Validation of Closed-Loop Behaviour From Noisy Frequency Response Measurements

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Abstract

It is shown how noisy closed-loop frequency response measurements may be used to obtain pointwise in frequency bounds on the possible difference between an unknown closed-loop system and a nominal model of the closed-loop. To this end, the ν-gap metric framework for robustness analysis plays a central role.

Keywords: ν-gap metric, robust performance, controller validation

1 Preliminaries and Notation

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, $\mathbb{C}^n$ the space of $n \times 1$ complex vectors and $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ the open disc of radius $\rho > 0$. The symbol $\overline{\mathbb{D}}_\rho$ is used to denote the closure of $\mathbb{D}_\rho$ and for convenience, the sets $\mathbb{D}_1$ and $\overline{\mathbb{D}}_1$ are denoted by $\mathbb{D}$ and $\overline{\mathbb{D}}$, respectively. Given $\rho \geq 1$, let $\mathcal{H}_{\infty,\rho} := \{f : \mathbb{C} \mapsto \mathbb{C} \mid f \text{ is analytic in } D_\rho \text{ and } \|f\|_{\infty,\rho} := \sup_{z \in D_\rho} |f(z)| < \infty\}$ and for convenience, denote $\mathcal{H}_{\infty,1}$ and $\|f\|_{\infty,1}$ by $\mathcal{H}_\infty$ and $\|f\|_\infty$ respectively. The ball of radius $\gamma > 0$ in $\mathcal{H}_{\infty,\rho}$ is denoted by $\overline{\mathcal{B}}\mathcal{H}_{\infty,\rho}(\gamma) := \{ f \mid f \text{ is analytic in } D_\rho \text{ and } \|f\|_{\infty,\rho} := \sup_{z \in D_\rho} |f(z)| \leq \gamma\}$. Given an $f \in \overline{\mathcal{B}}\mathcal{H}_{\infty,\rho}(\gamma)$ it can be shown that each term f_k of the impulse response of the system corresponding to multiplication by the frequency domain symbol f, is bounded as $|f_k| \leq \gamma\rho^{-k}$. Given a matrix Q, the notation Q^T, Q^* and $\bar\sigma(Q)$ is used to represent the transpose, complex conjugate transpose and maximum singular value of Q, respectively. Finally, diag (x_i) denotes a diagonal matrix with x_i $(i = 1, 2, \ldots, n)$ along its diagonal.

2 Introduction

As many modern techniques for control system design are model based, it is of practical interest to know in what sense a system model should be accurate. Indeed, significant research effort has been devoted to answering such questions over the last few decades. Within the context of feedback compensator design, the gap and ν-gap metric frameworks for robustness analysis [1, 2] are particularly useful. In fact, these metrics induce the coarsest topology with respect to which both feedback stability and closed-loop performance are robust properties. This is established within a general linear setting in [3], using the following inequalities:

> Given linear systems P_1, P_2 and C such that the standard feedback configurations $[P_1, C]$ and $[P_2, C]$ are both stable, let
>
> $$H(P_i, C) := \begin{bmatrix} (I - CP_i)^{-1} & -C(I - P_iC)^{-1} \\ P_i(I - CP_i)^{-1} & -P_iC(I - P_iC)^{-1} \end{bmatrix}.$$
>
> Then
>
> $$\begin{aligned} \text{gap}(P_1, P_2) &\leq \|H(P_1, C) - H(P_2, C)\| \\ &\leq \|H(P_1, C)\| \, \|H(P_2, C)\| \, \text{gap}(P_1, P_2), \end{aligned} \quad (1)$$
>
> where gap(P_1, P_2) denotes the gap metric distance between P_1 and P_2, and $\|\cdot\|$ denotes the ℓ_2 induced norm.

For linear time-invariant (LTI) systems the bounds in (1) hold pointwise in frequency $\varphi := e^{j\omega}$, with gap(P_1, P_2) replaced by the chordal distance $\kappa(P_1(\varphi), P_2(\varphi))$ between the stereographic projection of the frequency responses $P_i(\varphi)$ onto the Riemann sphere [2, 4] – i.e.

$$\begin{aligned} \kappa(P_1(\varphi), P_2(\varphi)) &\leq \bar\sigma\left(H(P_1(\varphi), C(\varphi)) - H(P_2(\varphi), C(\varphi))\right) \\ &\leq \frac{\kappa(P_1(\varphi), P_2(\varphi))}{\rho(P_1(\varphi), C(\varphi)) \cdot \rho(P_2(\varphi), C(\varphi))}, \end{aligned} \quad (2)$$

where $\rho(P_i(\varphi), C(\varphi)) := 1/\bar\sigma\left(H(P_i(\varphi), C(\varphi))\right) \leq 1$. Furthermore,

$$\begin{aligned} \arcsin \rho(P_2(\varphi), C(\varphi)) &\geq \arcsin \rho(P_1(\varphi), C(\varphi)) \\ &- \arcsin \kappa(P_1(\varphi), P_2(\varphi)) \end{aligned} \quad (3)$$

for all $\varphi = e^{j\omega}$ and $\omega \in [0, 2\pi)$. Also note that $\sup_{\omega \in [0,2\pi)} \rho(P_i(e^{j\omega}), C(e^{j\omega})) =: b(P_i, C)$ is the generic performance measure employed in the $\mathcal{H}_\infty$ loop-shaping paradigm for design [5, 4]. The bounds in (1), (2) and (3) clearly indicate that gap-like metrics capture the important difference between open-loop systems from the perspective of closed-loop behaviour.

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Given a nominal model P_m of a true plant P_t, suppose that a feedback compensator C is known to stabilise both P_m and P_t. In addition to this, a handle on the actual behaviour of P_t in closed-loop with C is typically of interest. To this end, two approaches could be taken: (i) One could try to identify $H(P_t, C)$ at frequencies of interest from closed-loop measurements; or (ii) since the nominal closed-loop $[P_m, C]$ is known, one could try to determine $\kappa(P_m(\varphi), P_t(\varphi))$ at frequencies of interest and then use the bounds in (2) and (3). The problem with approach (i) is that any sensible technique for identifying $H(P_t, C)$ should involve constraints to reflect the relationships between the blockwise elements of $H(P_t, C)$; for example, the quotient of the 21-block and the 11-block, which is known *a priori* to be C. Such constraints are difficult to deal with numerically. Furthermore, identifying $H(P_t, C)$ only yields information pertinent to closed-loop behaviour of P_t with the particular controller C. Approach (ii) on the other hand, can be handled numerically (as will be shown shortly) and moreover, if $\kappa(P_m(\varphi), P_t(\varphi))$ were determined to be large at a particular frequency, the following conclusion could be made: For *any* controller C_1 that stabilises both the true plant and model, the closed-loop $[P_t, C_1]$ would differ significantly from the nominal $[P_m, C_1]$. Such a situation would suggest that a better nominal model of the plant may be required for model-based feedback compensator design. Motivated by all this, the remainder of this paper is dedicated to outlining a numerical technique for determining a sensible estimate of $\kappa(P_m(\varphi), P_t(\varphi))$ from noisy closed-loop frequency response measurements. Work that is related in terms of assessing closed-loop performance from measured-data/identified-sets, but distinct in terms of the approach taken, can be found in [6, 7] and the references therein.

3 Determining $\kappa(P_m(\varphi), P_t(\varphi))$

For the sake of notational simplicity, the SISO case is discussed here. The MIMO case follows similarly with appropriate notational modifications.

For the problem introduced above the *a priori* information is a model P_m of an unknown true system P_t, and a controller C which stabilises both P_m and P_t. Since it is assumed that C stabilises P_t, frequency response samples of

$$X_t = \begin{bmatrix} I \\ P_t \end{bmatrix} (I - CP_t)^{-1}$$

can be measured at any frequencies of interest. Techniques for achieving this are discussed in [8, 9]. Note that, unless C is itself stable, X_t is not necessarily a coprime factorisation of P_t over $\mathcal{H}_\infty$. However, at any frequency ω_i that does not correspond to a pole of C on the unit circle, X_t is left-invertible by $[\, I \ -C(e^{j\omega_i}) \,]$ and hence, the range of $X_t(e^{j\omega_i})$ is the graph of $P_t(e^{j\omega_i})$. Correspondingly, at any such frequency ω_i, the chordal distance

$$\kappa(P_m(e^{j\omega_i}), P_t(e^{j\omega_i})) = \inf_{Q \in \mathbb{C}} \overline{\sigma}(G_m(e^{j\omega_i}) - X_t(e^{j\omega_i})Q), \quad (4)$$

where $G_m(e^{j\omega_i})$ denotes the value of any *normalised* right graph symbol for P_m at the frequency ω_i. Such a graph symbol can be constructed from any normalised right coprime factorisation $P_m = N_m D_m^{-1}$ as follows: $G_m = \left[\begin{smallmatrix} D_m \\ N_m \end{smallmatrix}\right]$. See [4] for further details.

Now, the *a posteriori* information is a vector of (not necessarily uniformly spaced) noisy frequency response samples

$$\overline{X} = \left[X_1 \, X_2 \, \ldots \, X_n\right]^T,$$

where $X_i = X_t(e^{j\omega_i}) + v_i$, $X_t \in \overline{B}\mathcal{H}_{\infty,\rho}(\gamma)$, $\omega_i \in [0, \pi)$ and $\|v_i\| \leq \epsilon$ for $i = 1, 2, \ldots, n$ and some specified ϵ, ρ and γ. Note that the measured data is to be explained in terms of two components – noise v_i and true system behaviour X_t. The value ϵ bounds the level of data one is prepared to attribute to noise. Since parameters $\rho > 1$ and $\gamma > 0$ such that $X_t \in \overline{B}\mathcal{H}_{\infty,\rho}(\gamma)$ can be determined from additional measured data,[1] it is also sensible to constrain the partitioning of data into noise and true system behaviour in these terms. In light of this, and bearing in mind the objective of estimating $\kappa(P_m(\varphi), P_t(\varphi))$, consider the following constrained optimisation problem:

$$\begin{aligned} &\min_{\hat{X}_t} \max_i \left(\inf_{Q_i \in \mathbb{C}} \overline{\sigma}(G_m(e^{j\omega_i}) - \hat{X}_t(e^{j\omega_i})Q_i) \right) \\ &= \min_{\hat{X}_t} \max_i \kappa(P_m(e^{j\omega_i}), \mathrm{Quot}(\hat{X}_t(e^{j\omega_i}))\,) \end{aligned} \quad (5)$$

subject to

$$\hat{X}_t(e^{j\omega_i}) = X_i - v_i, \quad \hat{X}_t \in \overline{B}\mathcal{H}_{\infty,\rho}(\gamma) \quad \text{and} \quad \|v_i\| \leq \epsilon, \quad (6)$$

where $\mathrm{Quot}(\left[\begin{smallmatrix} X_D \\ X_N \end{smallmatrix}\right]) := X_D X_N^{-1}$ and v_i are the decision variables in the optimisation. The purpose and the result of this optimisation may be explained as follows. Let λ be the minimum achieved by solving the above problem (assuming it exists and is unique). Then there exists a system $\hat{X}_t \in \overline{B}\mathcal{H}_{\infty,\rho}(\gamma)$ and bounded noise terms v_i defined pointwise in frequency with $\|v_i\| \leq \epsilon$, $i = 1, 2, \ldots, n$ such that the measured data can be interpolated as

$$X_i = \hat{X}_t(e^{j\omega_i}) + v_i$$

and

$$\max_i \kappa(P_m(e^{j\omega_i}), \mathrm{Quot}(\hat{X}_t(e^{j\omega_i}))\,) \leq \lambda$$

holds. Put another way, there is no system consistent with the *a priori* assumptions (in terms of ϵ, γ, ρ) and with the *a posteriori* data (in terms of $\overline{X}$) whose worst case chordal distance over $\{\omega_i\}$ is *better* than λ.

An approach to solving the optimisation problem along these lines is outlined below. Note that since the problem

[1] Recall that the k-th term of the impulse response of a function in $\overline{B}\mathcal{H}_{\infty,\rho}(\gamma)$ is bounded by $\gamma\rho^{-k}$.

is not simultaneously convex in the v_i's and Q_i's, an iterative approach is taken. The algorithm described here is closely related to an iterative identification algorithm proposed in [10] and also to Pick interpolation based worst case identification algorithms in [11] and [12].

Partitioning each $X_i = \left[\begin{smallmatrix} x_{1,i} \\ x_{2,i} \end{smallmatrix}\right]$ and $v_i = \left[\begin{smallmatrix} v_{1,i} \\ v_{2,i} \end{smallmatrix}\right]$:

1. Set $k = 1$ and $Q_i^{\star,k-1} = (X_m^*(e^{j\omega_i})X_m(e^{j\omega_i}))^{-\frac{1}{2}}$ for each $i = 1, 2, \ldots, n$, where $X_m := \left[\begin{smallmatrix} I \\ P_m \end{smallmatrix}\right](I - CP_m)^{-1}$ – this initial value for each $Q_{i,k-1}^\star$ is taken because in the case that P_t were actually P_m, it would make the argument of the infimum in (4) equal to zero.

2. Solve
$$\min_{\substack{v_{1,1},v_{1,2},\ldots,v_{1,n}\in\mathbb{C} \\ v_{2,1},v_{2,2},\ldots,v_{2,n}\in\mathbb{C}}} \lambda \tag{7}$$
subject to the affine matrix inequality constraints
$$\overline{\sigma}\left(G_m(e^{j\omega_i}) - \left(X_i - \begin{bmatrix} v_{1,i} \\ v_{2,i} \end{bmatrix}\right) Q_i^{\star,k-1}\right) \le \lambda, \tag{8}$$
$$\operatorname{diag}\left(\begin{bmatrix} \epsilon & \begin{bmatrix} v_{1,i} \\ v_{2,i} \end{bmatrix} \\ \begin{bmatrix} v_{1,i} \\ v_{2,i} \end{bmatrix} & \epsilon \end{bmatrix}\right) \ge 0, \tag{9}$$
$$\operatorname{diag}\left(\begin{bmatrix} 1 & \frac{x_{1,i}^*-v_{1,i}^*}{\gamma} \\ \frac{x_{1,i}-v_{1,i}}{\gamma} & 1 \end{bmatrix}\right) \ge 0, \tag{10}$$
$$\begin{bmatrix} E^{-1} & \operatorname{diag}\left(\frac{x_{1,i}^*-v_{1,i}^*}{\gamma}\right) \\ \operatorname{diag}\left(\frac{x_{1,i}-v_{1,i}}{\gamma}\right) & E \end{bmatrix} \ge 0, \tag{11}$$
$$\operatorname{diag}\left(\begin{bmatrix} 1 & \frac{x_{2,i}^*-v_{2,i}^*}{\gamma} \\ \frac{x_{2,i}-v_{2,i}}{\gamma} & 1 \end{bmatrix}\right) \ge 0, \tag{12}$$
$$\begin{bmatrix} E^{-1} & \operatorname{diag}\left(\frac{x_{2,i}^*-v_{2,i}^*}{\gamma}\right) \\ \operatorname{diag}\left(\frac{x_{2,i}-v_{2,i}}{\gamma}\right) & E \end{bmatrix} \ge 0, \tag{13}$$
where
$$E = \left[\frac{1}{1 - \frac{e^{j(\omega_i-\omega_j)}}{\rho^2}}\right] \quad \text{for } i, j = 1, 2, \ldots, n,$$
and denote by $\lambda_{k,k-1}^\star$ the minimum cost and by $v_{1,i}^{\star,k}$ and $v_{2,i}^{\star,k}$ for each $i = 1, 2, \ldots, n$ the values for each v_i at which this is achieved;

3. Given $v_{1,i}^{\star,k}$ and $v_{2,i}^{\star,k}$, solve the linear least squares problem
$$\min_{Q_i\in\mathbb{C}} \max_i \overline{\sigma}\left(G_m(e^{j\omega_i}) - \left(X_i - \begin{bmatrix} v_{1,i}^{\star,k} \\ v_{2,i}^{\star,k} \end{bmatrix}\right) Q_i\right)$$
at each frequency ω_i, denoting by $\lambda_{k,k}^\star$ the minimum cost and by $Q_i^{\star,k}$ the value Q_i at which this is achieved;

4. If $|\lambda_{k,k}^\star - \lambda_{k-1,k-1}^\star|$ is less than some desired tolerance then stop otherwise set $k = k + 1$ and go back to step 2.

By virtue of Pick's interpolation theorem the constraints (10–13) in Step 2 above ensure the existence of analytic interpolants $f_1, f_2 : \mathbb{D} \mapsto \overline{\mathbb{D}}$ such that
$$f_1(\frac{e^{j\omega_i}}{\rho}) = \frac{x_{1,i} + v_{1,i}^{\star,k}}{\gamma} \quad \text{and} \quad f_2(\frac{e^{j\omega_i}}{\rho}) = \frac{x_{2,i} + v_{2,i}^{\star,k}}{\gamma}.$$
Correspondingly, $\hat{X}_t = \begin{bmatrix} \gamma f_1(\frac{z}{\rho}) \\ \gamma f_2(\frac{z}{\rho}) \end{bmatrix} \in \overline{B}\mathcal{H}_{\infty,\rho}(\gamma)$ interpolates each $X_i - \begin{bmatrix} v_{1,i}^{\star,k} \\ v_{2,i}^{\star,k} \end{bmatrix}$, as required. An attractive property of the above procedure is that its cost is always non-increasing.

Lemma 1 *For $k \ge 1$,*
$$\lambda_{k,k}^\star \le \lambda_{k,k-1}^\star \le \lambda_{k-1,k-1}^\star$$

Proof : The proof follows from definition of $\lambda_{k,k-1}$ and $\lambda_{k,k}$ in Steps 2 and 3 of the above procedure and the fact that $v_{1,i}^{\star,k}$ and $v_{2,i}^{\star,k}$ are feasible solutions for the $(k + 1)$-th iteration. ∎

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