STABILITY MARGIN VIA REFLECTION VECTORS

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Abstract

A new stability margin for discrete-time systems is proposed in the system characteristic polynomial coefficient space making use of, so-called, reflection vectors of monic Schur polynomials. Reflection vector margins give the distance to the stability boundary in directions of $2n$ reflection vectors of an n-th degree polynomial. The relations between the reflection vectors and its roots on the unit circle are obtained. An iterative procedure for stability radius determination is proposed.

1 Introduction

Some serious problems of so-called robust stability arise when the parameters of systems are not exactly known [1,3].That is why several stability margins are defined in different domains: gain and phase margin in frequency domain, minimal distance from imaginary axis (or unit circle) in pole domain, stability radius in system parameter domain.

For interval or polytopic type of parametric uncertainties some kind of stability margin can be obtained by the Kharitonov theorem [4] or edge theorem[2].

An alternative approach is to use the boundary crossing theorem to define the stability radius in polynomial coefficient space. You need to determine the distances to the real pole boundary and to complex poles boundary and select the minimal of them. The first task is simple but the second one is quite complicated for high order systems because the sweeping over the frequency range [1] or over the complex poles phase domain [3] is needed.

In this paper the reflection coefficient stability criteria [7] for discrete-time systems is used to define a Schur stability margin in polynomial coefficient space. The reflection vectors of an n-th order system will be introduced as $2n$ specific points on the stability boundary. The line segments between an arbitrary Schur polynomial (a point in coefficient space) and its reflection vectors will be Schur stable. So the minimal distance between a polynomial and its reflection vectors can be used as a stability margin for linear discrete-time systems. Starting from the crucial reflection vector by the use of line segments on the stability boundary the stability radius in the polynomial coefficients space can be found. By this procedure the nearest point on the stability boundary and the critical direction will be determined too.

The paper is organized as follows. In section 2 we recall the stability condition via reflection coefficients and introduce reflection vectors of a monic Schur polynomial. In section 3 the relations between the number (and sign) of the reflection vector and its roots will be studied. Section 4 is devoted to the the stability radius determination by an iterative procedure.

2 Reflection coefficients of Schur polynomials

A polynomial $a(z)$ of degree n with real coefficients $a_i \in \mathcal{R}$, $i = 0, ..., n$

$$a(z) = a_n z^n + ... + a_1 z + a_0$$

is said to be Schur if all its roots are placed inside the unit circle. A linear discrete-time dynamical system is stable if its characteristic polynomial is Schur, i.e. if all its poles lie inside the unit circle.

Besides the unit circle criterion some other criteria are known for checking the stability of a linear system. It is interesting to mention that the well-known Jury's stability test leads precisely to the stability hypercube of reflection coefficients . In the following we use the stability criterion via reflection coefficients.

Let us recall the recursive definition of reflection coefficients $k_i \in \mathcal{R}$ of a polynomial $a(z)$ [7]:

$$k_i = -a_i^{(i)}, \tag{1}$$

$$a_i^{(n)} = \frac{a_{n-i}}{a_n}, \qquad i = 1, ..., n; \tag{2}$$

$$a_j^{(i-1)} = \frac{a_j^{(i)} + k_i a_{i-j}^{(i)}}{1 - k_i^2}, \qquad j = 1, ..., i-1. \tag{3}$$

Reflection coefficients are well-known in signal processing and digital filters. The stability criterion via reflection coefficient is as follows [7].

Lemma 1. A polynomial $a(z)$ will be Schur if and only if its reflection coefficients $k_i, i = 1, ..., n$ lie within the interval $-1 < k_i < 1$.

A polynomial $a(z)$ lies on the stability boundary if some $k_i = \pm 1, i = 1, ..., n$. For monic Schur polynomials, $a_n = 1$, there is a one-to-one correspondence between the vectors $a = (a_0, ..., a_{n-1})^T$ and $k = (k_1, ..., k_n)^T$.

The transformation from reflection coefficients k_i to polynomial coefficients $a_{i-1}, i = 1, ..., n$ is multilinear. For monic polynomials we obtain from (1)-(3)

$$\begin{aligned} &a_i = a_{n-i}^{(n)}, \\ &a_i^{(i)} = -k_i, \\ &a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \qquad i = 1, ..., n; j = 1, ..., i-1. \end{aligned}$$

Lemma 2.[6] Through an arbitrary stable point $a = [a_0, a_1, ..., a_{n-1}]$ in polynomial coefficient space you can put n stable line segments

$$conv[a^i(\pm 1)] = \{a | k_i \in (-1, 1)\}$$

where $conv[a^i(\pm 1)]$ denotes the convex hull obtained by varying the reflection coefficient k_i between -1 and 1 , $i = 1, ..., n$.

Now let us introduce the reflection vectors of a monic polynomial $a(z)$. They will be useful for introducing a new stability margin in the polynomial coefficient space.

Definition. Let us call the vectors

$$a^i(1) = (a | k_i = 1), i = 1, ..., n$$

positive reflection vectors and

$$a^i(-1) = (a | k_i = -1), i = 1, ..., n$$

negative reflection vectors of a monic polynomial $a(z)$.

It means, reflection vectors are the extreme points of the Schur stable line segment $conv[a^i(\pm 1)]$ through the point a defined by Lemma 2. Due to the definition and the Lemmas 1 and 2 the following assertions hold:

1) every Schur polynomial has $2n$ reflection vectors $a^i(1)$ and $a^i(-1)$, $i = 1, ..., n$;

2) all the reflection vectors lie on the stability boundary ($k_i = \pm 1$);

3) the line segments between reflection vectors $a^i(1)$ and $a^i(-1)$ are Schur stable.

3 Roots of reflection vectors

In this section we study the reflection vectors placement on the stability boundary. It means that every reflection vector has one or more roots on the unit circle. The question is: how many roots and of what type (real or complex)? The following theorem gives the answer to these questions.

Theorem 1. Reflection vectors $a^i(\pm 1)$, $i = 1, ..., n$ of a monic Schur polynomial $a(z)$ have i roots on the unit circle. The type of roots $r(j)$, $j = 1, .., i$ is as follows:

1) the positive reflection vector $a^i(1)$ has

- for i even $r(1) = 1$,
  $r(2) = -1$
  and $(i-2)/2$ pairs of complex roots $r(j)$,
  $j = 3, .., i$ on the unit circle,
- for i odd $r(1) = 1$,
  and $(i-1)/2$ pairs of complex roots $r(j)$,
  $j = 2, .., i$ on the unit circle,

2) the negative reflection vector $a^i(-1)$ has

- for i even $i/2$ pairs of complex roots $r(j)$,
  $j = 1, .., i$ on the unit circle,
- for i odd $r(1) = -1$,
  and $(i-1)/2$ pairs of complex roots $r(j)$,
  $j = 2, .., i$ on the unit circle.

The proof is given in [6].

4 Stability radius via reflection vectors

Now we can introduce some kind of a stability margin via reflection vectors of a Schur polynomial.

Definition : Let us call the distance between a Schur stable polynomial $a(z)$ and its reflection vector $a^i(\pm 1)$, $i = 1, ..., n$ the stability margin in coefficient space in direction of i-th reflection vector or simply *i-th reflection vector margin* and denote it by $d_i(\pm 1)$.

Taking into account the background of reflection vectors (according to Theorem 1) we can claim that the most attractive reflection vectors are the first of them. Indeed:

- the first positive reflection vector margin $d_1(1)$ gives us the distance to the real positive root boundary,
- the first negative reflection vector margin $d_1(-1)$ gives us the distance to the real negative root boundary,
- the second negative reflection vector margin $d_2(-1)$ gives us the distance to the complex root boundary,
- the second positive reflection vector margin $d_2(1)$ gives us the distance to the two different real root boundary ($r_1 = 1, r_2 = -1$),
- the third positive reflection vector margin $d_3(1)$ gives us the distance to the real positive and complex root boundary ($r_1 = 1, r_{2,3} = \alpha \pm 1\beta\mathbf{i}, \alpha^2 + \beta^2 = 1$),
- etc.

As a matter of course the reflection vector margins do not give the minimal distances to real and complex root boundaries, i.e.

$$d_{min} \geq \rho,$$

$$d_1(1) \geq \rho_{+1},$$

$$d_1(-1) \geq \rho_{-1}$$

where ρ, ρ_{+1} and ρ_{-1} are the stability radius and the minimal distances to the positive and negative real root boundaries of $a(z)$. However, the minimal distances to real and complex root boundaries can be easily found by a simple search procedure in directions of reflection vectors.

1. For a given Schur polynomial $a(z)$ find the reflection vectors $a^1(1)$, $a^1(-1)$ and $a^2(-1)$.
2. Choose one of these reflection vectors as a starting point for iterative procedure $b(0) = a^{i^*}(j^*)$, $i^* \in \{1, 2\}$, $j^* \in \{-1, 1\}$.

3. Find the reflection vectors $b^i(l)(\pm 1)$, $i = 1, ..., n$; $i \neq i^*$.

4. Put $n - 1$ line segments $B^i(l) = conv\{[b(l)]^i(\pm 1)\}$, $i = 1, ..., n$; $i \neq i^*$ through the point $b(l)$. All the line segments $B^i(l)$ lie on the stability boundary.

5. Find $b(l+1)$ as the nearest point of all line segments $B^i(l)$, $i = 1, ..., n$; $i \neq i^*$ to the point a.

6. If $|b(l+1) - b(l)| > \epsilon$ for some given small $\epsilon > 0$ return to step 4.

7. If $|b(l+1) - b(l)| \leq \epsilon$ put $\rho_{i^*}(j^*) = |a - b(l)|$ and return to step 2.

8. The stability radius of the point a is

$$\rho = min_{i,j} \rho_{i^*}(j^*)$$

and the nearest point on the stability boundary is $b^{i^*}(j^*)(l)$.

Remark : The above procedure gives an alternative way for the stability radius determination. The convergence rate is not high because the search directions are determined by reflection vectors and the stability region is approximated by straight lines. But in addition to the stability radius we get by this procedure some useful information about the stability region (distances to the stability boundary in several directions, many points and line segments on the stability boundary). This information can be used for robust controller design via pole placement.

Example 1: Let $n = 2$. Then the stability region in the polynomial coefficient space $a = (a_1, a_0)$ is the triangle AGH (Fig.1). Let us find the stability margins for the polynomial $a(z) = z^2 + 0.75z + 0.5$ (point F in Fig.1). According to Lemma 2 we can put 2 stable line segments through the point F. By varying the first reflection coefficient k_1, $-1 < k_1 < 1$, we get the line segment AB and by varying the second reflection coefficient k_2, $-1 < k_2 < 1$, we get the line segment CD. By definition the second order polynomial $a(z)$ has 4 reflection vectors :

$$\begin{array}{lcl} a^1(1) & = & [\ -1.5 \quad 0.5\ ], \quad (point\ \ C), \\ a^1(-1) & = & [\ 1.5 \quad 0.5\ ], \quad (point\ \ D), \\ a^2(1) & = & [\ 0 \quad -1\ ], \quad (point\ \ A), \\ a^2(-1) & = & [\ 1 \quad 1\ ], \quad (point\ \ B) \end{array}$$

and the stability margins in the directions of reflection vectors are determined by the line segments FC , FD , FA and FB respectively.

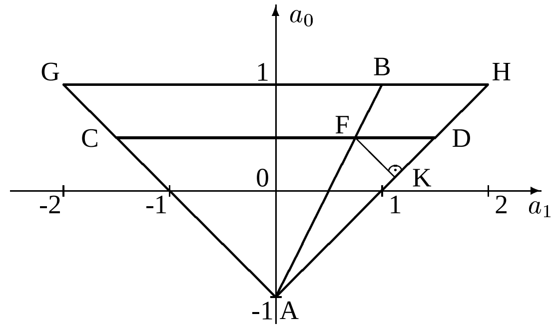

Fig.1 Stability region and stability margins in directions of reflection vectors ($n = 2$)

To find the minimal distance to the negative real root boundary we start from the first negative reflection vector (point D), $b(0) = [1.5 \quad 0.5]$. By varying the second reflection coefficient k_2, $-1 < k_2 < 1$, we get the line segment AH. The point $K = [1.125 \quad 0.125]$ with reflection coefficients $k^K = [-1 \quad -0.125]$ is the nearest point on the negative real root boundary and the distance to the negative real root boundary is $\rho_- = 0.53$.

Similarly, starting from the first positive reflection vector (pointC) we can find the minimal distance to the positive real root boundary (line segment AG) $\rho_+ = 1.591$.

Starting from the second negative reflection vector (point B) we get the minimal distance to the complex root boundary (line segment GH) $\rho_c = 0.5$. So the stability radius is

$$\rho = min(\rho_+, \rho_-, \rho_c) = 0.5.$$

Example 2: Let us now consider the example of Bhattacharyya [3, pp.136-138] for $n = 4$

$$a(z) = z^4 + 0.3z^3 + 0.4z^2 + 0.2z + 0.1.$$

The reflection coefficients of $a(z)$ are

$$k^a = [-0.1714 - 0.3246 - 0.1717 - 0.1].$$

Because $|k_i^a| < 1$, $i = 1, ..., 4$, $a(z)$ is a Schur polynomial and we can find its reflection vectors $a^i(\pm 1)$ and reflection vector margins $d_i(\pm 1)$ as follows:

$$\begin{array}{lcl} a^1(1) & = & [\ -1.2516 \quad 0.1069 \quad 0.0448 \quad 0.1\ ], \\ a^1(-1) & = & [\ 1.3974 \quad 0.6073 \quad 0.3097 \quad 0.1\ ], \\ a^2(1) & = & [\ -1.1545 \quad -1.0999 \quad 0.1545 \quad 0.1\ ], \\ a^2(-1) & = & [\ 0.5317 \quad 1.1646 \quad 0.2232 \quad 0.1\ ], \\ a^3(1) & = & [\ -0.1975 \quad 0.1073 \quad -1.0097 \quad 0.1\ ], \\ a^3(-1) & = & [\ 0.6517 \quad 0.6069 \quad 1.0551 \quad 0.1\ ], \\ a^4(1) & = & [\ 0.1111 \quad 0 \quad -0.1111 \quad -1\ ], \\ a^4(-1) & = & [\ 0.4545 \quad 0.7272 \quad 0.4545 \quad 1\ ]. \end{array}$$

$$\begin{aligned}
d_1(1) &= 1.5866,\\
d_1(-1) &= 1.1222,\\
d_2(1) &= 1.5679,\\
d_2(-1) &= 0.7993,\\
d_3(1) &= 1.3403,\\
d_3(-1) &= 0.9474,\\
d_4(1) &= 1.2256,\\
d_4(-1) &= 1.0028.
\end{aligned}$$

Starting from the reflection vectors $a^1(1)$, $a^1(-1)$ and $a^2(-1)$ the following minimal distances to real positive, real negative and complex pole boundary have been found after 5 iterations $\rho_{+1} = 1.0$, $\rho_{-1} = 0.5$, $\rho_c = 0.4987$. It confirms the result given in [3]. The stability radius is

$$\rho = 0.4987$$

and the critical point b on the stability boundary is

$$b = [\ 0.2335 \quad 0.746 \quad 0.2818 \quad -0.2434\ ]\ .$$

The reflection coefficients of $b(z)$ are $k^b = [\ 0.0194 \quad -1.0 \quad -0.36 \quad 0.2434\ ]$. By Theorem 1 $b(z)$ has a pair of complex roots on the unit circle. Indeed, the roots of $b(z)$ are

$$\begin{aligned}
r_1 &= 0.3756,\\
r_2 &= -0.648,\\
r_{3,4} &= 0.0194 \pm 0.9998\mathbf{i}.
\end{aligned}$$

5 Conclusions

A new kind of stability margin for discrete-time systems is proposed in the system characteristic polynomial coefficient space making use of, so-called, reflection vectors of monic Schur polynomials. It is shown, first, that reflection vectors are placed on the stability boundary with specific roots placement depending on the reflection vector number and the argument sign and, second, that the line segments between an arbitrary Schur polynomial and its reflection vectors are Schur stable.

Even though the reflection vector margins do not give the minimal distance to the stability boundary nevertheless they are quite informative: in addition to distances they give also the directions of crucial points. An iterative procedure is given for stability radius determination via reflection vectors.

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