$ and $H(t_f) = 0$)

$$-\dot{V} = (A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T - A_{12}Q_{22}^{-1}A_{12}^T P)^T V, \tag{26}$$

and

$$\dot{H} = V^T A_{12}Q_{22}^{-1}A_{12}^T V. \tag{27}$$

■

The controllability of (A_{11}, A_{12}) is sufficient to ensure the controllability of $(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T, A_{12})$. Moreover, the positivity condition on Q ensures that $Q_{22} > 0$ (so that Q_{22}^{-1} exists) and $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T > 0$. Then, there exist unique full rank matrix C with elements in $I\!R$ such that $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T = C^T C$ and the pair $(C, A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T)$ is observable [12]. From (20)-(24) and by analogy with Theorem 1 under constraint $Z(t_f) = 0$, the optimal vector Z_2 is a function of vector Z_1 and has the following form

$$\begin{aligned} Z_2 &= -(Q_{22}^{-1}A_{12}^T P(t) - Q_{22}^{-1}A_{12}^T V(t)H(t)^{-1} \\ &\quad V(t)^T + Q_{22}^{-1}Q_{12}^T)Z_1. \end{aligned} \tag{28}$$

Let $S(Z,t)$ defined by

$$\begin{aligned} S(Z,t) &= Z_2 + (Q_{22}^{-1}A_{12}^T P(t) - Q_{22}^{-1}A_{12}^T V(t)H(t)^{-1} \\ &\quad V(t)^T + Q_{22}^{-1}Q_{12}^T)Z_1. \end{aligned} \tag{29}$$

Equation $S(Z,t) = 0$ describes the desired dynamics which satisfy the finite time stabilization of vector $[Z_1^T \; Z_2^T]^T$ to zero and minimize the quadratic cost function (19). The *optimal switching manifold* is defined as

$$\mathcal{S} = \{x \in \mathcal{X} \mid S(Z,t) = 0\} \tag{30}$$

on which system (15) is forced to slide on via the discontinuous control v.

4.2 Controller design

We focus the attention to the design of the discontinuous vector control law which drives and constrains the system (12) to lie on $\mathcal{S}$ in finite time.

Theorem 3 *Consider the extended nonlinear system (12) with a relative degree r with respect to the sliding variable $\sigma(x,t)$. Suppose that hypotheses H_2 and H_3 are fulfilled and the system is minimum phase. Let $S \in I\!R$ a function defined as*

$$\begin{aligned} S = \sigma^{(r-1)} + (Q_{22}^{-1}A_{12}^T P - Q_{22}^{-1}A_{12}^T V H^{-1} V^T \\ + Q_{22}^{-1}Q_{12}^T) \cdot \left[\sigma \;\; \dot{\sigma} \;\; \cdots, \;\; \sigma^{(r-2)}\right]^T \end{aligned} \tag{31}$$

with the matrix A_{12} defined by (16), $P(t)$ the unique non-negative definite solution of the differential matrix Riccati equation (25) (with a given $P(t_f) = P_f$), V and H the solutions of equations (26) and (27) and Q is a symmetrical and positive matrix defined by (18). Then, the control input u whose the $(r-\rho)^{th}$ time derivative is

$$v = u^{(r-\rho)} = -\alpha \, \mathrm{sign}(S(\sigma, \dot{\sigma}, \cdots, \sigma^{(r-1)}, \; t)) \tag{32}$$

with

$$\alpha \ge \frac{C_0 + \Theta}{K_m} \text{ and} \tag{33}$$

$$\Theta > \mathrm{Max}\left(\left|\Psi \cdot \begin{bmatrix} \dot{\sigma} \\ \ddot{\sigma} \\ \vdots \\ \sigma^{(r-1)} \end{bmatrix} + \Delta \cdot \begin{bmatrix} \sigma \\ \dot{\sigma} \\ \vdots \\ \sigma^{(r-2)} \end{bmatrix}\right|\right) \tag{34}$$

where

$$\begin{aligned} \Psi &= Q_{22}^{-1}A_{12}^T P - Q_{22}^{-1}A_{12}^T V H^{-1} V^T + Q_{22}^{-1}Q_{12}^T \\ \Delta &= Q_{22}^{-1}A_{12}^T \cdot (\dot{P} - \dot{V}H^{-1}V^T - V(\dot{H^{-1}})V^T \\ &\quad -VH^{-1}(\dot{V^T})) \end{aligned} \tag{35}$$

with $\dot{P}$, $\dot{V}$ and $\dot{H}$ defined respectively by (25)-(26)-(27), leads to the establishment of r^{th} order sliding mode with respect to σ by attracting each trajectory in finite time. The convergence time is t_f.

■

Proof. The finite time stabilization to zero of vector $Z = [Z_1^T \; Z_2^T]^T = [\sigma \; \dot{\sigma} \; \cdots \; \sigma^{(r-1)}]^T$ via the minimization of (16) is realized by sliding on the optimal switching manifold

$$\begin{aligned} \mathcal{S} = \{x \in \mathcal{X} \mid \sigma^{(r-1)} + (Q_{22}^{-1}A_{12}^T P(t) - Q_{22}^{-1}A_{12}^T V(t) \\ H(t)^{-1}V(t)^T + Q_{22}^{-1}Q_{12}^T) \cdot [\sigma \, \dot{\sigma} \cdots \sigma^{(r-2)}\,]^T = 0\} \end{aligned} \tag{36}$$

The design of a switching control function follows the conventional path [12]: the variable structure control v takes the form

$$v = -\alpha \ \mathrm{sign}(S) \tag{37}$$

where the gain α is selected to satisfy the sliding mode condition [12]

$$\dot{S}\cdot S < 0. \tag{38}$$

One gets

$$\dot{S} = \beta + [\varphi + \gamma \cdot (-\alpha \cdot \mathrm{sign}(S))] \tag{39}$$

with β given by

$$\beta = \Psi \begin{bmatrix} \dot{\sigma} \\ \ddot{\sigma} \\ \vdots \\ \sigma^{(r-1)} \end{bmatrix} + \Delta \begin{bmatrix} \sigma \\ \dot{\sigma} \\ \vdots \\ \sigma^{(r-2)} \end{bmatrix} \tag{40}$$

where

$$\begin{aligned} \Psi &= Q_{22}^{-1}A_{12}^T P - Q_{22}^{-1}A_{12}^T V H^{-1} V^T + Q_{22}^{-1}Q_{12}^T \\ \Delta &= Q_{22}^{-1}A_{12}^T \cdot (\dot{P} - \dot{V}H^{-1}V^T - V(\dot{H^{-1}})V^T \\ &\quad -VH^{-1}(\dot{V^T})). \end{aligned} \tag{41}$$

To satisfy (38), the $(r-\rho)^{th}$ time derivative of the control $v = u^{(r-\rho)} = -\alpha \cdot \mathrm{sign}(S)$ must dominate in (39). It means formally that inequality $\gamma \cdot \alpha > \beta + \varphi$ implies $\dot{S}S < 0$ in finite time. A sufficient condition is

$$\min(|\gamma|)\cdot\alpha > \mathrm{Max}(|\beta|) + \mathrm{Max}(|\varphi|). \tag{42}$$

Since the vector $[\sigma\ \dot{\sigma}\ \cdots\ \sigma^{(r-2)}\ \sigma^{(r-1)}]^T$, $P(t)$, $V(t)$ and $H(t)$ are bounded functions, then function β can be bounded by a positive real number Θ. From (42), one derives that gain α has to be tuned so that $\alpha > (\Theta + C_0)/K_m$ to ensure (38).

5 An academic example

This part displays the control of a simple kinematic model of a car [7, 9]. It has been chosen to illustrate the control strategy previously exposed, and also to compare the results obtained by our approach to the results proposed in [7]. Then, the trajectories, the initial values of the state variables and the simulations have been made, as accurately as possible, in the same conditions as [7]. The car model is

$$\begin{aligned} \dot{x}_1 &= w\cdot\cos(x_3) \\ \dot{x}_2 &= w\cdot\sin(x_3) \\ \dot{x}_3 &= w/l\cdot\tan(x_4) \\ \dot{x}_4 &= u \end{aligned} \tag{43}$$

where x_1 and x_2 are the cartesian coordinates of the rear-axle middle point, x_3 the orientation angle and x_4 the steering angle. u is the control input. w is the longitudinal velocity ($w = 10\ m/s$), and l the distance between the two axles ($l = 5\ m$). The goal is to steer the car from a given initial position to the trajectory $x_2 = g(x_1) = 10\ \sin(0.05x_1) + 5$; all the state variables are assumed to be measured in real time. Let define the output $\sigma(x) = x_2 - g(x_1)$. The relative degree of (43) with respect to σ equals 3. In order to avoid chattering phenomena, we propose to steer σ to zero using 4^{th} order sliding mode control as in [7]. In this case, the control derivative $v = \dot{u}$ is viewed as the control instead of u, which is considered as a new state coordinate. Let Z_1 and Z_2 denote

$$Z_1 = \begin{bmatrix} \sigma \\ \dot{\sigma} \\ \ddot{\sigma} \end{bmatrix} \quad \text{and} \quad Z_2 = [\sigma^{(3)}]. \tag{44}$$

Then, according to Section 3, the 4^{th} order sliding mode with respect to σ is equivalent to the finite time stabilization to zero of the following system

$$\begin{aligned} \dot{Z}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot Z_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot Z_2 \\ &:= A_{11}\cdot Z_1 + A_{12}\cdot Z_2 \\ \dot{Z}_2 &= \varphi + \gamma\cdot v. \end{aligned} \tag{45}$$

Since the state variables are bounded, functions φ and γ are bounded with $\gamma > 0$. By using the control design of the previous section for the synthesis of a 4^{th} order sliding mode controller, and from (31), one gets

$$\begin{aligned} S &= \sigma^{(3)} + (Q_{22}^{-1}A_{12}^T P(t) - Q_{22}^{-1}A_{12}^T V(t)H(t)^{-1}V(t)^T \\ &\quad + Q_{22}^{-1}Q_{12}^T)\cdot[\sigma\ \dot{\sigma}\ \ddot{\sigma}\ ]^T. \end{aligned} \tag{46}$$

The choice of weighting matrix Q is as follows

$$Q_{11} = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $Q_{22} = 1000$. The choice of Q is made to obtain realistic value of the input. The weighting matrix $P(t_f) = P_f$ has been taken equal to $0_{3\times 3}$, and t_f equals 10 sec. The control input v is defined as

$$v = \dot{u} = -\alpha\cdot\mathrm{sign}(S). \tag{47}$$

Gain α must satisfy condition (33) and is tuned as $\alpha = 2$. The sampled time ($\tau = 10^{-3}$s) is the same as in [7], whereas the initial state variables conditions are more constraining. Figure 1 displays the tracking of $g(x_1)$ by x_2; note the absence of chattering phenomena. In Figure 2, the convergence to zero of σ and its first three derivatives is put into evidence; this convergence is done in the time interval defined by t_f. Figure 3 displays the state variable x_4, which is the physical input [7] and on which no chattering appears. The comparison between these results and [7] allows to conclude that the present approach, applied to this specific example, seems to be more efficient from a "chattering" point of view (see for example the time derivatives of σ). Note also two additional advantages of this approach : the relative simplicity of the control law (only t_f and Q must be tuned), and the possibility to state *a priori* the convergence time.

6 Conclusion

A methodology for the design of a robust higher order sliding mode controller with a simple structure for a class of SISO nonlinear uncertain systems has been presented. The problem is formulated in an uncertain linear context to allow the synthesis of a control law which uses the good features of optimal linear quadratic control. The controller is able to steer to zero in finite time the output function of any uncertain smooth SISO minimum-phase dynamic system with known relative degree. The effectiveness of the method is shown through simulation results of a car control.

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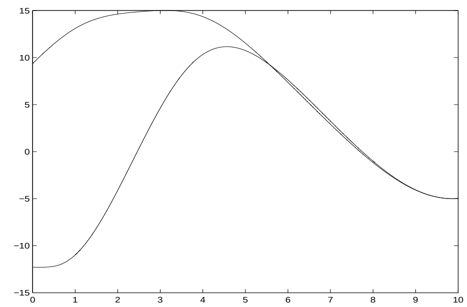

Figure 1: Reference $g(x_1)$ (m) and current trajectory x_2 (m) versus x_1 (m).

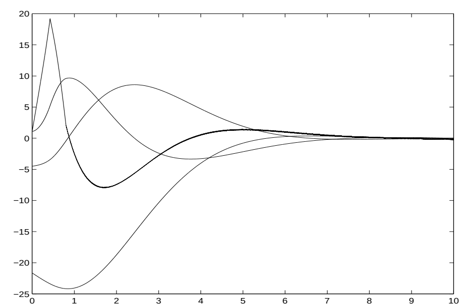

Figure 2: Surface and its 3 first time derivatives versus time (sec).

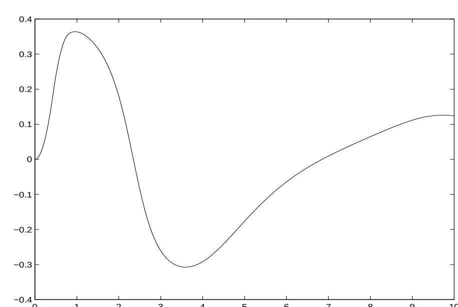

Figure 3: Steering angle x_4 (rad) versus time (sec).