ROBUST HYBRID LQ CONTROLLER

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Abstract

In this paper design of switching controllers for linear systems with analog uncertainty is considered. The controllers are LQ controllers and switching sequence is determined by minimization of suitable defined priority function. The priority function includes switching penalty term which introduces cautiousness in switching discrete state. First in the paper are finded conditions for weighting matrices in the LQ criterion under which nonswitching LQ controller is stable in the presence of uncertainty. After that for system with switching LQ controllers previously defined, the robust asymptotic stability is established.

1. Introduction

The hybrid dynamical system (HDS) is a dynamical system that involves the interaction of discrete and continuous dynamics. Such dynamical systems contain variables that take values form a continuous set (the set of real numbers) and variables that take values from a discrete set (the set of symbols). The field of HDS is now good established discipline [1]-[3]. Well known example of hybrid system is the dynamic system described by a set of ordinary differential equations with discontinuous right-hand sides [5]. Another examples are sliding mode control [5] and sampled-data system [6].

The important part of hybrid system is event driven dynamics which can be described using timed automata, max-plus algebra or Petry nets [7]. In the field of HDS main tool for analysis and design are: representation theory, supervisory control, computer simulation and verfication. From the clasical control theory point of wiew HDS can be interpreted as a switching control between analog feedback loops [8]. One, also can to consider HDS as special form of adaptation [9]. It is possible to distengwish a few different approaches for design of a hybrid controllers. One possible approach is dwelling-time switching strategy [8]. Relatively long dwell-time cause loss of performance. Another possibility is to make state space partition [10] which devide whole state space into a finite set of regions so that design problem can be reduced to finite automata. In that case complexity is very high problem. Finally, design problem of hybrid controller can be put in the frame of system performance [9].

Based on ideas from [9] in [11] the design of switching controllers for linear systems is considered. The controllers are LQ controllers and switching sequence is determined by minimization of suitable defined priority function. In the [12] the system with analog uncertainty is considered. The uncertainty belongs to the compact set. Switching sequence for desired feedback is determined by minimization of nominal peformance and switching penelty term. Control systems under investigation are general nonlinear systems.

In this paper we will consider linear system with analog uncertainty. The uncertain matrices depend continuosly on the uncertainty vector which is Lebesque measurable within an compact set. For such system is proposed robust LQ controller. Using these results we then will construct robust hybrid LQ controller. The switching sequence is determined by minimization of specific functional. Owing the property of functional the switching will be decided only if the worst-case performance of the system for new discrete state is better then the best performance of the system for current discrete sttate. Finally, in the form of theorem robust stabulity of the closed-loop hybrid system is established.

2. Contol systems without unmodeled dynamics

Continuous part of the system without unmodeled dynamic has the form

$$\dot{x}(t) = Ax(t) + \mathrm{B}(t), \tag{1}$$

where $x \in R^n$ and $u \in R^p$ are state and control signal of the systems respectively. Discrete part of the system is descrete event system [7]

$$m^+(t) = \Phi(m(t), \sigma(t)) \tag{2}$$

where $m(t)$ is discrete state variable, $\sigma(t)$ is discrete input and $\Phi(\cdot\,,\cdot)$ is function which described behaviour of $m(t)$. It is important to note that

$$m^+(t) = m(t_{n+1})\ ,\ m(t) = m(t_n), < t_{n+1} \tag{3}$$

Let us denote with $\sum$ set of events (or symbols). For descrete input σ is valid the constraint

$$\sigma(t) \in \sum \tag{4}$$

Systems (1) and (2) are coupled and model for hybrid system has the form [11]

$$\begin{aligned} &\dot{x}(t) = A_m x(t) + B_m u(t) \\ &m^{+}(t) = \Phi(x(t)\ ,\ m(t)\ ,\ u(t)\ ,\ \sigma(t)) \\ &m = 1,2,...,l \end{aligned} \tag{5}$$

Switching sequence m will be described leter

.

Let us consider the first relation of hybrid system (5) for fixed m. Optimal controller for this case is given by minimization of associated performance index [13]

$$J(x(t_0), u(\cdot); t_0) = \int_{t_0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right)dt \tag{6}$$

Optimal controller is determined with algebraic Riccati equation

$$P_m A_m + A_m^T P_m - P_m B_m R^{-1} B_m^T P_m + Q = 0 \tag{7}$$

Optimal analog law of control for fixed m is

$$u(t) = -R^{-1} B_m^T P_m x(t) \tag{8}$$

In the next section we will consider design problem, for fixed m, in the presence of unmodeled dynamics.

3. Robust LQ controller for fixed *m*

In this section a design of a convencional linear quadratic (LQ) state feedback for lonear uncertain system is considered. For fixed m a dynamic system (analog part) will be described in the next form

$$\dot{\hat{x}}(t) = ((A_m + \Delta A_m(\omega(t)))x(t) + (B_m + \Delta B_m(\omega(t))))u(t) \tag{9}$$

where A_m and B_m are the nominal system and input matrices, respectivelly, of appropriate dimensions and $\Delta A_m(\cdot)$ and $\Delta B_m(\cdot)$ are uncertain matrices. The matrices depend continuosly from the uncertainty vector $\omega(t)$ which is Lebesque measurable and within an allowable bounding set $\Omega \in R^p$ for all $t \in [0, \infty]$. The main gool of this section is the construction of weighting matrices Q and R in the criterion (6) so that the robust control can stabilize the system with unmodeled dynamics (9) for all $\omega \in \Omega$. For that purpose we will prove next theorem

Theorem 1. Let the for fixed m for dynamic control system (7)-(9) and weighting matrices Q and R in criterion (6) are valid

1° (A_m, B_m) is controlable

2° $\left(A_m, \sqrt{Q}\right)$ is observable pair

3° $\Omega \in R^p$ is a compact set

4° There are continuous mappings

$$D(\cdot): \omega \to R^{m \times n} \quad , \quad E(\cdot): \omega \to R^{m \times n}$$

such that

$$\begin{aligned} \Delta A_m(\omega) &= B_m D(\omega) \\ \Delta B_m(\omega) &= B_m E(\omega) \end{aligned}$$

for all $\omega \in \Omega$

5° $E(\omega) + E^T(\omega) + I > 0 \quad , \quad \forall \omega \in \Omega$

6° R is the positive definite matrix

7° $Q > D^T(\omega) R D(\omega) / \beta \quad , \quad \forall \omega \in \Omega$

where

$$\beta = \min_{\omega \in \Omega} \lambda_{\min} \left\{ E(\omega) + E^T(\omega) + I \right\} > 0$$

Then the control law (8) stabilize uncertain system (9)

Proof: According with condition 1° and 2° one can conclude [13] that algebraic Riccati equation (7) has a unique positive definite matrix solution P_m. Lyapunov function has the form

$$V(x) = x^T(t) P_m x(t) \tag{10}$$

The first derivative of function $V(x)$ is

$$\begin{aligned} &\dot{V}(x) = \dot{x}(t)^T P_m x(t) + x^T(t) P_m \dot{x}(t) = x(t) \cdot \\ &\left\{ (A_m + B_m D(\omega(t)))^T P_m x(t) + P_m (A_m + B_m D(\omega(t))) \right\} x(t) + \\ &+ u^T(t)(I + E(\omega(t)))^T B_m^T P_m x(t) + x(t) P_m B_m (I + E(\omega(t))) u(t) \end{aligned} \tag{11}$$

Using relation (8) we have

$$\begin{aligned} &u^T(t)(I + E(\omega(t)))^T B_m^T P_m x(t) + x(t) P_m B_m (I + E(\omega(t))) u(t) = \\ &= -u^T(t)(I + E(\omega(t)))^T R u(t) - u^T(t) R (I + E(\omega(t))) u(t) = \\ &= -u^T(t) \left\{ (I + E(\omega(t)))^T R + R (I + E(\omega(t))) \right\} u(t) \end{aligned} \tag{12}$$

Also, by relation (7) and (8) one can get

$$x^T(t)\left\{(A_m + B_m D(\omega(t)))^T P_m x(t) + P_m (A_m + B_m D(\omega(t)))\right\}x(t) =$$
$$= x^T(t)\left\{A_m^T P_m + P_m A_m + D^T(\omega(t)) B_m^T P_m + P_m B_m D(\omega(t))\right\}x(t) =$$
$$= x^T(t)\left\{P_m B_m R^{-1} P_m - Q + D^T(\omega(t)) B_m^T P_m + P_m B_m D(\omega(t))\right\}\cdot$$
$$\cdot\, x(t) = -u(t) B_m^T P_m x(t) + x^T(t) Q x(t) - x^T(t) D^T(\omega(t)) R u(t) +$$
$$+ x^T(t) P_m B_m D(\omega(t)) x(t) = u^T(t) R u(t) - x^T(t) Q x(t) -$$
$$- x^T(t) D^T(\omega(t)) R u(t) - u^T(t) R D(\omega(t)) x(t) \tag{13}$$

From (11)-(13) and adding and subtracting term

$$\frac{x^T(t) D^T(\omega(t)) R D(\omega(t))}{\beta} \tag{14}$$

one can get

$$\dot V(x) = -u^T(t)\left\{R(I + E(\omega(t))) + (I + E(\omega(t)))^T R - (1+\beta)R\right\}\cdot$$
$$u(t) - (D(\omega(t))x(t) + \beta u(t))^T R (D(\omega(t))x(t) + \beta u(t)) / \beta - \tag{15}$$
$$- x^T\left\{\beta Q - D^T(\omega(t)) R D(\omega(t))\right\}x(t)/\beta$$

From relation (15) and assumption 5°-7° of theorem fallows

$$\dot V(x) < 0 \tag{16}$$

It means that uncertain system satisfied the matching conditions 3°-5° of theorem, stabilizable by LQ controller defined by relations (7) and (8). So theorem is proved ▪

Remark 1. In the assumption 4° of theorem the structure of uncertaintes is restriced. That is known in the literature as the matching conditions [14]. This assumption can be relaxed using the measure of mismatching treshold [15].

Remark 2. For above set of uncertaintes in [16] is proved that LQ controller with the prescribed degree of stability can exponentially stabilize uncertain system.

In the next section we will consider hybrid LQ controller with analog uncertaintes in the form described in this section.

4. Robust hybrid LQ controller

In this part of the paper we will consider hibrid LQ controller where

$$m(t) = \in \{1,2,\ldots,l\}$$

is a a picewise constant function of time, called a switching signal. System (9) can be rewritten in the next form

$$\dot x(t) = (A_m + \Delta A_m(\omega(t)))x(t) + (B_m + \Delta B_m(\omega(t)))u(t) =$$
$$= A_m x(t) + B_m u(t) + \Delta(x(t), m(t), u(t), \omega(t)) \tag{18}$$

where

$$\Delta x(t), m(t), u(t), \omega(t) = \Delta A_m x(t) + \Delta B_m(\omega(t))u(t) \tag{19}$$

The nominal trajectory $x*(t)$ of the system (18) is the one governed by the nominal hybrid system

$$\dot x*(t) = A_m x(t) + B_m u(t) \quad , \quad m = 1,2,\ldots,l \tag{20}$$

The nominal performance is given by

$$J_m^*(x(t_0), u(\cdot); t_0) = \int_{t_0}^{\infty} \left(x*^T(t) Q x*(t) + u^T(t) R u(t)\right)dt \tag{21}$$

Modeling error $\Delta(x(t), m(t), u(t), \omega(t))$ perturb the system trajectories from the nominal ones. The corresponding performance value is

$$J_m(x(t_0), u(\cdot); t_0) = \int_{t_0}^{\infty} \left(x^T(t) Q x(t) + u^T(t) R u(t)\right)dt \tag{22}$$

for all $\omega \in \Omega \subset R^p$. As one can see the criterion (22) cannot be determined a priori since it depends from $\Delta(x(t), m(t), u(t), \omega(t))$ and because cannot be used as an index for design. But nominal performance index $J_m(x(t_0), u(\cdot); t_0)$ is aveilable and, also, the worst-case performance error

$$e_m(x(t), t) = \sup_{\omega(t) \in \Omega} \left| J_m(x(t_0), u(\cdot), t_0) - J_m^*(x(t_0), u(\cdot), t_0)\right| \tag{23}$$

is aveilable (in the information sense [17]). That means that the computational issues is very important topic for future research and will not be considered here.

Now we can formulate hybrid control law.

A) The analog feedback

$$P_m A_m + A_m^T P_m - P_m A_B R^{-1} B_m^T P_m + Q = 0 \tag{24}$$

$$u(t) = -R^{-1} B_m^T P_m x(t) \tag{25}$$

B) The discrete feedback: The priority function is defined in the next form (at time t_+ the new discrete state m_1 is determuned as in the next formula)

$$m_1 = \arg\min\left\{J_{m_1}^*(x(t_0), u(\cdot); t_0) + SP(m, m_1)\right\} \tag{26}$$

where swithching penalty term $SP(m, m_1)$ is defined as

$$SP(m, m_1) = \begin{cases} 0 & , m_1 = m \\ e_m(x(t_+)) + e_{m_1}(x(t_+)) + \varepsilon_0 & , m_1 \neq m \end{cases} \tag{27}$$

where $\varepsilon_0 > 0$ is a constant.

From the last relation follows that switching will be decided only if the worst case performance for the system m_1 is better than the best performance of the system m. The switching penalty term will impose cautiousness in switching discrete states [9]

In the following theorem we will prove robust asymptotic stability of the closed-loop hybrid system. Assumptions will be explained after the proof of theorem.

Theorem 2. Suppose that for hybrid dynamic system (18), (24)-(27) are valid

A) $$\left\|\left(A_m - B_m R^{-1} B_m^T P_m\right)x(t)\right\| \le$$
$$\le k_1 x^T(t)\left(Q + P_m B_m R^{-1} B_m^T P_m\right)x(t) + c$$
$$k_1 > 0 \ , \ c > 0 \ , \ \left\|\left(A_m - B_m R^{-1} B_m^T P_m\right)x(t)\right\| \ge 1$$
$$m = 1,2,\dots,l$$

B) $$\|x(t)\| \le k_2 x^T(t)\left(Q + P_m B_m R^{-1} B_m^T P_m\right)x(t)$$
$$k_2 > 0 \ , \ \|x(t)\| \ge 1 \ , \ m = 1,2,\dots,l$$

Then

$$\|x\|_\infty \le (k_1 + k_2) q_J + r_{\Delta_1} + c + 2$$

where

$$q_J = \min\left[J_m^*(x(0),0) + e_m(x(0),0)\right]$$

$$r_{\Delta_1} = \sup_{\omega(t)\in\Omega} \left\|\Delta_1(x(t), m(t), \omega(t))\right\|$$

$$\Delta_1(x(t), m(t), \omega(t)) = \Delta\left(x(t), m(t), -R^{-1} B_m^T P_m x(t), \omega(t)\right) \ ▪$$

Proof: From relation (18)-(25) for any $\tau \in [t, t+1]$ we have

$$x(t) = x(\tau) - \int_t^\tau \left(A_m - B_m R^{-1} B_m^T P_m\right)x(\theta)d\theta + \int_t^\tau \Delta_1(x(\theta), m(\theta), \omega(\theta))d\theta \quad (28)$$

Further follows

$$\|x(t)\| \le \|x(\tau)\| + \int_t^\tau \left\|\left(A_m - B_m R^{-1} B_m^T P_m\right)x(\theta)\right\|d\theta + \int_t^\tau \left\|\Delta_1(x(\theta), m(\theta), \omega(\theta))\right\|d\theta \quad (29)$$

Let us introduce the next sets

$$\Omega_1 = \left\{\theta \in [t, t+1] : \left\|\left(A_m - B_m R^{-1} B_m^T P_m\right)x(t)\right\| < 1\right\} \quad (30)$$

$$\Omega_2 = \tau \in [t, t+1] - \Omega_1$$

Now from (29) and condition A) of theorem follows

$$\begin{aligned} \|x(t)\| &\le \|x(\tau)\| + \int_{\Omega_1} d\theta + \\ &+ \int_{\Omega_2} \left[k_1 x^T(\theta)\left(Q + P_m B_m R^{-1} B_m^T P_m\right)x(\theta) + c\right]d\theta + \\ &\int_{\Omega} \left\|\Delta_1(x(\theta), m(\theta), \omega(\theta))\right\|d\theta \le \\ &\le \|x(\tau)\| + 1 + c + k_1 J(x(t), t) + r_{\Delta_1} \end{aligned} \quad (31)$$

According to definition of unmodeled dynamic (Theorem 1) r_{Δ_1} is bounded value.

Noting that from assumption B) of theorem one can conclude

$$\|x(\tau)\| \le 1 + k_2 x^T(t)\left(Q + P_m B_m R^{-1} B_m^T P_m\right)x(t) \ , \ \forall x \quad (32)$$

From form of the control law (24)-(27) follows that swithching from curent discrete state m to the new m_1 is possible when the nominal performance level

$$J_{m_1}^*(x(t_+), t_+) \quad (33)$$

is better than

$$J_{m_1}^*(x(t_+), t_+) - e_{m_1}(x(t_+), t_+) - e_m(x(t_+), t_+) \quad (34)$$

Namely

$$J_{m_1}^*(x(t_+), t_+) + e_m(x(t_+), t_+) + e_{m_1}(x(t_+), t_+) + \varepsilon < \\ < J_{m_1}^*(x(t_+), t_+) \quad (35)$$

From last relation fllows

$$J_{m_1}^*(x(t_+), t_+) + e_{m_1}(x(t_+), t_+) < J_{m_1}^*(x(t_+), t_+) - \\ - e_m(x(t_+), t_+) \quad (36)$$

It means that switching will occur only if the worst-case performance for the system m_1 is better then the best performance of the sistem *m*. From (36) follows that the resulting performance of the hybrid control system is bounded by the robust worst-case performance of non-switching control system, i.e.

$$J(x(t),t) \le \min_{m \in m(t)} \left[J^*_{m_1}(x(0),0) + e_m(x(0),0) \right] = q_J \quad (37)$$

From the relation (31) and (37) one can get

$$\|x(t)\| \le \int_t^{t+1} \left(\|x(\theta)\| + 1 + c + k_1 q_J + r_{\Delta_1} \right) d\theta \le$$

$$\le \int_t^{t+1} \left(1 + k_2 x^T(\theta) \left(Q + P_m B_m R^{-1} B_m^T P_m \right) x(\theta) \right) d\theta + \quad (38)$$

$$+ 1 + c + k_1 q_J + r_{\Delta_1} \le k_2 q_J + 2 + c + k_1 q_J + r_{\Delta_1} =$$

$$= (k_1 + k_2) q_J + 2 + c + r_{\Delta_1}$$

Since the right-hand side is independent of t we have

$$\|x\|_\infty \le (k_1 + k_2) q_J + 2 + c + r_{\Delta_1} \quad (39)$$

Theorem is proved ▪

Remark 3. In this remark we will comment the assumption A) and B) of Theorem 2. It is well known fact that optimally designed controllers via Riccati equations always guarantee stability. That fact suggest that if system performance indices are appropriately selected, optimality of performance or boundedness of performance, can provide stability and robustness. Such idea is used in this paper. According with the general definition of performance dominant conditions [9] for linear system (considered in this paper) without the unmodeled dynamic (ralation (5)) and index of performance (ralation (6)), we have

$$A_m x(t) + B_m u(t) \le k_1 \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) + c \quad (40)$$

$$\|x(t)\| \le k_2 \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) \quad (41)$$

From the first relation follows that any finite escaping of states will show up in the performance measure and from the second relation follows that large persistent $x(t)$ values must be detected in index of performance avoiding the situation in which the tail of $x(t)$ remains large even when the index performance is bounded.

Using control law (relation (25) and relation (40) and (41)) we easy can get assumptions A) and B) of Theorem 1.

Remark 4. The optimal performance for quadratic performance index

$$J_m(x(t),u(t),t) = \int_t^\infty \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt$$

is given with

$$J_m(x(t)) = \frac{1}{2} x^T(t) K_m x(t)$$

In that case, the worst-case performance errors has the form

$$e_m(x(t)) = \sup_{\omega(t) \in \Omega} \left| x^T(t) K_m x(t) - J^*_m(x(t)) \right|$$

From last relation follows corresponding changes in relations (26) and (27).

The theory of hybrid control systems is powerful tool for complex system. Recently, such kind of control strategy is used for distributed control systems (control over networks [18] and [19]). Also, one can use the hybrid systems for design the quantized fedback law (possibility of making discrete on-line adjustment of quintizer parameters [20]).

5. Conclusions

In this paper the problem of design of robust hybrid LQ controller is considered. The main motivation for such type of controllers is performance improvement of feedback system. In practice exist system which impossible to control with the controllers with fixed structure. Proposed switching controller, in this paper, in the presence of analog unmodeled dynamic guarantee robust stability of feedback system. Further invistegations is directed to the case when in the hybrid system description exists discrete uncertaintes.

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