THE LOCAL OUTPUT REGULATION PROBLEM: CONVERGENCE REGION ESTIMATES

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Abstract

In this paper, the problem of local output regulation is considered. The presented results answer the question: given a controller solving the local output regulation problem, how to estimate the set of admissible initial conditions for which this controller makes the regulated output converge to zero. Theoretical estimation results and an estimation procedure explaining the application of these results in practice are presented. An example of the disturbance rejection problem for a mechanical system (TORA system) is given as an illustration.

1 Introduction

In this paper, we consider the problem of asymptotic regulation of the output of a dynamical system, which is subject to disturbances generated by an external system. This problem is known as the output regulation problem. Many problems in control theory can be considered as particular cases of this problem: tracking of a class of reference signals, rejecting a class of disturbances, partial stabilization and controlled synchronization. For nonlinear systems, solutions to the *local* output regulation problem were given in [10, 6]. In [10], necessary and sufficient conditions for the solvability of the problem in some neighborhood of the origin were obtained and a procedure for designing a controller that solves the problem was presented. That paper was followed by publications regarding the local approximate output regulation problem [7, 8, 15] and other aspects of the output regulation problem for nonlinear systems: regulation in the presence of uncertainties, adaptive, semiglobal and global output regulation (see [2, 3, 11, 14, 9] and references therein). At the same time, one problem regarding the *local* output regulation problem remained open: given a controller solving the problem in *some* neighborhood of the origin, how to determine (or estimate) this neighborhood of admissible initial conditions? Without answering this question, solutions to the local output regulation problem may not be satisfactory from an engineering point of view.

One answer to that question was given in [13]. In that paper, a procedure for estimating the set of admissible initial conditions was proposed. That result was based on the notion of convergent systems developed by B.P. Demidovich [4, 5]. Roughly, a convergent system is a system that, being excited by a bounded signal, has a unique asymptotically stable bounded response.
In this paper, we modify the approach developed in [13] in order to obtain improved estimation results. This work is also inspired by the results of B.P. Demidovich. More information related to the notion of convergent systems can be found in [16], [13] and in the paper [1] on incremental stability of dynamical systems.

The paper is organized as follows. In Section 2, we recall the problem of local output regulation and formulate the problem of estimating the set of admissible initial conditions. In Section 3, we describe the ideas that are used to find the estimates and formulate a technical result supporting these ideas. Section 4 contains the main results on the estimation problem as well as an explanation of their application in practice given in the form of an estimation procedure. In Section 5, the procedure is applied to a disturbance rejection problem in the TORA system (see [17], [18], [12] for details about the TORA system). Conclusions are contained in Section 6. The proofs of all results are given in the Appendix.
The notations used in the paper are the following. $\mathcal{A}^T$ is the transpose of matrix $\mathcal{A}$ and $\mathcal{A}^{-T} = (A^{-1})^T$. The norm of a vector is denoted as $|x| = (x^T x)^{1/2}$. For a positive definite matrix $P = P^T > 0$ we define the vector norm $|\cdot|_P$ as $|x|_P := \sqrt{x^T P x}$. An ellipsoid $E_P(\mathcal{R})$ is defined by $E_P(\mathcal{R}) := \{x \in \mathbb{R}^n : |x|_P < \mathcal{R}\}$. An open ball is denoted $B_w(r) := \{w : |w| < r\}$. $\|P\|$ is the operator norm of the matrix P induced by the vector norm $|\cdot|$. By I we denote the identity matrix. The largest eigenvalue of a symmetric matrix $J = J^T$ is denoted $\Lambda(J)$. $\mathcal{D}F_x(x)$ is the Jacobian matrix of $F(x)$.

2 Estimation problem statement

First, we recall the problem of local output regulation. Following [10], consider systems modelled by equations of the form

$$\dot{x} = f(x, u, w), \quad (1)$$
$$e = h(x, w), \quad (2)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^l$ and exogenous input $w \in R^m$ generated by the exosystem

$$\dot{w} = s(w). \quad (3)$$

The exogenous signal $w(t)$ can be viewed as a disturbance in equation (1) or as a reference signal in (2). It is assumed that $f(0,0,0) = 0$, $h(0,0) = 0$, $s(0) = 0$; functions f, h, s are C^k functions for some large k. We assume that exosystem (3) is *neutrally stable*. Neutral stability means that the equilibrium $w = 0$ is Lyapunov stable in forward and backward time [3]. An important representative of neutrally stable exosystems is a linear harmonic oscillator.

The local state-feedback output regulation problem is formulated in the following way. Given a nonlinear system of the form (1), (2) and a neutrally stable exosystem (3), find, if possible, a feedback $u = \beta(x,w)$, $\beta(0,0) = 0$, such that

A) The system

$$\dot{x} = f(x, \beta(x,0), 0) \tag{4}$$

has an asymptotically stable linearization at $x = 0$,

B) There exists a neighborhood $X \times W$ of $(0,0)$ such that for each initial condition $(x(0), w(0)) \in X \times W$ the solution of

$$\begin{aligned} \dot{x} &= f(x, \beta(x,w), w), & (5)\\ \dot{w} &= s(w) & (6) \end{aligned}$$

satisfies $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$.

A controller solving the local output regulation problem makes the output e tend to zero at least for small initial conditions $(x(0), w(0))$. Without specifying the region $X \times W$ of admissible initial conditions, such solution may not be satisfactory from an engineering point of view. Thus, we come to the following **estimation problem:** *given the closed-loop system (5) and the exosystem (6), estimate the region of admissible initial conditions $X \times W$ for which the regulated output $e(t) = h(x(t), w(t))$ tends to zero on solutions $(x(t), w(t))$ starting in $X \times W$.*

In this paper, we assume that the closed-loop system (5) has the form

$$\dot{x} = f(x, \beta(x,w), w) =: F(x) + a(w). \tag{7}$$

This is the case, for example, for systems of the form $\dot{x} = f(x) + Bu + d(w)$ in closed-loop with a controller of the form $u = g(x) + c(w)$. This assumption is made to simplify the subsequent analysis and to make the ideas behind this analysis more transparent. It should be noted that the results presented in this paper can be extended with slight technical modifications to the case of system (5) with a general right-hand side.

Estimation results are presented in Section 4. In order to avoid different formulations for different controllers solving the local output regulation problem, the results are based on certain properties of the closed-loop system (7) that are the same for all the controllers. These properties are (see [10], [2]):

$\mathcal{A}$) *The Jacobian matrix $\mathcal{D}F_x(0)$ is Hurwitz,*

$\mathcal{B}$) *There exists a mapping $x = \pi(w)$ defined in a neighborhood $\mathcal{W}$ of the origin, with $\pi(0) = 0$ and such that*

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w)s(w) &= F(\pi(w)) + a(w),\\ 0 &= h(\pi(w), w) \end{aligned} \tag{8}$$

for all $w \in \mathcal{W}$.

In fact, a controller $u = \beta(x,w)$ solves the local output regulation problem if and only if the closed-loop system (7) satisfies conditions $\mathcal{A}$) and $\mathcal{B}$) (see [2] for details). So, we assume that these conditions hold and that the mapping $\pi(w)$ is known. The last assumption is not restrictive, because in many cases the mapping $\pi(w)$ is required for the construction of the controller.

To simplify the subsequent analysis, it is assumed that the closed-loop system (7) and the mapping $\pi(w)$ are defined globally for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ (i.e. $\mathcal{W} = \mathbb{R}^m$). If this assumption does not hold, one should restrict all the subsequent results to the sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^m$ for which $F(x) + a(w)$ and $\pi(w)$ are well-defined.

3 Ideas and preliminaries

Before formulating the estimation results, let us first have a look at the dynamics of the extended closed-loop system (5), (6) near the origin. Under the neutral stability assumption on the exosystem (6) and properties $\mathcal{A}$) and $\mathcal{B}$) of the closed-loop system, the manifold $x = \pi(w)$ is a locally attractive invariant center manifold and on this manifold the regulated output $e = h(x,w)$ is equal to zero (see [2] for details). This implies that for a small trajectory $w(t)$ of the exosystem (6), any solution $x(t)$ of the closed-loop system (5) starting close enough to the origin converges to the solution $\bar{x}(t) := \pi(w(t))$ on the center manifold. Thus, $e(t) = h(x(t), w(t)) \to h(\pi(w(t)), w(t)) \equiv 0$ (see (8)) and output regulation is attained. This dynamics can be described in a different, yet equivalent way: all solutions of the closed-loop system (5) starting close enough to the origin converge one to another and among them there is a solution $\bar{x}(t) = \pi(w(t))$ on which $e(t) \equiv 0$.

Such a reformulation suggests a way to estimate the set of admissible $(x(0), w(0))$ for which the output regulation occurs. First, find a set $\mathcal{C} \subset \mathbb{R}^n$ such that any two solutions $x_1(t)$ and $x_2(t)$ of the closed-loop system (7) lying in $\mathcal{C}$ for all $t \geq 0$ converge to each other: $|x_1(t) - x_2(t)| \to 0$ as $t \to \infty$. We call such set $\mathcal{C}$ a *convergence set*. Second, find a set $\mathcal{Y} \subset \mathbb{R}^{n+m}$ of initial conditions $(x(0), w(0))$ such that any trajectory $(x(t), w(t))$ starting in this set satisfies the conditions $x(t) \in \mathcal{C}$ and $\pi(w(t)) \in \mathcal{C}$ for all $t \geq 0$. Then, by the properties of the set $\mathcal{C}$, any trajectory $(x(t), w(t))$ starting in $\mathcal{Y}$ satisfies $x(t) \to \pi(w(t))$ as $t \to +\infty$ and hence $e(t) = h(x(t), w(t)) \to h(\pi(w(t), w(t)) \equiv 0$. So, the set $\mathcal{Y}$ is an estimate of the set of initial conditions $(x(0), w(0))$ for which output regulation occurs. Below, a lemma that states sufficient conditions for a set $\mathcal{C}$ to be a convergent set is given.

Lemma 1 *Suppose $\mathcal{C} \subset \mathbb{R}^n$ is a convex set satisfying the Demidovich condition*

$$\sup_{x \in \mathcal{C}} \Lambda(P\mathcal{D}F_x(x) + \mathcal{D}F_x^T(x)P) < 0 \tag{9}$$

for some positive definite matrix $P = P^T > 0$. Then, any two solutions $x_1(t)$ and $x_2(t)$ of system (7) lying in $\mathcal{C}$ for all $t \geq t_0$

satisfy

$$|x_1(t) - x_2(t)| \le C\epsilon^{-\beta t}|x_1(t_0) - x_2(t_0)|, \qquad (10)$$

where $\beta > 0$ and $C > 0$ do not depend on the particular solutions $x_1(t)$ and $x_2(t)$.

Lemma 1 is a direct corollary of a result of Demidovich from [4], where it was shown that condition (9) implies

$$(x_1 - x_2)^T P(f(x_1, t) - f(x_2, t)) \le -\beta|x_1 - x_2|_P^2$$

for all $x_1, x_2 \in \mathcal{C}$, $t \in \mathbb{R}$ and for some $\beta > 0$ independent of the particular x_1, x_2 and t. Thus, if at some instant t_* any two solutions of system (7) $x_1(t)$ and $x_2(t)$ lie in $\mathcal{C}$, then the function $V(t) := 1/2|x_1(t) - x_2(t)|_P^2$ satisfies

$$\frac{dV}{dt}(t_*) \le -2\beta V(t_*). \qquad (11)$$

This implies (10).

By a proper choice of the matrix $P > 0$ we can guarantee, that condition (9) is satisfied at least for $\mathcal{C}$ being some neighborhood of the origin. Indeed, since $\mathcal{D}F_x(0)$ is a Hurwitz matrix, we can find such $P > 0$ that the matrix $P\mathcal{D}F_x(0) + \mathcal{D}F_x^T(0)P$ is negative definite. By continuity, $P\mathcal{D}F_x(x) + \mathcal{D}F_x^T(x)P$ is negative definite at least for small x. Hence, condition (9) is satisfied for $\mathcal{C}$ being some neighborhood of the origin.
In practice, it is more convenient to look for a convergent set $\mathcal{C}$ in some parameterized family of sets. So, in the sequel we assume that such family is given by the expression $\mathcal{C}(\mathcal{R}) = \{x \in \mathbb{R}^n : U(x) < \mathcal{R}\}$, where the function $U(x)$ satisfies the following three conditions:

1) $U(x + y) \le U(x) + U(y)$
2) $\exists\, d > 0 : U(x) \le d|x|_P \;\forall\, x \in \mathbb{R}^n$
3) $The\ set\ \{x : U(x) < \mathcal{R}\}\ is\ convex.$

Property **3)** is required for Lemma 1 and properties **1)** and **2)** will be required in subsequent results. Examples of such functions are $U(x) = |x|_P$ and $U(x) = |x^i|$, where x^i is the i-th component of the vector x.

If the matrix $P > 0$ is chosen such that $P\mathcal{D}F_x(0) + \mathcal{D}F_x^T(0)P < 0$ and $U(x) = |x|_P$ (or $U(x)$ is any other vector norm on $\mathbb{R}^n$) then condition (9) is satisfied for $\mathcal{C}(\mathcal{R}) = \{x \in \mathbb{R}^n : U(x) < \mathcal{R}\}$ at least for small $\mathcal{R}$. If $F(x)$ has the form $F(x) = Ax + \phi(x^i)$, it is convenient to choose $U(x) = |x^i|$, because $P\mathcal{D}F_x(x) + \mathcal{D}F_x^T(x)P$ depends only on x^i. In this case, condition (9) is also satisfied for $\mathcal{C}(\mathcal{R}) = \{x \in \mathbb{R}^n : U(x) = |x^i| < \mathcal{R}\}$ at least for small $\mathcal{R}$. If $P > 0$ and $U(x)$ are chosen as described above, we can find the maximal convergence set $\mathcal{C}(\mathcal{R})$ from the specified family by increasing the parameter $\mathcal{R}$ until (9) is violated.

4 Estimation Results

Prior to formulating the main results, we introduce the following functions:

$$m(w_0) := \sup_{t \ge 0} U(\pi(w(t, w_0))), \qquad (12)$$

$$k(w_0) := \sup_{t \ge 0} |\pi(w(t, w_0))|_P, \qquad (13)$$

where $w(t, w_0)$ is a solution of the exosystem (3) satisfying $w(0, w_0) = w_0$. The function $m(w_0)$ allows to establish whether the solution $\pi(w(t, w_0))$ lies in a set of the form $\mathcal{C} = \{x : U(x) < \mathcal{R}\}$ for all $t \ge 0$. The function $k(w_0)$ indicates how large can the solution $\pi(w(t, w_0))$ be in the $|\cdot|_P$-norm.
The following theorem gives an estimate of the admissible initial conditions set in the form of a neighborhood of the center manifold $x = \pi(w)$.

Theorem 1 *Consider the closed-loop system (7) and the exosystem (3). Suppose for some $P = P^T > 0$ the Demidovich condition (9) is satisfied for the set $\mathcal{C} := \{x : U(x) < \mathcal{R}\}$, where $\mathcal{R} > 0$ and the function $U(x)$ satisfies conditions* **1)-3)**. *Then, any trajectory $(x(t), w(t))$ starting in the set*

$$\mathcal{Y} := \{ (x_0, w_0) : m(w_0) < \mathcal{R},\ |x_0 - \pi(w_0)|_P < \frac{1}{d}(\mathcal{R} - m(w_0))\} \qquad (14)$$

satisfies

$$|x(t) - \pi(w(t))| \le Ce^{-\beta t}|x(0) - \pi(w(0))| \qquad (15)$$

for some $\beta > 0$ and $C > 0$ independent of $(x(t), w(t))$, and $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$.

The inequality $m(w_0) < \mathcal{R}$ in the definition of $\mathcal{Y}$ specifies the set of admissible w_0. It means that if a solution $w(t)$ of the exosystem (3) starts in w_0, then $\pi(w(t))$ does not leave the set $\mathcal{C} = \{x : U(x) < \mathcal{R}\}$. The second inequality in the definition of $\mathcal{Y}$ specifies for each admissible w_0 a set of such x_0 that the solution $x(t)$ starting in x_0 does not leave the convergence set $\mathcal{C}$.
If we want the closed-loop system (7) and the exosystem (3) to start in the set $\mathcal{Y}$, we need to guarantee that, first, the exosystem starts in a point w_0 in the set $\mathcal{M} = \{w_0 : m(w_0) < \mathcal{R}\}$ and, second, that the closed-loop system (7) starts in the set $\mathcal{E}(w_0) = \{x_0 : (x_0, w_0) \in \mathcal{Y})\}$. As can be seen from Fig. 1, the set $\mathcal{E}(w_0)$ may be different for different values of w_0. Thus,

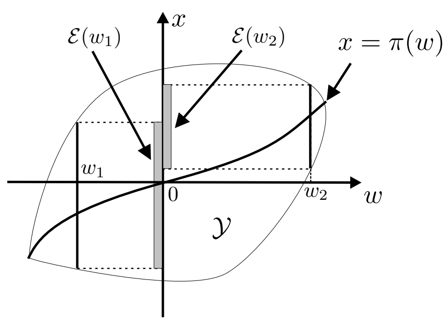

Fig.1 The sets $\mathcal{Y}$ and $\mathcal{E}(w_0)$: for different w_1 and w_2, the sets $\mathcal{E}(w_1)$ and $\mathcal{E}(w_2)$ may be different.

the knowledge of w_0 is important. In practice, however, we may not know the value of w_0. For example, if the exosystem generates disturbances, then, knowing the level of disturbances, we can establish that $w_0 \in \mathcal{M}$, but we do not know the exact value of w_0. In order to cope with this difficulty, in the next result we find sets X_0 and W_0 such that in whatever point $w_0 \in W_0$ the exosystem is initialised, if the closed-loop system starts in $x_0 \in X_0$, then the output regulation occurs. Mathematically, this means that we find a set of admissible initial conditions (x_0, w_0) in the form of a direct product $X_0 \times W_0$. Relation between the sets $\mathcal{Y}$ and $X_0 \times W_0$ is shown in Fig. 2. Prior to formulating the result, define the functions

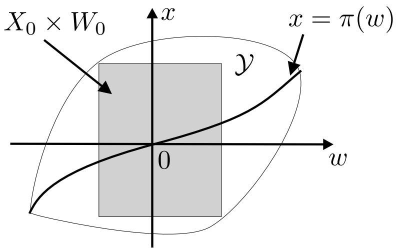

Fig.2 Relation between the sets $\mathcal{Y}$ and $X_0 \times W_0$.

$$\alpha(r) := \sup_{|w_0| \le r} (m(w_0) + dk(w_0))$$

and

$$R(r) := (\mathcal{R} - \alpha(r))/d.$$

The function $\alpha(r)$ is nondecreasing and $\alpha(0) = 0$. Let $r_* > 0$ be the largest number such that $\alpha(r) < \mathcal{R}$ for all $r \in [0, r_*)$. Such r_* exists due to stability of the trivial solution $w(t) \equiv 0$ and the fact that $\pi(0) = 0$. Indeed, stability implies that

$$\sup_{|w_0| \le r,\ t \ge 0} |w(t, w_0)| \to 0, \quad \text{as } r \to 0$$

and, by continuity of $\pi(w)$,

$$\sup_{|w_0| \le r} k(w_0) = \sup_{|w_0| \le r,\ t \ge 0} |\pi(w(t, w_0))|_P \to 0, \quad \text{as } r \to 0.$$

Due to property **2)** of the function $U(x)$ and the definitions of $m(w)$ and $k(w)$, it holds that $m(w) \le dk(w)$. Hence, $\alpha(r) \le 2d \sup_{|w_0| \le r} k(w_0) \to 0$ as $r \to 0$. So, the inequality $\alpha(r) < \mathcal{R}$ is satisfied at least for small $r > 0$. Hence, there exists the largest $r_* > 0$ such that $\alpha(r) < \mathcal{R}$ for all $r \in [0, r_*)$. The next theorem gives estimates of the admissible initial conditions set in the form of a direct product $X_0 \times W_0$.

Theorem 2 *Under the conditions of Theorem 1, any trajectory $(x(t), w(t))$ starting in the set $E_P(R(r)) \times B_w(r)$ for any $r \in [0, r_*)$ satisfies*

$$|x(t) - \pi(w(t))| \le Ce^{-\beta t} |x(0) - \pi(w(0))|$$

for some $\beta > 0$ and $C > 0$ independent of $(x(t), w(t))$, and $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$.

Theorem 2 provides estimates $X_0 \times W_0$ of the admissible initial conditions set. For practical application, it is convenient to unify all the steps necessary for finding $X_0 = E_P(R(r))$ and $W_0 = B_w(r)$ in the following estimation procedure.

Procedure 1 .
i) *Find a positive definite matrix P such that*

$$P\mathcal{D}F_x(0) + \mathcal{D}F_x^T(0)P < 0. \tag{16}$$

Such P exists, because $\mathcal{D}F_x(0)$ is Hurwitz.
ii) *Choose a function $U(x)$ satisfying conditions* **1)–3)** *and find the maximal $\mathcal{R}$ such that condition (9) is satisfied for $\mathcal{C} = \{x : U(x) < \mathcal{R}\}$ (see Section 3 for details). Compute the minimal constant d such that $U(x) \le d|x|_P$.*
iii) *Compute the function $\alpha(r) := \sup_{|w_0| \le r}(m(w_0) + dk(w_0))$ for increasing r starting from $r = 0$ until $\alpha(r)$ reaches $\mathcal{R}$ in some point r_*. Compute $R(r) := (\mathcal{R} - \alpha(r))/d$.*
Then, for every $r \in [0, r_*)$ the set $E_P(R(r)) \times B_w(r)$ is an estimate of the set of admissible initial conditions (x_0, w_0).

The matrix inequality in step **i**) admits multiple positive definite solutions P. At the moment it is an open question how to choose P in order to obtain the best (in some sense) estimates. Step **iii**) is computationally most expensive, since it requires integration of the exosystem. Yet, in the case of the exosystem being a linear harmonic oscillator

$$\dot{w}_1 = \Omega w_2, \quad \dot{w}_2 = -\Omega w_1, \tag{17}$$

the formula for $\alpha(r)$ simplifies to

$$\alpha(r) = \sup_{|w_0| \le r} (U(\pi(w_0)) + d|\pi(w_0)|_P). \tag{18}$$

5 Example

Let us illustrate the application of the estimation procedure. Consider the system shown in Fig. 3, called the TORA system (see [17], [18] for details about this system).

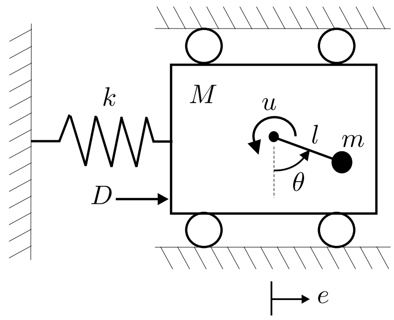

Fig.3 TORA system: M – cart mass, k – spring stiffness, m – eccentric mass, u – control torque, e – horizontal displacement, D – disturbance force.

The control problem is to find a control law for the torque u such that the horizontal displacement e tends to zero regardless of the harmonic disturbance force D of known frequency, but unknown amplitude and phase. This is a disturbance rejection problem. A tracking problem, in which there is no disturbance force $D(t)$ and the output $e(t)$ has to track a prescribed reference signal, was solved globally in [12].

In certain coordinates and after some feedback transformations, the TORA system can be described by the following equations

[18]:

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -x_1 + \epsilon \sin x_3 + \mu D \\ \dot{x}_3 &=& x_4 \\ \dot{x}_4 &=& v, \\ e &=& (\lambda(x_1 - \epsilon \sin x_3), \end{array} \tag{19}$$

where $\epsilon := \sqrt{m/(m+M)}$, $\mu := 1/(kl\epsilon)$, $\lambda := l\epsilon$ and v is a new control input. The disturbance force D is the output of a linear harmonic oscillator

$$\dot{w}_1 = \Omega w_2, \quad \dot{w}_2 = -\Omega w_1, \quad D = w_1. \tag{20}$$

The above control problem is a particular case of the output regulation problem. For small initial conditions $(x(0), w(0))$ this problem is solved by a controller of the form $v = c(w) + K(x - \pi(w))$, where the mappings $\pi(w) \in \mathbb{R}^4$ and $c(w) \in \mathbb{R}$ are defined by the formulae

$$\begin{array}{ll} \pi_1(w) := -\frac{\mu w_1}{\Omega^2} & \pi_3(w) := -\arcsin(\frac{\mu w_1}{\Omega^2 \epsilon}) \\ \pi_2(w) := -\frac{\mu w_2}{\Omega} & \pi_4(w) := -\frac{\mu \Omega w_2}{\sqrt{\Omega^4 \epsilon^2 - \mu^2 w_1^2}}, \end{array} \tag{21}$$

$$c(w) := \frac{\mu \Omega^2 w_1 (\Omega^4 \epsilon^2 - \mu^2 (w_1^2 + w_2^2))}{(\sqrt{\Omega^4 \epsilon^2 - \mu^2 w_1^2})^3}, \tag{22}$$

and the matrix K is such that the closed-loop system has an asymptotically stable linearization at the origin. Indeed, for such controller the closed-loop system has the form $\dot{x} = F(x) + a(w)$ and it satisfies conditions $\mathcal{A}$) and $\mathcal{B}$) with the specified $\pi(w)$. Thus, for all small initial conditions $(x(0), w(0))$ the trajectory $(x(t), w(t))$ satisfies $e(t) \to 0$ as $t \to \infty$. Let us apply Procedure 1 to estimate this set of admissible $(x(0), w(0))$ for the following values of parameters: $\epsilon = 0.5$, $\mu = 0.04$, $\Omega = 1$, $K = (12, -4, -8, -5)$.

In Procedure 1, we pick a matrix $P = P^T > 0$ such that $P\mathcal{D}F_x(0) + (\mathcal{D}F_x(0))^T P = -Q$, where Q is diagonal matrix $diag(2, 8, 1, 1)$. For convenience, P is normalized such that $\|P\| = 1$. The function $U(x)$ is chosen $U(x) := |x_3|$. Such choice of $U(x)$ is made because $F(x)$ equals a linear part plus a nonlinearity depending on x_3 (see Section 3). Completing the remaining steps of the procedure, we obtain the estimates of the admissible initial conditions set: $E_P(R(r)) \times B_w(r)$, where $R(r)$ is shown in Fig. 4. Note, that the mappings $\pi(w)$ and $c(w)$ and, thus, the closed-loop system are defined only for $|w_1| < \Omega^2 \epsilon / \mu$. For the given values of the system parameters this constraint is given by $|w_1| < 12.5$. The obtained estimates satisfy this condition. The estimates are rather conservative. According to simulations, for a fixed level of disturbance r, output regulation still occurs for $x(0) \in E_P(\bar{R}(r))$ with $\bar{R}(r)$ at least 4 times larger than $R(r)$. One possible reason for such conservativeness is a bad choice of the matrix P. A different choice for P may result in a better estimates.

6 Conclusions

In this paper, we have considered the problem of estimating the set of admissible initial conditions for a solution to the local output regulation problem. Two estimation results have been presented: one gives the estimates in the form of a neighborhood $\mathcal{Y} \subset \mathbb{R}^{n+m}$ of the center manifold; another one gives estimates in the form of a direct product $X_0 \times W_0$ of two neighborhoods of the origin $X_0 \subset \mathbb{R}^n$ and $W_0 \subset \mathbb{R}^m$. In both cases, trajectories starting in the estimated sets tend to the output-zeroing center manifold uniformly exponentially. The application steps of the last result are unified in an estimation procedure. This procedure is illustrated by application to the example of the disturbance rejection problem in the TORA system. Further investigation is needed to make the estimates less conservative.

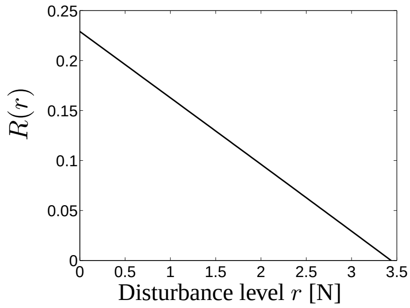

Figure 1: $R(r)$ and r for the estimates $E_P(R(r)) \times B_w(r)$.

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Appendix

Proof of Theorem 1. Notice, that it is sufficient to show that the set $\mathcal{Y}$ is invariant and that for any $(x, w) \in \mathcal{Y}$ it holds that $U(x) < \mathcal{R}$ and $U(\pi(w)) < \mathcal{R}$. If this is the case, then for every $(x(t), w(t))$ starting in $\mathcal{Y}$ the solutions $x(t)$ and $\pi(w(t))$ belong to the convergence set $\mathcal{C} = \{x : U(x) < \mathcal{R}\}$ for all $t \geq 0$. Thus, the statement of the theorem follows from Lemma 1.

Let us show that the set $\mathcal{Y}$ is such that for any $(x, w) \in \mathcal{Y}$ it holds that $U(x) < \mathcal{R}$ and $U(\pi(w)) < \mathcal{R}$. Let $(x, w) \in \mathcal{Y}$. By definition of $\mathcal{Y}$, $U(\pi(w)) \leq m(w) < R$. Successively applying properties 1) and 2) of the function $U(x)$ and the definition of $\mathcal{Y}$, we obtain: $U(x) \leq U(\pi(w)) + U(x - \pi(w)) \leq U(\pi(w)) + d|x - \pi(w)|_P < m(w) + \mathcal{R} - m(w) = \mathcal{R}$.

The invariance of $\mathcal{Y}$ is proved by contradiction. Let $(x_0, w_0) \in \mathcal{Y}$ and suppose that $t_1 > 0$ is the first instant when the trajectory $(x(t), w(t))$, starting in (x_0, w_0), leaves the set $\mathcal{Y}$. This is equivalent to the following two statements:

$$|x(t) - \pi(w(t))|_P < \frac{1}{d}(\mathcal{R} - m(w(t))) \quad \forall\, t \in [0, t_1),$$

$$|x(t_1) - \pi(w(t_1))|_P = \frac{1}{d}(\mathcal{R} - m(w(t_1))).$$

Since $m(w(t)) \leq m(w_0) < \mathcal{R}$ for all $t \geq 0$, we obtain

$$|x(t_1) - \pi(w(t_1))|_P \geq \frac{1}{d}(\mathcal{R} - m(w_0)).$$

Thus, the function $V(t) = 1/2|x(t) - \pi(w(t))|_P^2$ satisfies the inequality: $V(0) < V(t_1)$. Necessarily, there exists a time instant $t_* \in (0, t_1)$ such that the derivative of $V(t)$ is positive at t_*: $dV/dt(t_*) > 0$. Notice, that the set $\mathcal{C} = \{x : U(x) < \mathcal{R}\}$ satisfies the conditions of Lemma 1 and both $x(t_*)$ and $\pi(w(t_*))$ belong to this set. Hence, by Lemma 1 (see formula (11)) $dV/dt(t_*) \leq -2\beta V(t_*) \leq 0$. Thus, we come to a contradiction. So, the set $\mathcal{Y}$ is indeed invariant. This completes the proof. □

Proof of Theorem 2. It is sufficient to show that $E_P(R(r)) \times B_w(r) \subset \mathcal{Y}$ for any $r \in [0, r_*)$. Then, the statement of Theorem 2 follows from Theorem 1.

Suppose $x_0 \in E_P(R(r))$ and $w_0 \in B_w(r)$ for some fixed $r \in [0, r_*)$. According to the definition of $\mathcal{Y}$, we first need to show that $m(w_0) < \mathcal{R}$. By the definition of $\alpha(r)$ and the choice of w_0, it holds that $m(w_0) + dk(w_0) \leq \alpha(r)$. Thus, $m(w_0) \leq \alpha(r)$ and, by the choice of r, $\alpha(r) < \mathcal{R}$. This implies $m(w_0) < \mathcal{R}$.

Second, we show that $|x_0 - \pi(w_0)|_P < (\mathcal{R} - m(w_0))/d$. The triangle inequality implies

$$|x_0 - \pi(w_0)|_P \leq |x_0|_P + |\pi(w_0)|_P. \tag{23}$$

By the choice of x_0 and by the definition of $R(r)$,

$$\begin{aligned} |x_0|_P < R(r) \quad &= (\mathcal{R} - \alpha(r))/d \\ &= (\mathcal{R} - \sup_{|w| \leq r}(m(w) + dk(w)))/d \\ &\leq (\mathcal{R} - m(w_0))/d - k(w_0). \end{aligned}$$

By definition of $k(w_0)$ it holds that $|\pi(w_0)|_P \leq k(w_0)$. Substituting these inequalities in (23), we obtain $|x_0 - \pi(w_0)|_P < (\mathcal{R} - m(w_0))/d$. This completes the proof.□