ON FEEDBACK STABILIZATION OF A CLASS OF SYSTEMS WITH TIME DELAYS

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Abstract

In this paper, the problem of stabilization of systems with delays is addressed. By using the Lyapunov approach, we deduce general conditions for stabilizing the closed-loop system and derive stabilizing state feedback control laws.

1 Introduction

Several control processes encountered in practice, for example in biology, mechanic or chemistry (see [10, 14]) involve delays. Their presence may affect the performances of control laws or even be a source of instability. During the last decades, the problem of stabilizability of control systems and the design of stabilizing feedback has been the subject of many papers, see ([1, 2, 3, 4, 7, 8, 9, 13, 16, 17, 18, 19, 20]), and the references therein. The problems of stabilization and controller design for linear systems with delays has been extensively studied and is still under investigation. Very few works, however, have been performed to deal with the stabilization of nonlinear systems with delays. It is due to the difficulty derived by the infinite dimensionality of the state combined with the nonlinear structure of the differential equations.

The purpose of this paper is to present results on the stabilizability problem of equilibrium positions of nonlinear systems with delays by means of state feedback. Specifically, we present a rigorous developement of sufficient conditions and propose feedback controllers for these systems. The approach developed is inspired by the classical result, well-known as the Jurdjevic and Quinn method [8], dedicated to the problem of stabilization of nonlinear systems. One of the main features of the proposed approach concerns the use of an extension of the usual Lie derivative. Such extensions have been considered for example in [15] in order to investigate the input-output linearization problem for retarded nonlinear systems involving time-delays in the state. For the same type of problem, in [5, 6] the authors propose to overcome the difficulties generated by the presence of the delay by introducing a suitable mathematical formalism and by introducing an associated Lie derivative definition. In order to simplify the presentation, we will first treat the single delay case before to consider the case of multiple delays.

The organization of the paper is as follows. In Section 2 we describe the class of systems considered and recall some basic notions. In Section 3, we state and prove our main results. Finally, Section 4 gives conclusions.

2 System description and preliminaries

We first consider systems of the form :

$$\begin{cases} \dot{x}(t) & = & f(x(t)) + ug(x(t), x(t-h)) \\ x(t) & = & \phi(t), \quad t \in [-h, 0] \end{cases} \tag{1}$$

where f and g are smooth vector fields with $f(0) = g(0,0) = 0$. In the following, $x(t) \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the input vector. h is a positive scalar and represents the delay. The function $\phi(t) \in \mathcal{C} = \mathcal{C}([-h,0], \mathbb{R}^n)$ represents the initial condition. $\mathcal{C}([-h,0], \mathbb{R}^n)$ is the banach space of continuous function mapping $[-h, 0]$ into $\mathbb{R}^n$, with the norm $\|\phi\| = \sup\limits_{t\in[-h,0]} |\phi(t)|$. The euclidean norm of $\phi(t) \in \mathbb{R}^n$ is denoted by $|\phi(t)|$.

We assume that there exists a Lyapunov function V, such that

$$\langle f(\phi(0)), \nabla V(\phi(0))\rangle \leq 0 \quad \forall \phi \in \mathcal{C}([-h,0], \mathbb{R}^n) \tag{2}$$

where ∇ denotes the gradient and $\langle ., . \rangle$ designates the scalar product.

We denote by $\overline{V}$, the Lyapunov functional defined by $\overline{V}(\phi) = V(\phi(0))$.

We introduce δ, the delay operator defined for any function $a(.)$ by :

$$\delta a(t) = a(t-h). \tag{3}$$

We set :

$$\delta^0 a(t) = a(t)$$

and recursively :

$$\delta^k a(t) = \delta(\delta^{k-1} a(t)), \qquad \forall k \geq 1.$$

For any function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we set :

$$\begin{aligned} F_\delta(x(t)) & = F(x(t), \delta x(t)) \\ & = F(x(t), x(t-h)) \end{aligned} \tag{4}$$

For a function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we define the Lie derivative of G_δ along the vector field F_δ as :

$$\begin{aligned} L_{F_\delta} G_\delta(x(t)) &= \frac{\partial G_\delta(x(t))}{\partial x(t)}\ F(x(t), x(t-h)) \\ &+ \frac{\partial G_\delta(x(t))}{\partial \delta x(t)}\ \delta F(x(t), x(t-h)). \end{aligned}$$

This can be rewritten as :

$$L_{F_\delta} G_\delta(x(t)) = \sum_{i=0}^{1} \frac{\partial G_\delta(x(t))}{\partial \delta^i x(t)}\ \delta^i F_\delta(x(t)).$$

By recurrence, we define :

$$L^k_{F_\delta} G_\delta(x(t)) = L_{F_\delta}(L^{k-1}_{F_\delta} G_\delta(x(t))), \quad \forall k \geq 1.$$

Note that when there is no delay, we recover the usual Lie derivative. We can also remark that with this notation, the condition (2) can be rewritten as

$$L_f V(\phi(0)) \leq 0, \quad \forall \phi \in \mathcal{C}([-h, 0], \mathbb{R}^n).$$

Before proceeding further, we will give some preliminary results. Consider the nonlinear delay systems of the general form

$$\dot{x}(t) = f(t, x_t) \tag{5}$$

where $f : \mathbb{R} \times \mathcal{C}([-h, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous with respect to the first argument, lipschitzian with respect to the second and satisfy $f(t, 0) = 0$ for all $t \in \mathbb{R}$.

For $t \geq \sigma - h$, we denote by $x(\sigma, \phi)(t)$, its solution at time t with initial data ϕ, specified at time σ, i.e., $x(\sigma, \phi)(\sigma + \eta) = \phi(\eta)$, $\forall \eta \in [-h, 0]$. For $\eta \in [-h, 0]$,

$$x_t(\eta) = x(t + \eta)$$

and represents the state of the delay system. For all $\delta > 0$, let us denote by $\mathcal{B}(0, \delta)$, the ball $\mathcal{B}(0, \delta) = \{\phi \in \mathcal{C}([-h, 0], \mathbb{R}^n) : \|\phi\| < \delta\}$. $\mathcal{A}$ will designate in the following, the class of scalar non decreasing functions α of $\mathcal{C}([0, \infty), \mathbb{R})$, satisfying $\alpha(s) > 0$ for $s > 0$ and $\alpha(0) = 0$.

Definition
The equilibrium solution, $x \equiv 0$ of the delay differential equation (5) is said to be :

1. stable, if for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma)$ such that $\phi \in \mathcal{B}(0, \delta)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \varepsilon)$ for $t \geq \sigma$.

2. asymptotically stable, if it is stable and there exists $b_0 = b_0(\sigma) > 0$ such that $\phi \in \mathcal{B}(0, b_0)$ implies $x_t(\sigma, \phi) \to 0$ as $t \to \infty$.

Definition
Let $U : \mathbb{R} \times \mathcal{B}(0, \delta) \to \mathbb{R}$ be a continuous functional such that $U(t, 0) = 0$. The functional $(t, \phi) \to U(t, \phi)$ is said to be positive definite, if there is a function α in $\mathcal{A}$ such that $U(t, \phi) \geq \alpha(|\phi(0)|)$, for all $t \in \mathbb{R}$, $\phi \in \mathcal{B}(0, \delta)$.

Theorem 2.1 (see[11])
If there exists a continuous positive definite functional $((t, \phi) \to U(t, \phi)) : \mathbb{R} \times \mathcal{B}(0, \delta) \to \mathbb{R}$ whose derivative $\dot{U}$ is negative on $\mathbb{R} \times \mathcal{B}(0, \delta)$. Then the trivial solution of (5) is stable.

Now, we state and prove our main result.

3 Main results

Theorem 3.1 *If the set*

$$\begin{aligned} W = \{\phi \in \mathcal{C} \,/\, L_f^{k+1} V(\phi(0)) \,=\, L_f^k L_{g_\delta} V(\phi(0)) = 0; \\ k \in \mathbb{N}\} \end{aligned} \tag{6}$$

is reduced to the origin, then the system (1) is globally asymptotically stabilizable at the origin.

Proof :

Let $\theta \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n;]0, \infty))$. We define u by

$$u = -\theta(\phi(0), \phi(-h)) L_{g_\delta} V(\phi(0))\,. \tag{7}$$

Then the closed-loop system(1) with (7) has the following form

$$\begin{aligned} \dot{x}(t) &= z(x(t), x(t-h)) = f(x(t)) \\ &-\theta(x(t), x(t-h)) L_{g_\delta} V(x(t))\ g(x(t), x(t-h)). \end{aligned} \tag{8}$$

Along trajectories of the system (1)(7) :

$$\begin{aligned} \dot{\overline{V}}(\phi) &= \dot{V}(\phi(0)) = L_f V(\phi(0)) \\ &-\theta(\phi(0), \phi(-h)) L_{g_\delta} V(\phi(0))^2 \leq 0, \quad \forall \phi \in \mathcal{C}. \end{aligned} \tag{9}$$

Then the system (1) (7) is stable.

Let $z_t(\phi)$, be the flow of the closed-loop system (1) (7). By LaSalle's invariance principle for differential delay systems (see [12]), $z_t(\phi)$ converges to the largest invariant set I contained in $\Omega = \{\phi \in \mathcal{C} \,/ \dot{\overline{V}}(\phi) \,= 0\}$.

Let $\phi \in I$. Using (9) we get :

$$L_f V(\phi(0)) = L_{g_\delta} V(\phi(0)) = 0.$$

For any t, for which $z_t(\phi)$ is defined, we have :

$$z_t(\phi) = x_t(\phi), \quad \forall t \geq 0$$

where $x_t(\phi)$ is the flow associated to the vector field f.

By invariance of I :

$$L_f V(x_t(\phi)(0)) = L_{g_\delta} V(x_t(\phi)(0)) = 0, \quad \forall t \geq 0.$$

Then :

$$L_f^2 V(\phi(0)) = \left. \frac{d}{dt} L_f V(x_t(\phi)(0)) \right|_{t=0} = 0.$$

and

$$L_f L_{g_\delta} V(\phi(0)) = \left. \frac{d}{dt} L_{g_\delta} V(x_t(\phi)(0)) \right|_{t=0} = 0.$$

By recurrence, one can show that :

$$L_f^{k+1} V(\phi(0)) = \left. \frac{d}{dt} L_f^k V(x_t(\phi)(0)) \right|_{t=0} = 0$$

and for all $k \geq 1$

$$L_f^k L_{g_\delta} V(\phi(0)) = \left. \frac{d}{dt} L_f^{k-1} L_{g_\delta} V(x_t(\phi)(0)) \right|_{t=0} = 0.$$

Finally, we obtain

$$L_f^{k+1} V(\phi(0)) = 0 \quad \text{and} \quad L_f^k L_{g_\delta} V(\phi(0)) = 0, \ \ \forall\, k \in \mathbb{N}.$$

Therefore ϕ is an element of W. Since $W = \{0\}$, the attractivity of the origin is proved.

This finishes the proof of the theorem .

The previous result can be extended to the case of multiple commensurate delays as follows.

We consider the system of the form :

$$\left\{ \begin{array}{l} \dot{x}(t) = f(x(t)) + ug(x(t), x(t-h), \ldots, x(t-mh)) \\ x(t) = \phi(t), \quad t \in [-mh, 0] \end{array} \right. \tag{10}$$

where f and g are smooth vector fields with

$$f(0) = g(0, \ldots, 0) = 0.$$

We introduce the following delay operators δ_i, $(i \in \mathbb{N})$ given by :

$$\delta_i x(t) = x(t - ih)$$

and we define the vector of delay operators τ_p as :

$$\tau_p = (\delta_1, .., \delta_m).$$

Let F be a function mapping $\mathbb{R}^{n\times(p+1)}$ into $\mathbb{R}^n$ and $F_{\tau_p}(x(t))$ defined by :

$$\begin{aligned} F_{\tau_p}(x(t)) &= F(x(t), x(t-h), \ldots, x(t-ph)) \\ &= F(x(t), \tau_p(x(t))). \end{aligned}$$

In the same manner, for a function G mapping $\mathbb{R}^{n\times(q+1)}$ into $\mathbb{R}^n$, we define a function G_{τ_q} by:

$$\begin{aligned} G_{\tau_q}(x(t)) &= G(x(t), x(t-h), \ldots, x(t-qh)) \\ &= G(x(t), \tau_q(x(t))). \end{aligned}$$

For a function $G : \mathbb{R}^{n\times(q+1)} \to \mathbb{R}^n$, we define the Lie derivative of G_{τ_q} along the vector field F_{τ_p} as :

$$L_{F_{\tau_p}} G_{\tau_q}(x(t)) = \frac{\partial G_{\tau_q}(x(t))}{\partial x(t)}\ F_{\tau_p}(x(t))$$

$$+ \textstyle\sum_{i=1}^q \displaystyle\frac{\partial G_{\tau_q}(x(t))}{\partial \delta^i x(t)}\ \delta_i F_{\tau_p}(x(t)).$$

By recurrence, we define :

$$L^k_{F_{\tau_p}} G_{\tau_q}(x(t)) = L_{F_{\tau_p}}(L^{k-1}_{F_{\tau_p}} G_{\tau_q}(x(t))), \ \ \forall k \geq 1.$$

Therefore we have the following result.

Theorem 3.2 *If the set*

$$\begin{aligned} \widetilde{W} = \{ & \phi \in \mathcal{C} \,/\, L_f^{k+1} V(\phi(0)) \ = \ L_f^k L_{g_\tau} V(\phi(0)) = 0; \\ & k \in \mathbb{N}\} \end{aligned} \tag{11}$$

with $\tau = (\delta_1, .., \delta_m)$, is reduced to the origin, then the system (10) can be made globally asymptotically stable at the origin.

Proof :

The proof of this result is analogous to the previous one with the delay operator δ given in (3) replaced by $\tau = (\delta_1, .., \delta_m)$.

Remark 3.1 *For sake of simplicity our main result is given for $u \in \mathbb{R}$. It is clear that a similar result, for $u \in \mathbb{R}^p$ $(p > 1)$, can be established.*

4 Conclusions

In this paper we have considered a problem of stabilization of nonlinear systems with time delays. We have used the Invariance Principle of LaSalle for differential delay systems, combined with an extension of the usual Lie derivative, in order to treat this problem. We have obtained sufficient conditions for guaranteeing the asymptotic stability of the closed-loop system and derived stabilizing state feedback control laws.

References

[1] A. Baccioti. “The local stabilizability for nonlinear systems”. *IMA Journal of Mathematical Control & Information* **5**, pp. 27-39, (1988).

[2] C.I. Byrnes, A. Isidori and J.C. Willems. "Passivity, feedback equivalence and the global stabilization of minimum phase nonlinear systems". *IEEE Transactions on Automatic Control* **AC-36** (11), pp. 1228-1240, November (1991).

[3] J.M. Coron. "Linearized control systems and application to smooth stabilization". *SIAM Journal of Control and Optimization*, **32**, pp. 358-386, (1994).

[4] J.P. Gauthier and G. Bornard. "Stabilisation des systèmes non linéaires", in: I.D.Landau, Ed., *Outils et Modèles Mathématiques pour l'Automatique et l'Analyse des systèmes* (Editions du CNRS, Paris), **Vol.1**, pp. 307-324, (1981).

[5] A. Germani, C. Manes and P. Pepe. "Linearization of input-output mapping for nonlinear delay systems via static state feedback". *Proceeding of CESA IMACS Multiconference on Computational Engineering in Systems Applications,* **Vol.1**, pp. 599-602, Lille, France, (1996).

[6] A. Germani, C. Manes and P. Pepe. "Linearization and decoupling of nonlinear delay systems". *Proceeding of 1998 American Control Conference*, Philadelphia, Pennsylvania, (1998).

[7] D.H. Jacobson. "Stabilization and optimal control nonlinear homogeneous-in-input". *Proceedings of the Conference on Directions in Decentralized Control, Many-Person Optimazation and Large-Scale Systems,(Boston, Massachusetts, September 1-3, 1975)*, Plenum Press, (1976).

[8] V. Jurdjevic and J.P. Quinn. "Controllability and stability". *J. Differential Equations*, **28**, pp. 381-389, (1978).

[9] N. Kalouptsidis and J. Tsinias. "Stability improvement of nonlinear systems by feedback". *IEEE Transaction on Automatic Control*, **AC-29** (4), pp. 364-367, (1984).

[10] V.B. Kolmanovskii and V.R. Nosov. Stability of Functional Differential Equations, New-York, Academic Press, (1986).

[11] V.B. Kolmanovskii and A. Myshkis. Applied Theory of Functional Differential Equations, Dordrecht, Kluwer Academic Publishers, (1992).

[12] J.P. LaSalle. "The Stability of Dynamical Systems". Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics. *Philadelphia*, (1976).

[13] K.K. Lee and A. Arapostathis. "Remarks on smooth feedback stabilisation of nonlinear systems". *Systems & Control Letters*, **10**, pp. 41-44, (1988).

[14] M. Malek-Zavarei and M. Jamshidi. Time Delay Systems: Analysis, Optimization and Applications, North-Holland, (1987).

[15] T. Oguchi, A. Watanabe and T. Nakamizo. "Input-output Linearization of Retarded Nonlinear Systems by an Extended Lie derivative". *Proceeding of the 37th IEEE Conference on Decision and Control*, Florida, pp. 1364-1369, December (1998).

[16] R. Outbib and G. Sallet. "Stabilizability of the angular velocity of a rigid body revisited". *Systems & Control Letters*, **18**, pp. 93-98, (1992).

[17] E. Ryan and J. Buckingham. "On asymptotically Stabilizing Feedback Control of Bilinear Systems". *IEEE Transaction on Automatic Control* **AC-28** (8), pp. 863-864, August (1983).

[18] M. Slemrod. "Stabilization of bilinear control systems with applications to nonconservative problems in elasticity". *SIAM Journal of Control and Optimization* **16** (81), pp. 131-141, January (1978).

[19] E.D. Sontag. Mathematical Control Theory. Deterministic Finite Dimensional System. Springer Verlag, New-York, (1989).

[20] J. Tsinias. "Sufficient Lyapunov like conditions for stabilization". *Math. of Control, Signals and Systems* **2** (81), pp. 343-357, (1989).