A REACHABLE THROUGHPUT UPPER BOUND FOR LIVE AND SAFE FREE CHOICE NETS VIA T-INVARIANTS

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Abstract

This paper focuses on the problem of computing a reachable upper bound for the throughput of transitions in live and safe free choice nets. Starting from the consideration that a live a safe free choice net can be viewed as an interconnection of live and safe marked graphs, previous results on the throughput upper bound computation for marked graphs are extended to live and safe free choice net case. We propose a decomposition in terms of marked graph components induced by minimal T-invariants in order to compute a reachable throughput upper bound for a live and safe free choice net. Such bound is expressed as linear combination of throughput upper bounds of the marked graph components induced by minimal T-invariants.

Keywords – Free choice nets, performance evaluation, throughput upper bound, T-invariants, net decomposition technique.

1 Introduction

In this paper we consider a subclass of Petri nets called Free Choice Nets (FCNs). The main feature of this subclass is that every arc from a place is either a unique outgoing arc or a unique incoming arc to a transition. We suppose that a timed activity is associated to each transition; in order to avoid the coupling between resolution of conflicts and duration of activities, transitions in conflict are supposed to be immediate and the conflict is solved according to routing rates associated to each subset of conflicting transitions (generalized stochastic Petri nets [1]). The problem to determine at steady state a reachable upper bound for the throughput of a transition of this net subclass is addressed. The bound computation requires less computation effort with respect to the exact solution. FCNs model not only concurrency and synchronization of activities as the Marked Graph (MG) subclass, for which the bound computation has been solved [2, 3], but also decisions. We restrict our analysis to live and safe FCNs. This net subclass [4] admits a cyclic behavior and thus the steady state performance makes sense. Furthermore, a live a safe FCN can be viewed as an interconnection of live and safe MGs [4, 5], allowing to extend previous results on the throughput upper bound computation for MGs to live and safe FCN case.

The key point of the approach followed in this paper is to consider a special FCN decomposition in terms of MG-components. If a live and safe FCN net exhibits a cyclic behavior, we prove that a T-invariant can be associated with the firing count vector of a firing cyclic sequence and viceversa. In addition, any T-invariant can be expressed as linear combination of minimal T-invariants and to each minimal T-invariant a MG-component can be associated. For safe FCNs, any MG-component can be activated, i.e. there exists a reachable marking under which the MG-component is live. Starting from these considerations, we express any cyclic sequence in terms of a number of MG-component activations and so we obtain a reachable throughput bound for a live and safe FCN from the MG-component bounds. We also provide a procedure to determine the number of activations at steady state of each MG-component by using routing rates.

When the net safeness is lost, the simultaneous firing of two conflicting transitions is possible. In this case the routing rate satisfaction is not sufficient to express the cyclic behavior of the FCN in terms of MG-components and the obtained bound is not reachable.

The paper is organized as follows. In Section 2 we provide a short background on Petri nets. In Section 3 the decomposition of a live and safe FCN in terms of MG-components induced by minimal T-invariants is discussed. In Section 4 a reachable throughput upper bound for live and safe FCN is devised and some considerations about the non-safe case are provided. In addition a manufacturing example is presented as case study. Conclusions are finally drawn in Section 5.

2 Background

A place/transition (P/T) net is a structure $N = \langle P, T, \mathbf{Pre}, \mathbf{Post}\rangle$ where: P is a set of n *places* represented by circles; T is a set of m *transitions* represented by bars; $P \cap T = \emptyset$, $P \cup T \neq \emptyset$; $\mathbf{Pre}$ ($\mathbf{Post}$) is the $\mid P \mid \times \mid T \mid$ sized, natural valued, pre-(post-)incidence matrix. For instance, $\mathbf{Pre}(p,t) = w$ ($\mathbf{Post}(p,t) = w$) means that there is an arc from p (t) to t (p) with weight w. A P/T net is called *ordinary* if all of its arc weights are 1's. For pre- and post-sets we use the conventional dot notation, e.g. ${}^{\bullet}t = \{p \in P \mid \mathbf{Pre}(p,t) \neq 0\}$. The incidence matrix $\boldsymbol{C}$ of the net is defined as $\boldsymbol{C} = \mathbf{Post} - \mathbf{Pre}$. A *marking* is a $m \times 1$ vector $\boldsymbol{m} : P \to \mathbb{N}$ that assigns to each place of a P/T net a non-negative integer number of tokens. A P/T *system* or *net system* $\langle N, \boldsymbol{m}_0\rangle$ is a P/T net N with an initial marking $\boldsymbol{m}_0$. A transition $t \in T$ is enabled at a marking $\boldsymbol{m}$ iff $\boldsymbol{m} \geq \mathbf{Pre}(\cdot,t)$. If t is enabled, then it may fire yielding a new marking $\boldsymbol{m}' = \boldsymbol{m} + \mathbf{Post}(\cdot,t) - \mathbf{Pre}(\cdot,t) = \boldsymbol{m} + \mathbf{C}(\cdot,t)$. The notation $\boldsymbol{m}[t > \boldsymbol{m}'$ means that an enabled transition t may fire at $\boldsymbol{m}$ yielding $\boldsymbol{m}'$. A *firing sequence* from $\boldsymbol{m}_0$ is a (possibly empty) sequence of transitions $\sigma = t_1...t_k$ such that $\boldsymbol{m}_0[t_1 > \boldsymbol{m}_1[t_2 > \boldsymbol{m}_2..[t_k > \boldsymbol{m}_k$. A marking $\boldsymbol{m}$ is reachable in $\langle N, \boldsymbol{m}_0\rangle$ iff there exists a firing sequence σ such that

$\boldsymbol{m}_0[\sigma > \boldsymbol{m}$. Given a net system $\langle N, \boldsymbol{m}_0 \rangle$ the set of reachable markings is denoted $R(N, \boldsymbol{m}_0)$. The function $\boldsymbol{\sigma} : T \rightarrow \mathbb{N}$, where $\boldsymbol{\sigma}(t)$ represents the number of occurrences of t in σ, is called firing count vector of the fireable sequence σ. If $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$, then we can write in vector form $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$. This is known as the *state equation* of the system.

Right annuller vectors of $\boldsymbol{C}$ are called *T-invariants* (i.e. $\boldsymbol{x} : T \rightarrow \mathbb{N}$, $\boldsymbol{x} \neq \boldsymbol{0} \mid \boldsymbol{C} \cdot \boldsymbol{x} = 0$). The *support* of a T-invariant $\boldsymbol{x}$ is defined as $\parallel \boldsymbol{x} \parallel = \{t \in T \mid \boldsymbol{x}(t) > 0\}$. A T-invariant $\boldsymbol{x}$ has a *minimal support* iff there exists no other invariant $\boldsymbol{x}'$ such that $\parallel \boldsymbol{x}' \parallel \subset \parallel \boldsymbol{x} \parallel$. A T-invariant is *canonical* iff the greatest common divisor of its components is 1. A T-invariant is said to be *minimal* iff it is canonical and has a minimal support. A T-invariant $\boldsymbol{x}$ is said to be positive iff $\boldsymbol{x} > \boldsymbol{0}$.

An ordinary net N is a Marked Graph (MG) if ${}^\bullet p = p^\bullet = 1, \forall p \in P$. An ordinary net N is a Free Choice Net (FCN) if $\forall p \in P, |p^\bullet| \leq 1$ or ${}^\bullet\{p^\bullet\} = \{p\}$. A P/T system is *live* when, from every reachable marking, every transition can ultimately occur. A P/T system is *safe* if $\boldsymbol{m}(p) \leq 1 \; \forall p \in P$.

3 A characterization of cyclic sequence firing count vector in terms of minimal T-invariants

In this section we first recall some previous results and then we prove that the cyclic behaviour of a live and safe free choice net can be characterized in terms of its minimal T-invariants.

Definition 1 *Let N' be the subnet of a net N generated by a non-empty set Σ of nodes. N' is a* MG-component *of N if:*

- *$\forall t \in \Sigma$, ${}^\bullet t \cup t^\bullet \subseteq \Sigma$, and*
- *N' is a strongly connected MG.*

Theorem 1 [5] *If a live free-choice net system $\langle N, \boldsymbol{m}_0 \rangle$ is safe then N is covered by strongly connected MG components and there is a marking $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ such that each MG component $\langle N_1, \boldsymbol{m}_1 \rangle$ is a live and safe MG, where $\boldsymbol{m}_1$ is $\boldsymbol{m}$ restricted to N_1.*

Let us consider a minimal T-invariant $\boldsymbol{x}$, we say that the subnet generated by ${}^\bullet \parallel \boldsymbol{x} \parallel \cup \parallel \boldsymbol{x} \parallel \cup \parallel \boldsymbol{x} \parallel^\bullet$ is *induced* by minimal T-invariant $\boldsymbol{x}$.

Theorem 2 [4] *Let N be a live and bounded free choice net.*

1. *Minimal T-invariants induce MG-components.*
2. *MG-components induce minimal T-invariants.*

Let $\langle N, \boldsymbol{m}_0 \rangle$ be a system. A marking $\boldsymbol{m}$ is a *home marking* of $\langle N, \boldsymbol{m}_0 \rangle$ iff $\boldsymbol{m} \in R(N, \boldsymbol{m}'), \forall \boldsymbol{m}' \in R(N, \boldsymbol{m}_0)$.

Corollary 1 *Given a live and safe free-choice net system $\langle N, \boldsymbol{m}_h \rangle$ where $\boldsymbol{m}_h$ is a home marking, each MG component $\langle N_1, \boldsymbol{m}_1 \rangle$ is a live and safe MG, where $\boldsymbol{m}_1$ is $\boldsymbol{m}_h$ restricted to N_1.*

Proof: Given a net system $\langle N, \boldsymbol{m}_0 \rangle$ each sequence σ such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}_0$ implies that $\boldsymbol{\sigma}$ is a T-invariant, that is a linear combination of minimal T-invariants. From theorem 2 it follows that each minimal T-invariant induces a MG-component. Theorem 1 proves the existence of a reachable marking under which all MG-components are live. If $\boldsymbol{m}_0$ is a home marking, each sequence σ such that $\boldsymbol{m}_0[\sigma >$ can always be completed with a sequence σ' in order to get $\boldsymbol{m}_0[\sigma\sigma' > \boldsymbol{m}_0$. Therefore all MG-components have to be live. ■

From now on we assume that n_T is the number of minimal T-invariants.

Theorem 3 [4] *Live and bounded free-choice systems have home markings.*

In the following theorem we prove that in a live and bounded free-choice systems any T-invariant can be associated with a fireable cyclic firing sequence, i.e. a sequence σ such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}_0$, assuming $\boldsymbol{m}_0$ a home marking. Such invariant can be expressed as linear combination of minimal T-invariants.

Theorem 4 *Let $\langle N, \boldsymbol{m}_0 \rangle$ be a live and safe free choice system and let $\boldsymbol{m}_0$ be a home marking. $\boldsymbol{y}$ is a T-invariant of $\langle N, \boldsymbol{m}_0 \rangle$ if and only if there exists a cyclic firing sequence σ such that $\boldsymbol{\sigma} = \boldsymbol{y}$. Moreover, σ includes a number of α_i firings of all the transitions that belong to the minimal T-invariant x_i such that $\boldsymbol{y} = \sum_{i=1}^{n_T} \alpha_i \boldsymbol{x}_i$.*

Proof: ($\Rightarrow$) From theorem 2 it follows that all minimal T-invariants have non-zero elements equal to one, since the unique T-invariant of each MG-component is a vector having all elements equal to one. Moreover, from corollary 1 all transitions of each MG-component will be enabled. Since any T-invariant $\boldsymbol{y}$ can be expressed as linear combination of minimal T-invariants, i.e. $\boldsymbol{y} = \sum_{i=1}^{n_T} \alpha_i \boldsymbol{x}_i$, it follows that, if $\boldsymbol{m}_0$ is a home marking, there exists a sequence σ such that $\boldsymbol{\sigma} = \boldsymbol{y}$. The firing of σ implies that any MG-component induced by a minimal T-invariant fires α_i times, being one the non-zero elements of $\boldsymbol{x}_i$.

($\Leftarrow$) If σ is a cyclic sequence, then $\boldsymbol{\sigma}$ is a T-invariant directly by definition. ■

4 A reachable throughput upper bound for transitions of live and safe FCNs

From now on we consider FCN systems $\langle N, \boldsymbol{m}_0 \rangle$ where $\boldsymbol{m}_0$ is a home marking.

4.1 Preliminary concepts

The introduction of a timing specification is essential in order to use Petri net models for performance evaluation of distributed systems. We consider nets with deterministically or stochastically timed transitions with one phase firing rule, i.e. a timed enabling (called the service time of the transition) followed by an atomic firing. The service time of transitions are supposed to be mutually independent and time independent.

Let us denote by $\overline{\boldsymbol{m}}$ and $\boldsymbol{\sigma}^*$ the *limit average marking* and the *limit vector of transition throughputs* defined as follows

$$\overline{\boldsymbol{m}} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \boldsymbol{m}_u \, du \quad (1)$$

$$\boldsymbol{\sigma}^* = \lim_{\tau\to\infty} \frac{\boldsymbol{\sigma}_\tau}{\tau} \quad (2)$$

The existence of the limits $\overline{\boldsymbol{m}}$ and $\boldsymbol{\sigma}^*$, which is called *weak ergodicity* of the marking and firing processes, is assured for live and bounded free choice nets as proved in the following theorem.

Theorem 5 **[7]** *Let $\langle N, \boldsymbol{m}_0\rangle$ be a live and bounded free choice net with deterministic or stochastic service times of transitions. Then, both the marking and the firing processes of $\langle N, \boldsymbol{m}_0\rangle$ are weakly ergodic.*

Theorem 4 let us to express the limit vector of transition throughputs $\boldsymbol{\sigma}^*$ as follows

$$\boldsymbol{\sigma}^* = \lim_{\tau\to\infty} \frac{\boldsymbol{\sigma}_\tau}{\tau} = \lim_{\tau\to\infty} \frac{\sum_{i=1}^{n_T} \alpha_i(\tau)\boldsymbol{x}_i}{\tau} = \sum_{i=1}^{n_T} [\lim_{\tau\to\infty} \frac{\alpha_i(\tau)}{\tau} \cdot \boldsymbol{x}_i] = \sum_{i=1}^{n_T} \overline{\alpha_i} \cdot \boldsymbol{x}_i$$

where $\overline{\alpha_i} = \lim_{\tau\to\infty} \frac{\alpha_i(\tau)}{\tau}$ is the average number of occurrences of the i-th MG-component MG_i associated to the minimal T-invariant $\boldsymbol{x}_i$. Notice that we are assuming that all cyclic sequences have finite length.

Now we briefly recall some results presented in [8].

In order to avoid the coupling between resolution of conflicts and duration of activities, we suppose that transitions in conflict are taken according to routing rates associated to immediate transitions (generalized stochastic Petri nets). In other words, $T' = \{t_1, ..., t_k\} \subset T$ that are in *generalized free (or equal) conflict* (i.e. having equal pre-incidence function: $\mathbf{Pre}(., t_1) = ... = \mathbf{Pre}(., t_k)$) are considered *immediate* (they fire in zero time), and the constants $r_1, ..., r_k \in \mathbb{N}^+$ (*routing rates*) are explicitly defined in the net interpretation in such a way that when $t_1, ..., t_k$ are enabled, transition $t_i \in T'$ fires with probability (or with long run rate, in the case of deterministic conflicts resolution policy) $r_i/(\sum_{j=1}^k r_j)$.

We can restrict to deterministic resolution policies, which for safe FCNs give the same performance than any probabilistic routing, in steady state.

The vector of *visit ratios*, normalized to transition t_j, is defined as follows

$$\boldsymbol{v}_{(j)} = \frac{1}{\boldsymbol{\sigma}^*(t_j)}\, \boldsymbol{\sigma}^* = \Gamma_{(j)}\, \boldsymbol{\sigma}^* \quad (3)$$

where $\Gamma_{(j)}$ is called the *mean interfiring time* of t_j, that is the inverse of its throughput.

Theorem 6 **[7]** *For any net system, a lower bound for the mean interfiring time $\Gamma_{(j)}$ of transition t_j can be computed by solving the following linear programming problem*

$$\Gamma_{(j)} \geq \quad max_{\boldsymbol{y}} \quad \boldsymbol{y}^T \cdot \mathbf{Pre} \cdot \boldsymbol{D}_{(j)} \qquad s.t. \quad \begin{cases} \boldsymbol{y}^T \cdot \boldsymbol{C} = 0 \\ \boldsymbol{y}^T \cdot \boldsymbol{m}_0 = 0 \\ \boldsymbol{y} \geq \mathbf{0} \end{cases} \quad (4)$$

where $\boldsymbol{D}_{(j)}$ is a vector defined as:

$\boldsymbol{D}_{(j)}(t_k) = \boldsymbol{v}_{(j)}(t_k) \cdot \boldsymbol{d}(t_k)$, being $\boldsymbol{d}(t_k)$ the time delay of transition t_k.

For a live and bounded marked graph, the bound derived from theorem 6 has been shown to be reachable [7] under the earliest firing policy, otherwise it is a lower bound; it is the same for all transitions and so we can speak of lower bound of the MG-component mean interfiring time. However, this is not true in the case of FCN systems.

For a live and bounded free choice nets, $\boldsymbol{v}_{(j)}$ can be computed in polynomial time, from the net structure and routing rates at conflicts, as follows

- the vector of visit ratios must be a right annuller of the incidence matrix, i.e.

$$\boldsymbol{C} \cdot \boldsymbol{v}_{(j)} = \mathbf{0}; \quad (5)$$

- the components of $\boldsymbol{v}_{(j)}$ must verify the following relations with respect to the routing rates for each subset of transitions $T' = \{t_1, ..., t_k\} \subset T$ in structural conflict

$$\begin{aligned} r_2\boldsymbol{v}_{(j)}(t_1) - r_1\boldsymbol{v}_{(j)}(t_2) = 0 \\ r_3\boldsymbol{v}_{(j)}(t_2) - r_2\boldsymbol{v}_{(j)}(t_3) = 0 \\ \cdots \\ r_k\boldsymbol{v}_{(j)}(t_{k-1}) - r_{k-1}\boldsymbol{v}_{(j)}(t_k) = 0. \end{aligned} \quad (6)$$

The above homogeneous system of equations can be expressed in a matrix form: $\boldsymbol{R}_{T'} \cdot \boldsymbol{v}_{(j)} = 0$, where $\boldsymbol{R}_{T'}$ is a $(k-1)\times m$ matrix. Now, by considering all structural conflict sets $T_1, ..., T_r$, it follows that $\boldsymbol{R} \cdot \boldsymbol{v} = 0$, where R is a $\delta \times m$ matrix (δ is the number of independent relations fixed by the routing rates)

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{T_1} \\ \vdots \\ \boldsymbol{R}_{T_r} \end{pmatrix} \quad (7)$$

Theorem 7 **[9]** *Let $\langle N, \boldsymbol{m}_0\rangle$ be a live and bounded free choice net. Let $\boldsymbol{C}$ be the incidence matrix of N, and $\boldsymbol{R}$ the matrix previously defined. Then, the vector of visit ratios $\boldsymbol{v}_{(j)}$ normalized, for instance, for transition t_j, can be computed from $\boldsymbol{C}$ and $\boldsymbol{R}$ by solving the following linear system of equations*

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{R} \end{pmatrix} \cdot \boldsymbol{v}_{(j)} = \mathbf{0}, \quad \boldsymbol{v}_{(j)}(t_j) = 1. \quad (8)$$

4.2 Computation of $\Gamma_{(j)}$ in a live and safe free choice net

Let $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_{n_T}\}$ be the set of minimal T-invariants on N and $\boldsymbol{X} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ ...\ \boldsymbol{x}_{n_T}]$ a block matrix built up by placing vectors $\boldsymbol{x}_i$ side by side. Of course $(\boldsymbol{C} \cdot \boldsymbol{x}_i = 0)_{i=1,...,n_T} \Rightarrow \boldsymbol{C} \cdot \boldsymbol{X} = \mathbf{0}$.

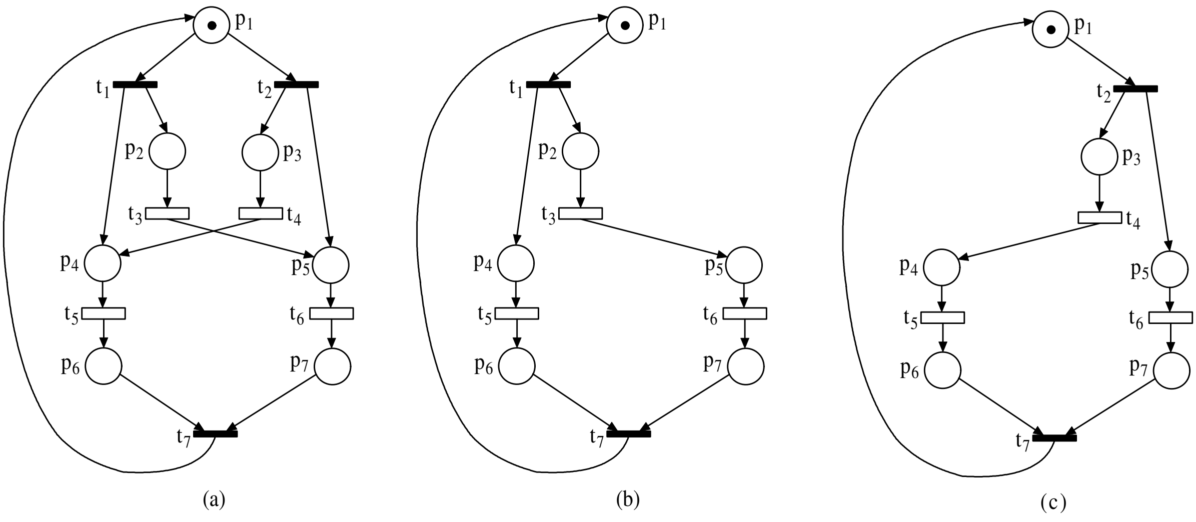

Figure 1: A live and safe free choice system (a) and its MG-components MG_1 (b) and MG_2 (c).

Since $\boldsymbol{\sigma}^* = \sum_{i=1}^{n_T} \overline{\alpha_i} \cdot \boldsymbol{x}_i = \boldsymbol{X} \cdot \overline{\boldsymbol{\alpha}}$, where $\overline{\boldsymbol{\alpha}} = [\overline{\alpha_1}, ..., \overline{\alpha_{n_T}}]^T$, it follows that

$$\boldsymbol{v}_{(j)} = \Gamma_{(j)}\ \boldsymbol{\sigma}^* = \Gamma_{(j)}\ \boldsymbol{X} \cdot \overline{\boldsymbol{\alpha}}.$$

From (8)

$$\Gamma_{(j)} \left(\begin{array}{c} \boldsymbol{C} \\ \boldsymbol{R} \end{array} \right) \cdot \boldsymbol{X} \cdot \overline{\boldsymbol{\alpha}} = \boldsymbol{0} \Longrightarrow \boldsymbol{R} \cdot \boldsymbol{X} \cdot \overline{\boldsymbol{\alpha}} = \boldsymbol{0}. \tag{9}$$

Although the $\overline{\alpha_i}$ terms have been defined with respect to an infinite time horizon, they can be computed considering a finite time horizon of length Δ after some considerations. We consider live and safe FCNs $\langle N, \boldsymbol{m}_0 \rangle$, where $\boldsymbol{m}_0$ is a home marking, which have finite length sequences σ such that $\boldsymbol{R} \cdot \boldsymbol{\sigma} = \boldsymbol{0}$, i.e. all routing rates are met; in order to compute the $\overline{\alpha_i}$ terms, the shortest of such sequences, called σ_{min}, can be considered. It follows that

$$\overline{\alpha_i} = \frac{q_i}{\Delta} = \frac{q_i}{\sum_{i=1}^{n_T} q_i \Gamma^{LB(i)}} \tag{10}$$

where Δ is the time duration of σ_{min}, q_i are defined as the number of activations of the i-th MG-component MG_i in σ_{min} and $\Gamma^{LB(i)}$ is the lower bound of MG_i mean interfiring time.

Directly from the definition of the q_i terms, it follows that they can be computed by solving the following IPP

$$\begin{array}{ll} min_{q_i} & \sum_{i=1}^{n_T} q_i \\ s.t. & \left\{ \begin{array}{l} (\boldsymbol{R} \cdot \boldsymbol{X}) \cdot \boldsymbol{q} = \boldsymbol{0} \\ \sum_{i=1}^{n_T} q_i \geq n_T \\ q_i \in \mathbb{N} \\ q_i \geq 1, \quad i = 1..n_T \end{array} \right. \end{array} \tag{11}$$

where q_i is the i-th component of vector $\boldsymbol{q}$.

4.3 Main Result

Theorem 8 *Let $\langle N, \boldsymbol{m}_0 \rangle$ be a live and safe free choice system. Let m_h be a home marking of $\langle N, \boldsymbol{m}_0 \rangle$. Let $\mathbb{X} = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_{n_T}\}$ be the set of minimal T-invariants on N. Let $t_j \in T$ be a transition whose mean interfiring time $\Gamma_{(j)}$ has to be calculated. Let us define $\mathcal{I}_{(j)} = \{i \in \{1, ..., n_T\} \mid t_j \in \|\boldsymbol{x}_i\|,\ \boldsymbol{x}_i \in \mathbb{X}\}$. A reachable lower bound $\Gamma_{(j)}^{LB}$ for the mean interfiring time $\Gamma_{(j)}$ of transition t_j can be computed as follows*

$$\Gamma_{(j)} \geq \Gamma_{(j)}^{LB} = \frac{1}{\sum_{k \in \mathcal{I}_{(j)}} q_k} \sum_{i=1}^{n_T} q_i \Gamma^{LB(i)} \tag{12}$$

where the q_i terms are solutions of the IPP (11) and $\Gamma^{LB(i)}$ is the lower bound of MG_i mean interfiring time.

Proof: Since $\boldsymbol{x}_k(t_j) \in \{0, 1\}$, if we define $\mathcal{I}_{(j)} = \{k \in \{1, ..., n_T\} \mid t_j \in \|\boldsymbol{x}_k\|,\ \boldsymbol{x}_k \in \mathbb{X}\}$, it follows that

$$\boldsymbol{\sigma}^*(t_j) = \sum_{k=1}^{n_T} \overline{\alpha_k} \cdot \boldsymbol{x}_k(t_j) = \sum_{k \in \mathcal{I}_{(j)}} \overline{\alpha_k}$$

$$\Gamma_{(j)} = \frac{1}{\boldsymbol{\sigma}^*(t_j)} = \frac{1}{\sum_{k \in \mathcal{I}_{(j)}} \overline{\alpha_k}}$$

From (10) we obtain

$$\Gamma_{(j)}^{LB} = \frac{1}{\sum_{k \in \mathcal{I}_{(j)}} q_k} \sum_{i=1}^{n_T} q_i \Gamma^{LB(i)}$$

and thus, depending on the firing policy adopted, we have that $\Gamma_{(j)}$ will be greater or equal then $\Gamma_{(j)}^{LB}$.

∎

Of course, a reachable upper bound for the throughput of t_j is $1/\Gamma_{(j)}^{LB}$.

Example 1 *Let us consider the system depicted in fig. 1(a). Let d_3, d_4, d_5, d_6 and d_7 be the mean service times associated with t_3, t_4, t_5, t_6 and t_7, respectively. Let t_1 and t_2 be* immediate *transitions (i.e. they fire in zero time). Let r_1 and r_2 be the routing rates defining the resolution of conflict at place p_1. Minimal*

T-invariants are

$$\boldsymbol{x}_1 = \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 1 & 1\end{bmatrix}^T$$
$$\boldsymbol{x}_2 = \begin{bmatrix}0 & 1 & 0 & 1 & 1 & 1 & 1\end{bmatrix}^T.$$

Fig. 1(b),(c) shows MG-components MG_1 and MG_2 induced by T-invariants $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively.

By applying theorem 8, a reachable lower bound for the mean interfiring time of t_7 can be computed as follows

$$\begin{aligned} q_1 = r_1, \quad q_2 = r_2;\\ q'_1 = \frac{r_1}{r_1 + r_2}, \quad q'_2 = \frac{r_2}{r_1 + r_2}; \end{aligned}$$

$$\begin{aligned} \Gamma^{LB}_{(7)} = q'_1 \Gamma^{LB(1)} + q'_2 \Gamma^{LB(2)} = \\ q'_1(max\{d_5,\ d_3 + d_6\} + d_7) + \\ (1 - q'_1)(max\{d_6,\ d_4 + d_5\} + d_7) = \\ max\{q'_1 d_3 + d_6,\ (1 - q'_1) d_4 + d_5, \\ q'_1 d_3 + (1 - q'_1) d_4 + (1 - q'_1) d_5 + q'_1 d_6, \\ q'_1 d_5 + (1 - q'_1) d_6\} + d_7. \end{aligned}$$

It is the exact mean interfiring time of t_7 (a reachable lower bound), while by solving the linear programming problem in theorem 6 we obtain the following lower bound

$$max\{(1 - q'_1) d_4 + d_5,\ q'_1 d_3 + d_6\} + d_7.$$

that is clearly lower than $\Gamma^{LB}_{(7)}$.

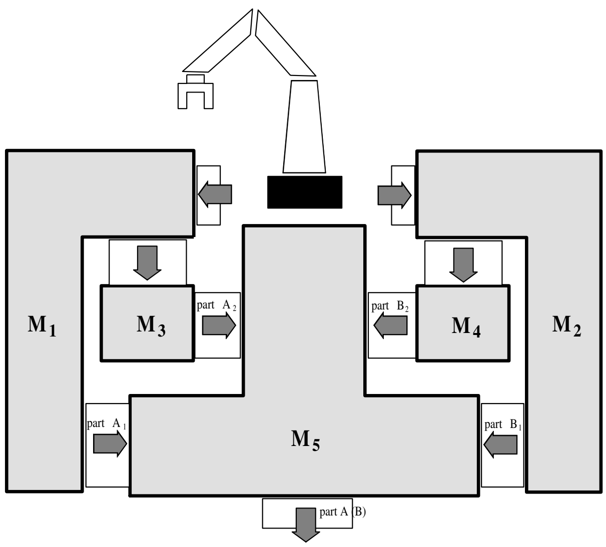

Figure 2: Layout of a manufacturing cell.

Example 2 *Fig. 2 shows the layout of a manufacturing cell. Each arriving raw part can be routed to machine M_1 or machine M_2 (it depends on local policies). Machine M_1 (M_2) produces two parts, one called part A_1 (B_1), and another is a non-terminal part. This last part is loaded into machine M_3 (M_4) which produces a part called A_2 (B_2). Finally, product A_1 and A_2 (B_1 and B_2) are assembled in machine M_5 in order to obtain*

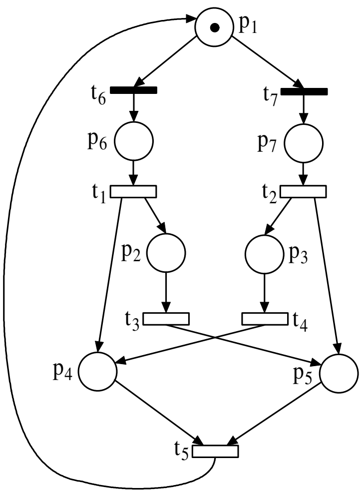

Figure 3: PN of the manufacturing cell.

the final product A (B). Only when the whole process is finished a new raw part can be loaded into machine M_1 or machine M_2. In fig. 3 is shown a Petri net system modelling the manufacturing cell. Notice that it is a live and safe free choice system. Let us assume the following deterministic conflict resolution policy at place p_1: select twice machine M_1, then once machine M_2, and repeat it. Let $d_1 = d_2 = d_3 = 1\ min$, $d_4 = 2\ min$ and $d_5 = 3\ min$ be the mean service times associated with M_1, M_2, M_3, M_4 and M_5, respectively. Let us compute an upper bound for the throughput of M_5. Minimal T-invariants are

$$\boldsymbol{x}_1 = \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix}^T$$
$$\boldsymbol{x}_2 = \begin{bmatrix}0 & 1 & 0 & 1 & 1 & 0 & 1\end{bmatrix}^T.$$

Making use of (12), we can compute $\Gamma^{LB}_{(5)}$ as follows

$$\begin{aligned} q_1 = r_1, \quad q_2 = r_2;\\ q'_1 = \frac{r_1}{r_1 + r_2}, \quad q'_2 = \frac{r_2}{r_1 + r_2}; \end{aligned}$$

$$\begin{aligned} \Gamma^{LB}_{(5)} = q'_1 \Gamma^{LB(1)} + q'_2 \Gamma^{LB(2)} = \\ q'_1(d_1 + d_3 + d_5) + q'_2(d_2 + d_4 + d_5) = \\ q'_1(d_1 + d_3) + (1 - q'_1)(d_2 + d_4) + d_5. \\ q'_1 = \frac{2}{3} \Rightarrow \Gamma^{LB}_{(5)} = 5.333\ min. \end{aligned}$$

An upper bound for the throughput of transition t_5 (machine M_5) is $1/\Gamma^{LB}_{(5)} = 0.1875\ min^{-1}$.

In the non-safe case our approach does not work as explained afterwards. By solving (12) for the net system in fig. 4 we obtain $\Gamma^{LB}_{(5)} = 0.5$, if $r_1/r_2 = 2$ and transitions t_3 and t_4 have time duration equal to one second. On the other hand, the actual mean interfiring time value is $\Gamma^{LB}_{(5)} = 0.409$. This result can be explained by observing that if the net system is not safe, under a net marking more than one conflicting transition may fires or, depending on the routing rates, or a single conflicting transition may fire more than one time. Let us consider the initial marking of the net system in fig. 4. We have that t_1, t_2, t_5 fire in zero time and, depending on the conflict resolution, the marking $\boldsymbol{m}_1 = \begin{bmatrix}0 & 2 & 1 & 1 & 0\end{bmatrix}^T$ or $\boldsymbol{m}_2 = \begin{bmatrix}0 & 1 & 2 & 0 & 1\end{bmatrix}^T$ can be reached. After a time 1 second is spent, t_5 fires two times and $\boldsymbol{m}_0$ is reached again. Finally we have that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}_0$ and three firings of t_5 are included in σ. Our approach do not consider the simultaneous firing of two conflicting transitions, but only two simultaneous firings of the same transitions. This is the reason why the sequence σ described above cannot be considered by our approach. Further research has to be carried on this topic.

5 Conclusions

For a live and safe free choice net it has been proved that any cyclic sequence can be associated with a firing count vector that is a linear combination of minimal T-invariants. Since minimal T-invariant induced subnets are marked graphs, called MG-components, it has been shown that a throughput upper bound of a transition in live and safe free choice nets can be expressed as linear combination of throughput bounds of minimal T-invariant induced MG-components. Reachable throughput upper bounds for marked graphs can be computed via a linear programming problem, and thus a reachable throughput upper bound of a transition in a live and safe free choice net can be derived.

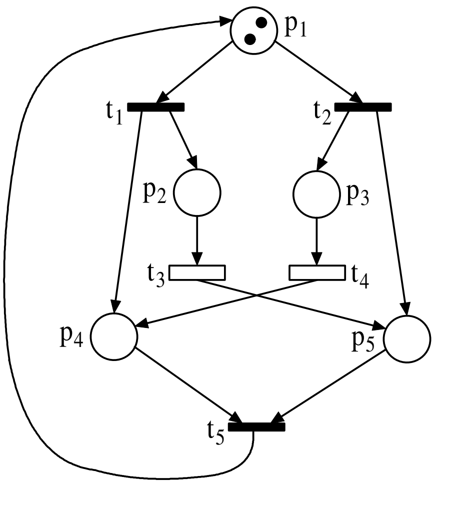

Figure 4: A non-safe FCN system.

References

[1] M. Ajmone Marsan, G. Balbo, G. Conte, S. Donatelli, and G. Franceschinis, *Modelling with Generalized Stochastic Petri Nets*, Wiley Series in Parallel Computing, John Wiley and Sons, 1994.

[2] J. Campos, G. Chiola, J. M. Colom, and M. Silva, "Properties and performance bounds for timed marked graphs," *IEEE Trans. on Circuits and Systems*, vol. 39, no. 5, pp. 386–401, May 1992.

[3] J. Magott, "Performance evaluation of concurrent system using Petri nets," *Information Processing Letters*, vol. 18, pp. 7–13, 1984.

[4] Jörg Desel and Javier Esparza, *Free Choice Petri nets*, Number 40. Cambridge University Press Cambridge Tracts in Theoretical Computer Science, 1995.

[5] Tadao Murata, "Petri nets: Properties, analysis and applications," *Proceedings of IEEE*, vol. 77, no. 4, pp. 541–580, April 1989.

[6] J. Esparza and M. Silva, "On the analysis and synthesis of free choice systems," in *Advances in Petri Nets 1990*, G. Rozenberg, Ed., vol. 483 of *LNCS*, pp. 243–286. Springer-Verlag, Berlin, 1991.

[7] J. Campos, G. Chiola, and M. Silva, "Properties and performance bounds for closed free choice synchronized monoclass queueing networks," *IEEE Trans. on Automatic Control*, vol. 36, no. 12, pp. 1368–1382, December 1991.

[8] Javier Campos and Manuel Silva, "Structural techniques and performance bounds of stochastic Petri net models," in *Advances in Petri Nets 1992*, G.Rozemberg, Ed., vol. 609 of *LNCS*, pp. 353–391. Springer-Verlag, Berlin, 1992.

[9] J. Campos, *Performance Bounds for Synchronized Queueing Networks*, Ph.D. thesis, Departamento de Ingeniería Eléctrica e Informática, Universidad de Zaragoza, Spain, October 1990.