OPERATOR FUNCTIONS IN THEORY OF NONLINEAR CONTINUOUS, DISCRETE AND RETARDED CONTROL SYSTEMS

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1 Introduction

In this paper we suggest a brief survey of some results on stability of continuous, discrete and retarded control systems which are based on recent estimates for the norms of operator-valued functions.

As it is well-known, one of the basic methods for investigation of solution stability is the Lyapunov functions (functionals) method [1, 6, 7]. By that method, many strong results are obtained. But finding Lyapunov functions is usually difficult. At the same time, by the combined usage estimates for norms of operator-valued functions with the method of linearization and the freezing method we establish explicit stability criteria. They make it possible to avoid the construction of Lyapunov's functions in appropriate situations. Some of the results presented below are new, and some of them are taken from [3, 5]. Moreover, as it is shown in [4], our results can be extended to distributed parameters systems.

2 Estimates for the norm of matrix functions

Throughout Sections 2-5 of the present paper, $\|.\|$ *means the Euclidean norm.*

Let $A = (a_{jk})$ be an $n \times n$-matrix (a linear operator in $\mathbf{C}^n$) and I be the unit matrix,.

The following quantity plays a key role in the sequel:

$$g(A) = (N^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}, \tag{2.1}$$

where $N(A)$ is the Frobenius (Hilbert-Schmidt) norm of A, and $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues taken with their multiplicities. The relations

$$g^2(A) \leq N^2(A) - |Trace\ A^2|,$$

$$g^2(A) \le \frac{1}{2} N^2(A^* - A)$$

and

$$g(e^{i\tau} A + zI) = g(A)$$

are true for all $\tau \in \mathbf{R}$ and $z \in \mathbf{C}$. To formulate the results, for a natural $n > 1$ introduce the numbers

$$\gamma_{n,k} = \sqrt{\frac{C^k_{n-1}}{(n-1)^k}} \ \ (k = 1, ..., n-1) \text{ and } \gamma_{n,0} = 1.$$

Here

$$C^k_{n-1} = \frac{(n-1)!}{(n-k-1)!k!}$$

are binomial coefficients. Evidently, for $n > 2$

$$\gamma^2_{n,k} = \frac{(n-2)(n-3)\dots(n-k)}{(n-1)^{k-1}k!} \le \frac{1}{k!} \ \ (k = 1, 2, ..., n-1). \tag{2.2}$$

Let A be a matrix and let $f(\lambda)$ be a scalar-valued function which is analytical on a neighborhood D of all the eigenvalues of A. We define the function $f(A)$ of A by

$$f(A) = \frac{1}{2\pi i} \int_L f(\lambda)(\lambda I - A)^{-1} d\lambda,$$

where $L \subset D$ is a closed smooth contour surrounding $\sigma(A)$

Theorem 2.1 *Let A be a linear operator in $\mathbf{C}^n$ and let f be a function regular on a neighborhood of the closed convex hull $co(A)$ of the eigenvalues of A. Then*

$$\|f(A)\| \le \sum_{k=0}^{n-1} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)| g^k(A) \frac{\gamma_{n,k}}{k!}.$$

For the proof see [2, 3]. This theorem and inequalities (2.2) yield

Corollary 2.2 *Let A be a linear operator in $\mathbf{C}^n$ and let f be a function regular on a neighborhood of the closed convex hull $co(A)$ of the eigenvalues of A. Then*

$$\|f(A)\| \le \sum_{k=0}^{n-1} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)| \frac{g^k(A)}{(k!)^{3/2}}.$$

Theorem 2.3 *Let A be a linear operator in $\mathbf{C}^n$. Then its resolvent $R_\lambda(A) = (A - \lambda I)^{-1}$ satisfies the inequality*

$$\|R_\lambda(A)\| \le \sum_{k=0}^{n-1} \frac{g^k(A)\gamma_{n,k}}{\rho^{k+1}(A, \lambda)} \text{ for all regular points } \lambda \text{ of } A,$$

where

$$\rho(A, \lambda) = \min_{k=1,\dots,n} |\lambda - \lambda_k(A)|.$$

For the proof see [2, 3]. The latter theorem and inequalities (2.2) yield

Corollary 2.4 *Let A be a linear operator in $\mathbf{C}^n$. Then*

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \text{ for all regular points } \lambda \text{ of } A.$$

Example 2.5 *Let A be a linear operator in $\mathbf{C}^n$. Then*

$$\|A^m\| \leq \sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A) \gamma_{n,k}}{(m-k)!k!}$$

for every integer m. Here $r_s(A) = \max_k |\lambda_k(A)|$ is the spectral radius of A. Corollary 2.2 gives us the inequality

$$\|A^m\| \leq \sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}}.$$

Example 2.6 *For a linear operator A in $\mathbf{C}^n$, Theorem 2.1 gives us the estimate*

$$\|exp(At)\| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} g^k(A) t^k \frac{\gamma_{n,k}}{k!} \quad (t \geq 0)$$

where $\alpha(A) = \max_{k=1,\ldots,n} Re\ \lambda_k(A)$. According to (2.2)

$$\|exp(At)\| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{g^k(A) t^k}{(k!)^{3/2}} \text{ for all } t \geq 0.$$

3 Nonlinear continuous systems with autonomous linear parts

Put $\Omega_r = \{h \in \mathbf{C}^n : \|h\| \leq r\}$ and consider in $\mathbf{C}^n$ the equation

$$\dot{x} = Ax + F(x,t) \quad (t \geq 0), \tag{3.1}$$

where A is a constant $n \times n$-matrix, and F maps $\Omega_r \times [0,\infty)$ into $\mathbf{C}^n$ with the property

$$\|F(h,t)\| \leq \nu \|h\| \text{ for all } h \in \Omega_r \text{ and } t \geq 0. \tag{3.2}$$

Here $\nu = const > 0$. Introduce the algebraic equation

$$z^n = \nu \sum_{j=0}^{n-1} \frac{g^j(A)}{\sqrt{j!}} z^{n-j-1}, \tag{3.3}$$

and denote by $z(\nu, A)$ the extreme right-hand (unique positive and simple) zero of that equation. Let

$$\Gamma(A) = \sum_{j=0}^{n-1} \frac{g^j(A)}{|\alpha(A)|^{j+1}\sqrt{j!}},$$

and

$$\chi(A) = \max_{t \geq 0} exp[\alpha(A)t] \sum_{j=0}^{n-1} \frac{g^j(A) t^j}{\sqrt{j!}}.$$

Theorem 3.1 *Under condition (3.2), let the matrix $A + z(\nu, A)I$ be a Hurwitz one. Then the zero solution of equation (2.1) is asymptotically stable. Moreover, the inequality*

$$\nu\Gamma(A) < 1 \tag{3.4}$$

is valid, and any vector x_0 satisfying the condition

$$(1 - \nu\Gamma(A))^{-1}\, \chi(A)\|x_0\| < r,$$

belongs to a region of attraction of the zero solution. Additionally, the solution $x(t)$ of (2.1) with $x(0) = x_0$ subordinates the estimate

$$\|x(t)\| \le \frac{\chi(A)\|x_0\|}{1 - \nu\Gamma(A)}\ (t \ge 0).$$

In particular, let (3.2) hold with $\Omega_r = \mathbf{C}^n$ (i.e. $r = \infty$). Then the zero solution of (3.1) is globally stable provided $A + z(\nu, A)I$ is a Hurwitz matrix.

For the proof see [3].

Certainly, we can use various estimates for the algebraic root $z(\nu, A)$, cf. Lemma 1.6.1 or Corollary 1.6.2 from [3].

4 Stability of continuous systems with time-variant linear parts

Let us consider in $\mathbf{C}^n$ the equation

$$\dot{x} = A(t)x + F(x, t)\ (t \ge 0), \tag{4.1}$$

where $A(t)$ is a piecewise continuous $n \times n$-matrix, and F maps $\Omega_r \times [0, \infty)$ into $\mathbf{C}^n$ with the following property: there exists a non-negative continuous function $\nu(t)$ bounded on $[0, \infty)$, such that

$$\|F(h, t)\| \le \nu(t)\|h\| \text{ for all } h \in \Omega_r \text{ and } t \ge 0. \tag{4.2}$$

Recall that $\Omega_r = \{h \in \mathbf{C}^n : \|h\| \le r\}$, and for any $n \times n$-matrix A denote

$$p(t, A) \equiv exp[\alpha(A)t] \sum_{k=0}^{n-1} \frac{g^k(A)t^k}{(k!)^{3/2}}$$

Put

$$q(t, s) \equiv \|A(t) - A(s)\|\ (t, s \ge 0), \text{ and } \chi_0 \equiv \sup_{t \ge 0} p(t, A(t)).$$

Theorem 4.1 *Let the conditions (4.2), $\chi_0 < \infty$, and*

$$\zeta(A(.), F) \equiv \sup_{t \ge 0} \int_0^t p(t - s, A(t))[q(t, s) + \nu(s)]ds < 1 \tag{4.3}$$

be fulfilled. Then the zero solution of equation (4.1) is asymptotically stable. In addition, any initial vector x_0 satisfying the inequality

$$\frac{\chi_0\|x_0\|}{1-\zeta(A(.),F)} < r, \tag{4.4}$$

belongs to the region of attraction of the zero solution. Moreover, under (4.4) the estimate

$$\|x(t)\| \leq \frac{\chi_0\|x_0\|}{1-\zeta(A(.),F)} \ (t \geq 0)$$

is true for a solution $x(t)$ with the initial vector x_0.

In particular, let inequality (4.2) hold with $\Omega_r = \mathbf{C}^n$, i.e. $r = \infty$. Then under condition (4.3), the zero solution of (4.1) is globally asymptotically stable.

Proof: As follows from Example 2.6, the inequality

$$\|exp[A(\tau)t]\| \leq p(t, A(\tau)) \ (t, \tau \geq 0) \tag{4.5}$$

is valid. We rewrite equation (4.1) in the form

$$dx/dt - A(\tau)x = [A(t) - A(\tau)]x + F(x,t),$$

regarding an arbitrary $\tau \geq 0$ as fixed. This equation is equivalent to the following one:

$$x(t) = exp[A(\tau)t]x(0)+$$

$$\int_0^t exp[(A(\tau)(t-s)][(A(s) - A(\tau))x(s) + F(x(s),s)]ds. \tag{4.6}$$

Since the solutions continuously depend on the initial vector, the inequality

$$\|x(t)\| \leq r \ (0 \leq t \leq t_0)$$

is true for a sufficiently small t_0. Due to (4.2) and (4.5), the latter inequality implies the relation

$$\|x(t)\| \leq p(t, A(\tau))\|x(0)\| + \int_0^t p(t-s, A(\tau))[q(\tau,s) + \nu(s)]\|x(s)\|ds \ (t \leq t_0).$$

Taking $\tau = t$, we get

$$\|x(t)\| \leq p(t, A(t))\|x(0)\| + \int_0^t p(t-s, A(t))[q(t,s) + \nu(s)]\|x(s)\|ds \ (t \leq t_0).$$

This relation according to the definitions of $\zeta(A(.),F)$ and χ_0 yields

$$\sup_{s \leq t_0}\|x(s)\| < \chi_0\|x(0)\| + \sup_{s \leq t_0}\|x(s)\|\zeta(A(.),F).$$

Consequently, due to (4.3)

$$\sup_{s \leq t_0}\|x(s)\| \leq \chi_0\|x(0)\|(1-\zeta(A(.),F))^{-1}$$

for a sufficiently small t_0. But condition (4.4) ensures this bound for all $t \geq 0$. That bound provides the Lyapunov stability. The asymptotic stability can be proved by a small perturbation of system (4.1) as claimed. □

5 Nonlinear discrete systems

Again put $\Omega_r = \{h \in \mathbf{C}^n : \|h\| \leq r\}$. Let A be an $n \times n$-matrix. Consider in $\mathbf{C}^n$ the equation

$$x_{k+1} = Ax_k + F_k(x_k) \tag{5.1}$$

where

$$F_k : \mathbf{C}^n \to \mathbf{C}^n \ (k = 0, 1, 2, ...)$$

are functions satisfying for a positive $r \leq \infty$ the conditions

$$\|F_k(h)\| \leq q\|h\| \ (q = const > 0, h \in \Omega_r). \tag{5.2}$$

For example, if

$$\|F_k(h)\| \leq m\|h\|^p \ (m = const > 0, h \in \mathbf{C}^n)$$

for a $p > 1$, then we have (5.2) with $q = mr^{p-1}$. Let the spectral radius $r_s(A)$ of A is less that one:

$$r_s(A) < 1. \tag{5.3}$$

Furthermore, recall that $g(A)$ and $\gamma_{n,k}$ are defined in Section 2.1. Put

$$\theta_2 \equiv \sum_{k=0}^{n-1} \frac{g^k(A)\gamma_{n,k}}{(1 - r_s(A))^{k+1}}$$

and

$$M_2 \equiv \sup_{m=0,1,\ldots} \sum_{k=0}^{n-1} C_k^m g^k(A)\gamma_{n,k} r_s^{m-k}(A)$$

where

$$C_k^m = \frac{m!}{(m-k)!k!}$$

are the binomial coefficient. Now we are in a position to formulate the main result of the present section

Theorem 5.1 *Let conditions (5.2) and (5.3) be fulfilled. In addition, let*

$$q\theta_2 < 1.$$

Then any solution of (1.1) is subject to the inequality

$$\|x_k\| \leq M_2\|x_0\|(1 - q\theta_2)^{-1} \ \ (k = 1, 2, ...),$$

provided

$$M_2\|x_0\|(1 - q\theta_2)^{-1} < r. \tag{5.4}$$

For the proof see [5]. Clearly, this theorem gives us the stability condition and the estimate for the region of attraction of the zero solution.

6 Nonlinear retarded systems

In this section the Euclidean norm is denoted by $\|.\|_{C^n}$ The space of all continuous functions defined on a segment $[a,b]$ with values in $\mathbf{C}^n$ and the sup-norm $\|.\|_{C[a,b]}$ is denoted by $C([a,b],\mathbf{C}^n)$. In addition, $L^2([a,b],\mathbf{C}^n)$ is the spaces of functions defined on $[a,b]$ with values in $\mathbf{C}^n$ and equipped with the norm

$$|w|_{L[a,b]} = [\int_a^b \|w(t)\|^2_{C^n} dt]^{1/2}.$$

Let us consider in $\mathbf{C}^n$ the equation

$$\dot{x}(t) = \int_0^\eta dR(\tau)x(t-\tau) + F(t,x_t) \quad (t \geq 0), \tag{6.1}$$

where F continuously maps $[0,\infty) \times C([-\eta,0],\mathbf{C}^n)$ into $\mathbf{C}^n$, and R has a bounded variation. Again take the initial condition

$$x(t) = \Phi(t) \text{ for } -\eta \leq t \leq 0. \tag{6.2}$$

It is assumed that for any $u \in L^2([-\eta,\infty),\mathbf{C}^n) \cap C([-\eta,\infty),\mathbf{C}^n)$ the inequality

$$\|F(.,u_t)\|_{L[0,\infty)} \equiv [\int_0^\infty \|F(t,u_t)\|^2_{C^n} dt]^{1/2} \leq q\|u\|_{L[0,\infty)} + M(u) \tag{6.3}$$

holds where $M(.)$ is a continuous (generally nonlinear) functional defined on space $L^2([-\eta,0],\mathbf{C}^n)$.

For instance, let there be constants $b_j > 0$ and $q_j \geq 0$ ($j = 1,\dots,m < \infty$), such that the relation

$$\|F(t,u_t)\| \leq \sum_{k=1}^m q_j \|u(t-h_j(t))\|_{C^n} \text{ for all } u \in L^2([-\eta,\infty),\mathbf{C}^n) \tag{6.4}$$

holds, where $h_j(t)$ are differentiable scalar-valued functions with the properties

$$1 - \dot{h}_j(t) \geq b_j > 0, \text{ and } 0 \leq h_j(t) \leq \eta \; (t \geq 0, \; j = 1,\dots,m). \tag{6.5}$$

Proposition 6.1 *Let relations (6.4) and (6.5) be fulfilled. Then condition (6.3) holds with*

$$q = \sum_{j=1}^m q_j b_j^{-1/2}, \tag{6.6}$$

and

$$M(u) = q[\int_{-\eta}^0 \|u(t)\|^2_{C^n} dt]^{1/2} = q\|u\|_{L[-\eta,0]} \; (u \in L^2([-\eta,0],\mathbf{C}^n)).$$

We need the characteristic matrix of the linear part of (6.1):

$$K(p) = \int_0^\eta exp(-ps)dR(s) - Ip \quad (p \in \mathbf{C}).$$

Put

$$\Gamma(K(p)) = \sum_{k=0}^{n-1} \frac{g^k(B(p))}{\sqrt{k!}d^{k+1}(K(p))} \quad (p \in \mathbf{C}) \text{ and } \Gamma_0(K) \equiv \sup_{\omega \in \mathbf{R}} \Gamma(K(\omega i)),$$

where $d(K(p))$ is the smallest modulus of eigenvalues of $K(p)$ (see Sections 10.1 and 10.3 from [3]). Now we are in a position to formulate the main result of this section.

Theorem 6.2 *Let all the zeros of* $detK(p)$ *lie in the open left half-plane. Let the conditions (6.3) and*

$$q\Gamma_0(K) < 1 \tag{6.7}$$

hold. Then any solution $x(t)$ *of equation (6.1) with a continuous initial function* Φ *belongs to* $L^2([0,\infty), \mathbf{C}^n)$*. Moreover, the bound*

$$\|x\|_{L[0,\infty)} \leq (1 - \Gamma_0(K)q)^{-1}(\Gamma_0(K)M(\Phi) + \|\phi\|_{L[0,\infty)}) \tag{6.8}$$

is valid, where $\phi(t)$ *is the solution of the equation*

$$\dot{\phi} = \int_0^\eta dR(\tau)\phi(t-\tau)$$

with the initial function Φ*.*

Definition 6.3 *We will say that the zero solution of equation (6.1) is absolutely stable in the class of nonlinearities satisfying the inequality*

$$\|F(.,u_t)\|_{L[0,\infty)} \equiv [\int_0^\infty \|F(t,u_t(t))\|^2_{C^n} dt]^{1/2} \leq q\|u\|_{L[-\eta,\infty)}$$

for any $u \in L^2([-\eta,\infty), \mathbf{C}^n) \cap C([-\eta,\infty), \mathbf{C}^n)$*, if under (6.3), the zero solution of (6.1) is globally asymptotically stable. Moreover, there is a positive constant* N *independent of the specific form of the function* F*, such that the inequality*

$$\|x(t)\| \leq N\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \ (t \geq 0)$$

holds for any solution of (6.1) with the initial condition (6.2).

Corollary 6.4 *Let all the zeros of* $detK(p)$ *lie in the open left half-plane, and condition (6.7) hold. Then the zero solution of equation (6.1) is absolutely stable in the class of nonlinearities (6.1).*

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