ROBUST REGULATION OF A CLASS OF NONLINEAR SYSTEMS USING SINGULAR PERTURBATION APPROACH

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Abstract

In this paper, we consider robust regulation of a class of nonlinear systems, via H_∞ controller using singular perturbation approach. First, using normal form equations, we eliminate the nonlinear part of the system matrix of equations of system and transform it to a linear diagonal form. Separating new equations to slow and fast subsystems, due to the singular perturbation approach and with the assumption of norm-boundedness of the fast dynamics, we can treat them as disturbance and design H_∞ controller for a system with a lower order than the original one that stabilizes the overall closed loop system. The proposed method is applied to a single link, flexible joint robot manipulator.

1. Introduction

In linear control theory the solution of state feedback H_∞ problem has proved to be instrumental while solutions to the resulting Hamilton-Jacobi-Isaac (HJI) inequalities in nonlinear H_∞ control problem are usually extremely difficult to obtain. On the other hand it is proved that if the H_∞ control problem for the linearized system is solvable then locally, one obtains a solution to the nonlinear H_∞ control problem [1]. Also using singular perturbation theory for systems with two (or more) time scales, one can circumvent most of the controller design difficulties for complex multi dimensional systems.
In this paper we consider the robust regulation of a class of nonlinear systems that is affine in input. We attempt to eliminate the nonlinear part of system using normal form theory [2], and then using the singular perturbation approach, separate the slow and fast modes of it. The idea that one can consider the fast dynamics of a singularly perturbed system as disturbances first is discussed in [3]. In fact if we have a system with both low and high frequency dynamics in its frequency response, it almost has a lower gain in high frequencies than the low frequencies, for example this is true for many mechanical systems that act as low pass filters.
With this idea, we can use the H_∞ method to design a robust controller for the slow subsystem as the nominal plant. In fact we consider one part of a system as uncertainty and then design the controller for the remaining certain part of system with an order less than the original one. It is important that choosing a part of system as uncertainty is not optional since the small gain theorem must be hold. The small gain theorem says that a system consists of two subsystems with g_1 and g_2 gains, is stable if $g_1.g_2 < 1$. Now if we suppose that one of these subsystems is the system related to slow dynamics with the gain g_1, and the other is related to the fast dynamics with the gain g_2, and also consider the subsystem with high frequency dynamics as uncertainty then the whole system is stable when $g_1.g_2 < 1$.
The only information we need is the H_∞ norm of uncertain subsystem and no need to any other dynamical information. With this method we consider the main system via a technique as a system with a dimension less than its actual dimension.
The main contribution of this paper is to use the normal form equations for eliminating the nonlinearities from the system matrix up to the desired degree; using singular perturbation approach to separating the dynamic modes of obtained Jordan form equations of system; using the idea stated in [3] for considering the fast modes of system as uncertainty and then designing an H_∞ controller for the slow part of system as the nominal system that stabilizes the whole closed loop system.

In section 2, use of normal form equations to eliminate the nonlinearities of system matrix is stated. In section 3 the H_∞ controller is designed for the nominal system, and in section 4 the designed H_∞ controller is applied to a single link flexible joint manipulator as simulation results.

2. System definition

Consider the system of nonlinear equation

$$\dot{x} = f(x) + g(x)u \qquad (1).$$

Using the Taylor expansion of $f(x)$ and $g(x)$ about the equilibrium point (without loss of generality at the origin) and noting that $f(0)=0$, we have:

$$\dot{x} = \frac{\partial f}{\partial x}(0)x + \tilde{f}(x) + (g^0 + \tilde{g}(x))u \qquad (2)$$

Where

$$\tilde{f}(x) = f(x) - \frac{\partial f}{\partial x}(0)x$$

$$\tilde{g}(x) = g(x) - g^0, g^0 = g(0) \qquad (3)$$

Now we use a similarity transformation to transform $\frac{\partial f}{\partial x}(0)$ into Jordan canonical form. With the transformation $x = Tw$, equation (2) will be:

$$\dot{w} = T^{-1}\frac{\partial f}{\partial x}(0)Tw + T^{-1}\tilde{f}(Tw) + T^{-1}(g^0 + \tilde{g}(Tw))u \qquad (4)$$

equation (4) Can be written as

$$\dot{w} = Jw + F(w) + (G + \tilde{G}(w))u \qquad (5)$$

We expand $F(w)$ and $\tilde{G}(w)$ by Taylor series, so that equation (5) becomes

$$\dot{w} = Jw + F_2(w) + F_3(w) + \ldots + F_{r-1}(w) + O(|w|^r)$$
$$+ (G + G_1(w) + G_2(w) + \ldots)u \qquad (6)$$

which $F_i(w)$ and $G_i(w)$ shows the order i term in w. We now perform a series of coordinate transformations to eliminate the nonlinearities [2]. The first is

$$y = w - h_2(w)$$
$$v = \alpha_2(w) + (1 + \beta_1(w))u \qquad (7)$$

where $h_2(w)$ and $\alpha_2(w)$ are second order functions in w and $\beta_1(w)$ is a first order function in w. Substituting equation (7) into equation (6) and using the assumption in [2] we will have

$$\dot{y} = Jy + Gv + [Jh_2(w) - Dh_2(w)Jw + F_2(w) - G\alpha_2(w)]$$
$$+ F_3(w) + \ldots + F_{r-1}(w) + O(|w|^r)$$
$$+ G\beta_1(w)u - Dh_2(w)Gu + G_1(w)u \qquad (8)$$

We can choose $h_2(y)$ and $\alpha_2(w)$ and $\beta_1(w)$ as below, so as simplify the second order terms in equation (8)

$$Dh_2(w)Jw - Jh_2(w) + G\alpha_2(w) = F_2(w)$$
$$G\beta_1(w)u + Dh_2(w)Gu = G_1(w)u \qquad (9)$$

We transform equation (8) using

$$y = w - h_3(w)$$
$$v = \alpha_r(w) + (I + \beta_{r-1}(w) \qquad (10)$$

where $h_3(w)$ is third order in w, and with the same procedure, equation (8) will be transformed to

$$\dot{y} = Jy + Gv \qquad (11)$$

with an error of order $r+1$. In order to eliminate all third order terms, we can choose $h_3(w)$, $\alpha_3(w)$ and $\beta_2(w)$ so that

$$Dh_3(w)Jw - Jh_3(w) + G\alpha_3(w) = F^1{}_3(w)$$
$$G\beta_2(w)u + Dh_3(w)Gu = G_2(w)u \qquad (12)$$

Equations (9) and (12) are special cases of a more general equations named homological equations

$$L_J h := Dh_r(y)Jy - Jh_r(y) + G\alpha_r(y) = F_r$$
$$G\beta_{r-1}(y)u + Dh_r(y)Gu = G_{r-1}(y) \qquad (13)$$

where h, α and β are unknown and F is the known vector field. This is a linear operator acting on a linear vector space [2].

To solve equation (13) uniquely, $(I + \beta(w))^{-1}$ must be exist, also the set of eigenvalues of J, $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ must be non-resonance, i.e. there is not relation of the form $\lambda_s = \sum_{i=1}^{n} m_i \lambda_i$ where $m_k \ge 0$ are integers and $\sum_{i=1}^{n} m_k \ge 2$.

The action $L_J(.)$ can be shown that is

$$L_J(h_k(w)) = Dh_k(w)Jw - Jh_k(w) + G\alpha_k(w)$$
$$= \left[\sum_{j=1}^{n} m_j\lambda_j - \lambda_i\right] h_k(w) + G\alpha_k(w) \qquad (14)$$

Thus in the case of non-resonance equation (13) can be solved uniquely and nonlinearities are eliminated by the nonlinear transformation. In the resonance case, all nonlinear terms except those terms associated with zero eigenvalues for $L_J(.)$, can be eliminated.

In the case of non-resonance we have

$$\dot{y} = Jy + Gv, \qquad (15)$$

where J is a diagonal matrix of system eigenvalues.

3. H_∞ Controller

Consider the system of equation (15). Suppose that we can recognize the slow and fast dynamics of system and decompose it into two subsystems as

$$\dot{X}_1 = \Lambda_{11} X_1 + B_1 u$$
$$\dot{X}_2 = \Lambda_{22} X_2 + B_2 u \qquad (16)$$

where y is a permutation of elements of X_1 (the slow dynamics of system) and of X_2 (the fast dynamics of system), and G is also a permutation of elements of B_1 and B_2, also for simplicity we replace v with u again (remain that in applying the designed controller on the main system, we must turn it in the main coordination) .

Now as mentioned earlier we consider the subsystem with high frequency dynamics as uncertainty Δ (figure 1). Suppose Δ, that shows the uncertain dynamics, is asymptotically stable and norm bounded, i.e. $\|\Delta\|_\infty \le \gamma_1 : \Delta(s) = (sI - \Lambda_2)^{-1} B_2$,

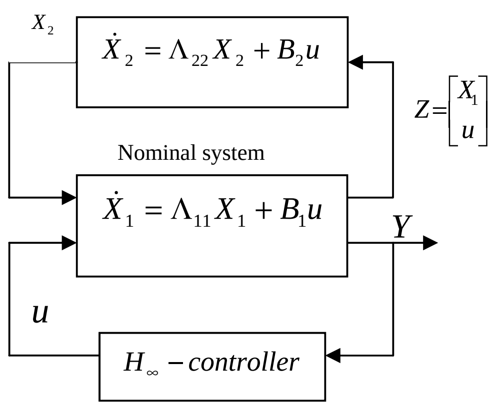

Figure 1. Block diagram of nominal system with uncertain dynamics

P_Δ is the generalized system and consists of Δ and the slow sub-system (nominal).

The H_∞ controller design problem for the system of figure 1, will be led to a H_∞ controller design for the nominal system that stabilizes it and (or) also perform another object such as reference tracking. To design this controller the only information about the fast subsystem needed is the H_∞ norm of it.

The nominal system can be shown as

$$P \sim \begin{bmatrix} \Lambda_{11} & 0 & B_1 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}, \qquad (17)$$

where the controlled output, Z is

$$Z = C_1 X_1 + D_{12} u := \begin{bmatrix} I \\ 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \qquad (18)$$

and the output is

$$Y = C_2 X_1 + D_{21} X_2 . \qquad (19)$$

Since X_2 that shows the fast modes of system and considered as the uncertain dynamics must be entered in the nominal system and can effect on it, we should apply a transformation $X = M\overline{X}$ to the system, then

$$\overline{X} = M^{-1} X, \quad M^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

and

$$\dot{\overline{X}} = \overline{\Lambda}\overline{X} + \overline{B} u \qquad (20)$$

where

$$\overline{\Lambda} = M^{-1} \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} M = \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} \\ 0 & \overline{\Lambda}_{22} \end{bmatrix},$$
$$\overline{B} = M^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}, \qquad (21)$$

then the new equation for the fast sub-system will be

$$\dot{\overline{X}}_2 = \overline{\Lambda}_{22} \overline{X}_2 + \overline{B}_2 u, \qquad (22)$$

The infinity norm of uncertainty block (fast sub-system) will be $\overline{\gamma}_1$; since

$$\overline{\Delta}(s) = M_{22}^{-1} (sI - \overline{\Lambda}_{22})^{-1} \overline{B}_2 =$$
$$M_{22}^{-1} (sI - M_{22} \Lambda_{22} M_{22}^{-1})^{-1} M_{22} B_2 = \Delta(s) \qquad (23)$$

thus $\overline{\gamma}_1 = \gamma_1$.

Also new equation for the nominal system will be

$$\overline{P} \sim \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} & \overline{B}_1 \\ \overline{C}_1 & D_{11} & D_{12} \\ \overline{C}_2 & \overline{D}_{21} & 0 \end{bmatrix}, \qquad (24)$$

$$Z = \overline{C}_1 \overline{X}_1 + D_{11} \overline{X}_2 + D_{12} u,$$
$$Y = \overline{C}_2 \overline{X}_1 + \overline{D}_{21} \overline{X}_2 , \qquad (25)$$

where

$$\overline{C}_1 = C_1 M_{11}^{-1}, \quad \overline{C}_2 = C_2 M_{11}^{-1},$$
$$D_{11} = -C_1 M_{11}^{-1} M_{12} M_{22}^{-1}, \overline{D}_{21} = D_{21} - C_2 M_{11}^{-1} M_{12} M_{22}^{-1} \qquad (26)$$

The H_∞ controller can be design via state feedback or output feedback. If we suppose that all states of the system are accessible, then we use state feedback. We must determine $\overline{\gamma}_2 = \min \gamma$, such that the eigenvalues of Hamiltonian matrix

$$H = \begin{bmatrix} \overline{\Lambda}_{11} & \gamma^{-2} \overline{\Lambda}_{12} \overline{\Lambda}_{12}^T - \overline{B}_1 \overline{B}_1^T \\ -\overline{C}_1^T \overline{C}_1 & -\overline{\Lambda}_{11}^T \end{bmatrix} \qquad (27)$$

don't place on the imaginary axis. Then we must find M, such that $\overline{\gamma}_1 . \overline{\gamma}_2 < 1$.

Theorem 1 [4]

Under assumption of stabilizability-detectability of reference [3], for a given $\gamma > 0$, there is an internal stabilizing controller such that $\|T_{zw}\|_\infty \le \gamma$, if and only if X_∞ is a p.s.d. solution of algebraic Riccati equation

$$\overline{\Lambda}_{11}^T X_\infty + X_\infty \overline{\Lambda}_{11} + X_\infty(\gamma^{-2}\overline{\Lambda}_{12}\overline{\Lambda}_{12}^T - \overline{B}_1\overline{B}_1^T)X_\infty + \overline{C}_1^T\overline{C}_1 = 0 \quad (28)$$

and the matrix $\overline{\Lambda}_{11} - (\overline{B}_1\overline{B}_1^T - \gamma^{-2}\Lambda_{12}\Lambda_{12}^T)X_\infty$ is a stability matrix. Then the related controller will be in the form

$$u(t) = -\overline{B}_1^T X_\infty \overline{X}_1(t) := K\overline{X}_1(t) \quad (29)$$

4. Simulation Results

Consider the equations of a flexible-joint robot manipulator [5] as follows

$$\dot{x} = f(x) + g(x)u$$

where $x = [x_{11}, x_{12}, x_{21}, x_{22}]^T$,

$$f(x) = \begin{bmatrix} x_{12} \\ \frac{-mgl}{I}\sin x_{11} - \frac{1}{I}x_{21} \\ \frac{1}{\varepsilon}x_{22} \\ \frac{-1}{\varepsilon}\frac{\alpha mgl}{I}\sin x_{11} + \frac{1}{\varepsilon}\frac{\alpha\beta}{J}x_{12} - \frac{\alpha(I+J)}{\varepsilon IJ}x_{21} - \frac{\beta}{J}x_{22} \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-\alpha}{\varepsilon J} \end{bmatrix}.$$

This system has an equilibrium point at the origin, also $f(0) = 0$. The first state shows the position and the second, the velocity of robot manipulator. The object is to design a controller that the position of manipulator reaches from zero state to zero.

Using the Taylor expansion about the equilibrium point we can write

$$\dot{x} = \frac{\partial f}{\partial x}(0)x + \tilde{f}(x) + g(x)u.$$

Using the transformation $x = Tw$ we transform $\frac{\partial f}{\partial x}(0)$ into Jordan canonical form

$$\dot{w} = Jw + F(w) + G(w)u,$$

$$J = diag\{-.1023 \pm 4.7767i, -.7727 \pm 44.8839i\},$$

$$G(w) = T^{-1}g(Tw) = [14.7478 \pm .2685i, -.1144 \pm 24.5273i]^T = G$$

Noting that the eigenvalues of J belong to two clusters (fast and slow), we can separate the whole system to two sub-system as follows,

The slow sub-system

$$\dot{w}_s = J_s w_s + F_s(w) + G_s u,$$
$$J_s = diag\{-.1023 \pm 4.7767i\},$$
$$F_s(w) = [F_{s1}(w), F_{s2}(w)]^T,$$
$$G_s = [G_{s1}, G_{s2}].$$

The fast sub-system

$$\dot{w}_f = J_f w_f + F_f(w) + G_f u,$$
$$J_f = diag\{-.7727 \pm 44.8839i\},$$
$$F_f(w) = [F_{f1}(w), F_{f2}(w)]^T,$$
$$G_f = [G_{f1}, G_{f2}].$$

The fast subsystem is asymptotically stable (noting its eigenvalues) and has bounded norm

$\|\Delta\|_\infty = 31.7253$.

Thus it can be considered as uncertainty block in figure 1.

The slow sub-system is

$$\dot{w}_s = J_s w_s + F_{3s}(w) + F_{5s}(w) + \ldots + G_s u,$$

where

$$F_{3s}(w) = \begin{bmatrix} \frac{-1}{6}(-11.64 - .26i)(x_{11})^3 \\ \frac{-1}{6}(-11.64 + .26i)(x_{11})^3 \end{bmatrix}_{x=Tw}$$

and $F_{5s}(w)$ contains the order of five terms, and so on.

Applying a coordinate transformation to eliminate the third order nonlinearities, we obtain

$$\dot{X}_f = J_f X_f + G_f u + O(|X|^5),$$
$$\dot{X}_s = J_s X_s + G_s u + O(|X|^5)$$

using transformation $X = M\overline{X}$, with

$$M_{11} = M_{22} = I, M_{12} = .0433, M_{22} = 0$$

and theorem 1, we apply H_∞ controller of equation (29) to the system, then the regulation of slow modes of system that represent the position and velocity of flexible joint robot manipulator is shown in figure 2. In figure 3, the behavior of fast dynamics that are modeled as uncertainty is shown. Figure 4 shows the controller output and also the disturbance attenuation.

5. Conclusions

In this paper the robust regulation of a class of nonlinear systems using singular perturbation approach is considered. First using the normal form theory the nonlinearity of system matrix is eliminated up to the desired degree. With the assumption about norm-boundedness of the fast dynamics and considering them as uncertainty, we have designed H_∞ controller for a reduced order system (slow subsystem) but the whole closed loop system will be stable.

6. References

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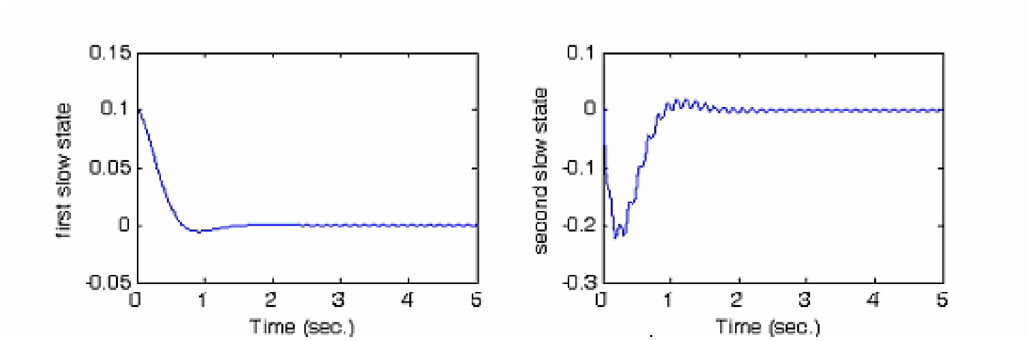

Figure 2. The dominant modes regulation

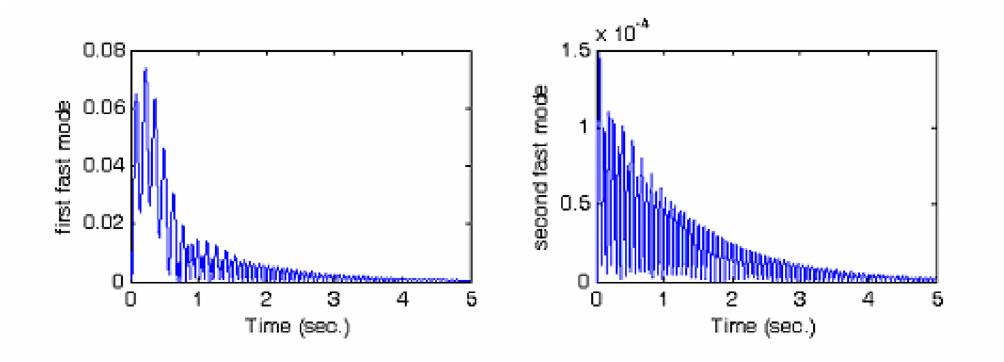

Figure 3. The absolute value of fast modes

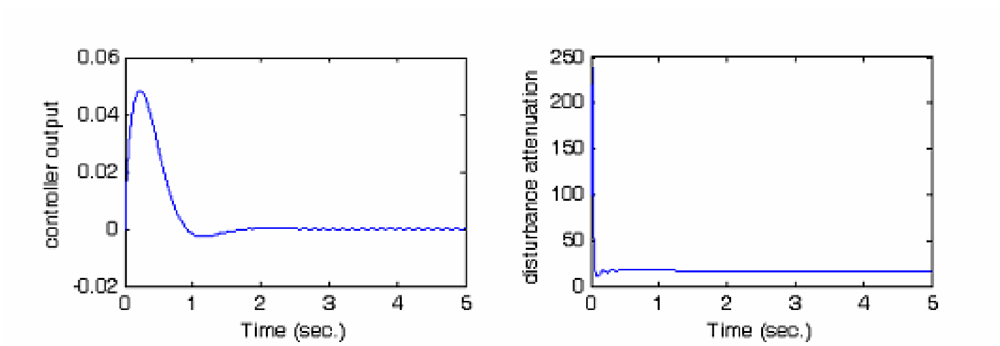

Figure 4. The controller output and the disturbance attenuation