FEEDBACKS FOR NON–AUTONOMOUS REGULAR LINEAR SYSTEMS

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Abstract

We introduce non–autonomous well–posed and (absolutely) regular linear systems as quadrupels consisting of an evolution family and output, input and input–output maps subject to natural hypotheses. In the spirit of G. Weiss' work these maps are represented in terms of admissible observation and control operators (the latter in an approximative sense) in the time domain. In this setting the closed–loop system exists for a canonical class of 'admissible' feedbacks, and it inherits the absolute regularity and other properties of the given system. In particular, one can iterate feedbacks.

1 Introduction

As a motivation, we first look at the finite dimensional non–autonomous linear system

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t), \quad t \geq s \geq 0, \\ y(t) &= C(t)x(t), \quad t \geq s \geq 0, \qquad x(s) = x_0, \end{aligned} \tag{1}$$

on the state space X with control operators $B(t) : U \to X$, observation operators $C(t) : X \to Y$, the control space U, and the observation space Y. Let $T(t,s)$, $t \geq s \geq 0$, be the evolution family (propagator) on X generated by $A(\cdot)$. Then the output of (1) with $u = 0$, the state of (1) with $x_0 = 0$, and the input–output operator of (1) are given by

$$\begin{aligned} (\Psi_s x_0)(t) &= C(t)T(t,s)x_0, \\ \Phi_{t,s} u &= \int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \\ (\mathbb{F}_s u)(t) &= C(t)\int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \qquad t \geq s. \end{aligned} \tag{2}$$

If one feeds back the output via $u(t) = \Delta(t)y(t)$, the resulting closed–loop system is described by the perturbed evolution equation

$$\begin{aligned} x'(t) &= [A(t) + B(t)\Delta(t)C(t)]\, x(t), \quad t \geq s, \\ x(s) &= x_0. \end{aligned} \tag{3}$$

Of course, $x(t) = T_\Delta(t,s)x_0$ solves (3), if T_Δ is generated by $A(t) + B(t)\Delta(t)C(t)$. This evolution family also satisfies the 'variation of constants formulas'

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t T(t,\tau)B(\tau)\Delta(\tau)C(\tau) \cdot T_\Delta(\tau,s)x\, d\tau, \tag{4}$$

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t T_\Delta(t,\tau)B(\tau)\Delta(\tau)C(\tau) \cdot T(\tau,s)x\, d\tau \tag{5}$$

for $t \geq s$ and $x \in X$. Identity (4) is the integrated version of (3). To derive (5), we perturb T_Δ by $-B(t)\Delta(t)C(t)$. There are formulas analogous to (4) and (5) relating the maps from (2) with the corresponding ones of the closed-loop system. These formulas are needed to show further properties of the closed–loop system. For instance, the closed–loop system is observable (controllable) if and only if the open–loop system is observable (controllable). We establish infinite dimensional versions of these results in Section 3.

If we pass to an infinite dimensional state space X, it is not clear anymore that (3) possesses differentiable solutions for 'many' initial values even if the Cauchy problem for $A(\cdot)$ is well–posed, cf. [4], [5, §VI.9], [6]. Nevertheless, the formulas (2) still work and there is an evolution family T_Δ fulfilling (4) and (5). Thus $x(t) = T_\Delta(t,s)x_0$ is the 'mild' solution of (3), [4]. However, point or boundary control and observation lead to input and output operators $B(t) : U \to \overline{X}_t$ and $C(t) : \underline{X}_t \to Y$ for spaces $\underline{X}_t \subsetneq X \subsetneq \overline{X}_t$, where $C(t)$ usually is not closable, see e.g. [2], [9]. In order to solve (4) in this more general setting, we may restrict ourselves to 'admissible' observation and control operators – roughly speaking those for which the expressions (2) make sense. Then we are also faced with the question whether the operators $B(t)$ and $C(t)$ are again admissible for the perturbed evolution family T_Δ, which is necessary to verify (5) or to iterate feedbacks.

The resulting perturbation problem (3) generalizes the settings of both the Desch–Schappacher theorem (where $\Delta(t) = C(t) = I$) and the Miyadera theorem (where $\Delta(t) = B(t) = I$) from semigroup theory, [5, §III.3], [12]. In the control literature there is a rich perturbation theory for the autonomous case (i.e., $A(t) = A$, $B(t) = B$, $C(t) = C$, $\Delta(t) = \Delta$). Linear systems belonging to the *Pritchard–Salamon class*, [11], were exhaustively treated in [3]. D. Salamon and G. Weiss introduced the larger class of *well–posed linear systems* in [14] and [17]–[20]. Here the semigroup T is given and the operators Φ, Ψ, and $\mathbb{F}$ are defined in an abstract way by certain algebraic relations. One can then construct admissible control and observation operators B and C and obtain formulas such as (2) if the system satisfies a quite natural *regularity* hypothesis. Weiss established a powerful feedback theory for regular systems in the Hilbert space situation, [21]. We refer to [2, §3.3], [10], [13], and in particular to O. Staffans' monograph [15] for further information.

For non–autonomous systems in variational form, there is the well–known approach due to J.L. Lions, see [1], [2, Chap.2], [9]. In a general setting, D. Hinrichsen, B. Jacob, and A.J. Pritchard, [6], [7], [8], constructed an evolution family solving (4) for initial values x contained in a dense subspace $\underline{X}$ of X under rather weak assumptions covering autonomous regular systems. But (5) and the admissibility of the perturbed system was investigated only in [7] requiring stronger hypotheses of Pritchard–Salamon type.

In the present work we combine the direct approach of Hinrichsen, Jacob, and Pritchard with some of Weiss' ideas: In Definition 2.3 we introduce 'Lebesgue extensions' of given observation operators $C(t)$, cf. [17], which allow the study of (4) and (5) for all $x \in X$ and to simplify several technical details of the proofs considerably. For similar reasons, we mostly work with *non–autonomous (absolutely) regular systems*, which are defined in the spirit of Weiss' work (see Definitions 2.8 and 2.9), as opposed to *admissible systems*, which have been used in [6], [7], [8] and are given directly by operators $B(t)$ and $C(t)$. In Theorem 2.4, Proposition 2.7, and Theorem 2.10 we then represent a given regular system similar as in (2).

It is known that (3) can only be solved if the feedback is not 'too large', [16, Exa.6]. We thus introduce a class of *admissible* feedbacks in Definition 3.1, cf. [15, §7.1], [21, §3]. In our main Theorem 3.2 we then establish the existence of an absolutely regular closed–loop system for a given absolutely regular non–autonomous system with admissible time varying feedback. We further derive analogues of (4) and (5) for the operators given in (2) and verify that the closed–loop system is controllable (or observable) if and only if the given system is controllable (or observable). Also, iterated feedbacks behave as one would expect.

However, the extension of Weiss' theory to the non–autonomous case is limited by two serious obstacles: One cannot apply transform methods and, in contrast to semigroups (see e.g. [5, §II.5]), we do not have a general extrapolation theory for evolution families. The first point excludes the use of transfer functions (being crucial in [21]), but leads us to arguments which work in a Banach space setting (as in [15, Chap.7]). The second point forces us to employ approximation formulas for the representation of control systems in Proposition 2.7. A similar problem occurs in the computation of the feedback system and in the context of (5), see Theorem 3.2.

Note. An extended version of this paper (with complete proofs) appeared in SIAM J. Control Optim. **41** (2002), 1141–1165.

2 Representation of regular systems

Throughout this paper X, Y, Z denote Banach spaces, $\mathcal{L}(X, Y)$ is the space of bounded linear operators, $\mathcal{L}(X, X) =: \mathcal{L}(X)$, $\mathcal{L}_s(X, Y)$ is endowed with the strong operator topology, and $p \in [1, \infty)$.

Definition 2.1 *A set* $T = (T(t,s))_{t\geq s\geq 0} \subseteq \mathcal{L}(X)$ *is an* evolution family *if*

(E1) $T(t,s) = T(t,r)T(r,s), \ T(s,s) = I,$

(E2) $(t,s) \mapsto T(t,s)$ *is strongly continuous, and*

(E3) $\|T(t,s)\| \leq Me^{w(t-s)}$

for $t \geq r \geq s \geq 0$ *and constants* $M \geq 1$ *and* $w \in \mathbb{R}$. *We also define* $(\mathbb{K}_s f)(t) = \int_s^t T(t,\tau) f(\tau)\, d\tau$ *for* $t \geq s \geq 0$ *and* $f \in L^1_{loc}([s,\infty), X)$.

Evolution families arise as solution operators of non–autonomous evolution equations, although not every evolution family solves such a problem, see e.g. [5, §VI.9] and the references therein. In the following we extend several definitions given by G. Weiss to the non–autonomous case.

Definition 2.2 *Let* T *be an evolution family on* X *and* $\Psi_s : X \to L^p_{loc}([s,\infty), Y)$, $s \geq 0$, *be linear operators satisfying*

$$\Psi_s x = \Psi_t T(t,s)x \quad \text{on } [t,\infty) \quad \text{and}$$
$$\int_s^{s+t_0} \|(\Psi_s x)(t)\|_Y^p \, dt \leq \gamma^p \, \|x\|_X^p$$

for $t \geq s \geq 0$, $x \in X$, *and some* $t_0 > 0$, $\gamma = \gamma(t_0) > 0$. *Then* $(T, \Psi) = (T, \{\Psi_s : s \geq 0\})$ *is a* non–autonomous observation system.

Definition 2.3 *We define for a non–autonomous observation system* (T, Ψ)

$$C(s)x := \lim_{\tau\searrow 0} C_\tau(s)x := \lim_{\tau\searrow 0} \frac{1}{\tau} \int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma \qquad (6)$$

for $x \in \underline{X}_s := \{x \in X :$ *the limit in* (6) *exists*$\}$. *We further set* $D(C(\cdot), s) := \{f \in L^p_{loc}([s,\infty), X) : f(t) \in \underline{X}_t$ *for a.e.* $t \geq s, \ C(\cdot)f(\cdot) \in L^p_{loc}([s,\infty), Y)\}$.

Our first representation theorem extends [17, Thm.4.5] to the non–autonomous case.

Theorem 2.4 *Let* (T, Ψ) *be a non–autonomous observation system and let* $C(s) : \underline{X}_s \to Y$ *be given as in Definition 2.3. Let* $x \in X$ *and* $t \geq s \geq 0$. *Then* $T(t,s)x \in \underline{X}_t$ *if and only if* $1/\tau \int_0^\tau (\Psi_s x)(t+\sigma)\, d\sigma$ *converges as* $\tau \searrow 0$. *If this is the case, then the limit equals* $C(t)T(t,s)x$. *Thus,* $(\Psi_s x)(t) = C(t)T(t,s)x$ *for a.e.* $t \in [s,\infty)$.

The theorem follows from the identity

$$\frac{1}{\tau}\int_t^{t+\tau}[\Psi_s x](\sigma)\,d\sigma = \frac{1}{\tau}\int_t^{t+\tau}[\Psi_t T(t,s)x](\sigma)\,d\sigma.$$

Definition 2.5 *Let T be an evolution family on X and $\Phi_{t,s} = \Phi(t,s) : L^p_{loc}([s,\infty),U) \to X$, $t \geq s \geq 0$, be linear operators satisfying*

$$\Phi_{t,s}u = \Phi_{t,r}(u|[r,\infty)) + T(t,r)\Phi_{r,s}u,\ t \geq r \geq s \geq 0, \quad (7)$$
$$\|\Phi_{t,s}u\|_X \leq \beta\,\|u\|_{L^p([s,t],U)}, \quad 0 \leq t-s \leq t_0,$$

for $u \in L^p(\mathbb{R}_+,U)$ and constants $t_0 > 0$, $\beta = \beta(t_0) > 0$. Then $(T,\Phi) = (T,\{\Phi_{t,s} : t \geq s \geq 0\})$ is called a non–autonomous control system.

One way to obtain a control system is indicated in the next definition, cf. [6], [7], [8].

Definition 2.6 *Let T be an evolution family on X and let $\overline{X}_t$, $t \geq 0$, be Banach spaces in which X is densely and continuously embedded. Assume that $T(t,s)$ has a locally uniformly bounded extension $\overline{T}(t,s) : \overline{X}_s \to \overline{X}_t$ (which then satisfies (E1) and is strongly continuous w.r.t. s). We call $B(t) \in \mathcal{L}(U,\overline{X}_t)$, $t \geq 0$, (T–)*admissible control operators *if the function $\overline{T}(t,\cdot)B(\cdot)u(\cdot)$ is integrable in $\overline{X}_t$,*

$$\left(\overline{\mathbb{K}}_s B(\cdot)u\right)(t) := \int_s^t \overline{T}(t,\tau)B(\tau)u(\tau)\,d\tau \in X,$$

and there are constants $t_0, \beta > 0$ such that

$$\|(\overline{\mathbb{K}}_s B(\cdot)u)(t)\|_X \leq \beta\,\|u\|_{L^p([s,t],U)}$$

for all $0 \leq s \leq t \leq s + t_0$ and $u \in L^p([s,t],U)$.

It is clear that $\Phi_{t,s}u := \left(\overline{\mathbb{K}}_s B(\cdot)u\right)(t)$ defines a control system. Conversely, in the autonomous case one can represent every control system by an admissible observation operator $B : U \to X_{-1}$, [18, Thm.3.9], where X_{-1} is the *extrapolation space* of the semigroup T, see e.g. [5]. Since there is no extrapolation theory for general evolution families, we have to proceed here in a different way. Let $u \in L^p_{loc}(\mathbb{R},U)$, $t \geq 0$, and $n \in \mathbb{N}$. We define $(B_n u)(t) = n\,\Phi(t,t-\frac{1}{n})u$, where $\Phi(t,s)u := \Phi(t,0)u$ if $s \leq 0$. It can be seen that $B_n u \in L^\infty_{loc}(\mathbb{R}_+,X)$. To approximate Φ, we introduce

$$\Phi^n(t,s)u = \Phi^n_{t,s}u := \int_s^t T(t,\tau)(B_n u)(\tau)\,d\tau \qquad (8)$$

for $t \geq s \geq 0$, $n \in \mathbb{N}$, and $u \in L^p_{loc}(\mathbb{R},U)$. These operators can be expressed by

$$\Phi^n(t,s)u = n\int_s^t (\Phi(t,\tau-\tfrac{1}{n})u - \Phi(t,\tau)u)\,d\tau \qquad (9)$$
$$= \Phi(t,s)u + nT(t,s)\int_{s-\frac{1}{n}}^s \Phi(s,\tau)u\,d\tau - n\int_{t-\frac{1}{n}}^t \Phi(t,\tau)u\,d\tau$$

due to (7). If we take a function $u \in L^p_{loc}([s,\infty),U)$ and extend it by 0 to $\mathbb{R}$, then

$$\Phi(t,s)u - \Phi^n(t,s)u = n\int_{t-\frac{1}{n}}^t \Phi(t,\tau)u\,d\tau. \qquad (10)$$

We further define operators $B_n(t) \in \mathcal{L}(U,X)$ by

$$B_n(t)z := (B_n u_z)(t) = n\Phi(t,t-\tfrac{1}{n})u_z\,,$$

where $u_z \equiv z$ for $z \in U$. The next result can be established using (9) and (10). In an approximative sense, it gives a replacement of the autonomous result mentioned above.

Proposition 2.7 *Let (T,Φ) be a non–autonomous control system, $n \in \mathbb{N}$, $0 \leq s \leq t \leq s + t_0$, $t_0 > 0$, $z \in U$, and $u \in L^p_{loc}(\mathbb{R},U)$. Then we have:*

(a) $\Phi^n_{t,s}u \to \Phi(t,s)u$, $\|\Phi^n(t,s)u\|_X \leq 2\beta(t_0)\|u\|_{L^p([s,t],U)}$.

(b) $(t,s) \mapsto \Phi(t,s)u$ *and* $t \mapsto B_n(t)z$ *are continuous in* X.

(c) $[\mathbb{K}_s B_n(\cdot)u](t) \to \Phi(t,s)u$ *and* $\|[\mathbb{K}_s B_n(\cdot)u](t)\|_X \leq \beta(t_0+1)\,\|u\|_{L^p([s,t],U)}$.

Here the limits as $n \to \infty$ are taken in X and are locally uniform in (t,s).

Definition 2.8 *Let (T,Φ) and (T,Ψ) be non–autonomous control and observation systems. If there are linear operators $\mathbb{F}_s : L^p_{loc}([s,\infty),U) \to L^p_{loc}([s,\infty),Y)$, $s \geq 0$, satisfying*

$$\mathbb{F}_s u = \Psi_t\Phi_{t,s}u + \mathbb{F}_t(u|[t,\infty)) \quad \text{on } [t,\infty) \quad \text{and}$$
$$\|\mathbb{F}_s u\|_{L^p([s,s+t_0],Y)} \leq \kappa\,\|u\|_{L^p([s,s+t_0],U)}$$

for $u \in L^p_{loc}([s,\infty),U)$, $t \geq s \geq 0$, and constants $t_0 > 0$, $\kappa = \kappa(t_0) > 0$, then $\Sigma = (T,\Phi,\Psi,\mathbb{F}) = (T,\Phi_{t,s},\Psi_s,\mathbb{F}_s)_{t\geq s\geq 0}$ is called a well–posed non–autonomous system, *and $\mathbb{F}_s$, $s \geq 0$, are called* input-output operators.

Definition 2.9 *A well–posed non–automonous system $\Sigma = (T,\Phi,\Psi,\mathbb{F})$ is called* regular (with feedthrough $D = 0$) *if*

$$\lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t u_z)(\sigma)\,d\sigma = 0$$

(in Y) and absolutely regular *if*

$$\lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau}\|(\mathbb{F}_t u_z)(\sigma)\|_Y^p\,d\sigma = 0$$

for all $t \geq 0$ and $z \in U$, where $u_z(s) := z$ for $s \geq 0$.

Again we generalize a representation theorem by Weiss, [19, Thm.4.5], to the non–autonomous case. The proof uses the approximation $C_\tau(s)$ from (6).

Theorem 2.10 *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a regular non–autonomous system, let $C(s)$ and $C_\tau(s)$ be given by Definition 2.3. Then $\Phi(\cdot, s)u \in D(C(\cdot), s)$ and $\mathbb{F}_s u = C(\cdot)\Phi(\cdot, s)u$ for $s \geq 0$ and $u \in L^p_{loc}([s, \infty), U)$. Moreover, $C_\tau(\cdot)\Phi(\cdot, s)u \to \mathbb{F}_s u$ in $L^p_{loc}([s, \infty), Y)$ as $\tau \searrow 0$ and $\|C_\tau(\cdot)\Phi(\cdot, s)u\|_{L^p([s,s+t_0],Y)} \leq c\,\|u\|_{L^p([s,s+t_0],U)}$ for $\tau \in (0, 1]$ and a constant $c = c(t_0)$ independent of u and s.*

The next result gives a different approximation of $\mathbb{F}_s$. Here we need absolute regularity and $p > 1$.

Proposition 2.11 *Let Σ be an absolutely regular non–autonomous system, $p \in (1, \infty)$, and $C(s)$ and $\Phi^n_{t,s}$ be given as in Definition 2.3 and (8). Then $\Phi^n(\cdot, s)u \in D(C(\cdot), s)$, $C(\cdot)\Phi^n(\cdot, s)u \to \mathbb{F}_s u$ in $L^p_{loc}([s, \infty), Y)$ as $n \to \infty$, and*

$$\|C(\cdot)\Phi^n(\cdot, s)u\|_{L^p([s,s+t_0],Y)} \leq 2\kappa(t_0)\,\|u\|_{L^p([s,s+t_0],U)}$$

for $u \in L^p_{loc}([s, \infty), U)$, $s \geq 0$, $n \in \mathbb{N}$, and $t_0 > 0$.

3 Feedbacks

Let Σ be a regular non–autonomous system, $C(s)$ be given by Definition 2.3, and $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y, U))$. For $x \in X$ and $s \geq 0$, we are looking for functions $x(\cdot) \in C([s, \infty), X) \cap D(C(\cdot), s)$ satisfying

$$x(t) = T(t, s)x + \Phi_{t,s}\Delta(\cdot)C(\cdot)x(\cdot), \qquad t \geq s, \qquad (11)$$

or, if $\Phi(\cdot, s)u = \overline{\mathbb{K}}_s B(\cdot)u(\cdot)$ for admissible control operators $B(s)$,

$$x(t) = T(t, s)x + \int_s^t \overline{T}(t, \tau)B(\tau)\Delta(\tau)C(\tau)x(\tau)\,d\tau,\ \ t \geq s.$$

As shown by [16, Exa.6], one cannot allow for every bounded feedback in (11), in general. (We note that this example gives rise to an absolutely regular autonomous system with $p = 1$ and $\Delta = B = I$.) This fact motivates the next concept.

Definition 3.1 *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a well–posed non–autonomous system. We call $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y, U))$ an* admissible feedback *for Σ if there is $t_0 > 0$ such that $I - \mathbb{F}(s + t_0, s)\Delta(\cdot)$, $s \geq 0$, have uniformly bounded inverses on $L^p([s, s + t_0], Y)$.*

Of course, $\Delta(\cdot)$ is admissible if

$$\|\Delta(\cdot)\|_\infty < [\inf_{t_0>0} \sup_{s\geq 0} \|\mathbb{F}(s + t_0, s)\|\,]^{-1} =: q.$$

The right hand side of this inequality equals ∞ if $B(t)$ and $C(t)$ are of 'lower order', see e.g. [3]. We point out that the invertibility of $I - \mathbb{F}(s + t_0, s)\Delta(\cdot)$ is in fact necessary for some properties of the feedback system.

Theorem 3.2 *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a regular non–autonomous system and $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y, U))$ be an admissible feedback. Then the following hold.*

(a) There is an evolution family T_Δ on X such that $T_\Delta(\cdot, s)x \in D(C(\cdot), s)$,

$$\|C(\cdot)T_\Delta(\cdot, s)x\|_{L^p([s,s+t_0],Y)} \leq \gamma'\,\|x\|\,,$$

$x(\cdot) = T_\Delta(\cdot, s)x$ is the unique solution of (11), and

$$T_\Delta(t, s)x = T(t, s)x + \Phi_{t,s}\Delta(\cdot)C(\cdot)T_\Delta(\cdot, s)x$$

for $t \geq s \geq 0$, $x \in X$, and a constant γ'. If, in addition, $\Phi(\cdot, s)u = \overline{\mathbb{K}}_s B(\cdot)u(\cdot)$ for T–admissible control operators $B(t)$, then

$$T_\Delta(t, s)x = T(t, s)x + \int_s^t \overline{T}(t, \tau)B(\tau)\Delta(\tau)C(\tau)T_\Delta(\tau, s)x d\tau.$$

(b) If the system is absolutely regular and $p \in (1, \infty)$, then

$$T_\Delta(t, s)x = T(t, s)x + \lim_{n\to\infty} \int_s^t T_\Delta(t, \tau)[B_n(\Delta(\cdot)\Psi_s x)](\tau)d\tau$$

for $t \geq s \geq 0$ and $x \in X$, where the limit is taken in X and is locally uniform in t. Moreover, $\Sigma^\Delta = (T_\Delta, \Phi^\Delta, \Psi^\Delta, \mathbb{F}^\Delta)$ is an absolutely regular system, where we set

$$\Psi^\Delta_s x = C(\cdot)T_\Delta(\cdot, s)x, \quad \Phi^\Delta_{t,s}u = \lim_{n\to\infty} [\mathbb{K}^\Delta_s B_n u](t),$$

$$\mathbb{F}^\Delta_s u = \lim_{n\to\infty} C(\cdot)\mathbb{K}^\Delta_s B_n u, \quad \mathbb{K}^\Delta_s f\,(t) = \int_s^t T_\Delta(t, \tau)f(\tau)\,d\tau$$

for $t \geq s \geq 0$, $x \in X$, $u \in L^p_{loc}([s, \infty), U)$, and $f \in L^p_{loc}([s, \infty), X)$, where the limits are taken in X and L^p_{loc} respectively.

Part (a) is a version of the results in [6], [7], [8]. Its proof relies essentially on the definition

$$T_\Delta(t, s)x := T(t, s)x + \Phi_{t,s}\Delta(\cdot)(I - \mathbb{F}(s+t_1, s)\Delta(\cdot))^{-1}\Psi_s x,$$

for $x \in X$. The proof of part (b) is much more involved and uses in particular the approximation and representation results of the previous section. In the autonomous case, see [15], [21], one can put the limits in (b) inside the integrals using extrapolation theory; and it is possible to compute the generator of the resulting semigroup T_Δ.

We now study the relationship between the open and the closed–loop system in more detail, see [15, Chap.7] and [21, §6] for similar results in the autonomous case. To put the formulas in a concise form, we define $\Psi(t, s)x = \mathbb{1}_{[s,t]}\Psi_s x$ and

$$\Sigma(t, s) = \begin{pmatrix} T(t, s) & \Phi(t, s) \\ \Psi(t, s) & \mathbb{F}(t, s) \end{pmatrix}$$

mapping $X \times L^p([s, t], U)$ to $X \times L^p([s, t], Y)$, $t \geq s \geq 0$.

Proposition 3.3 *Let Σ be an absolutely regular non–autonomous system, $p \in (1, \infty)$, $\Delta(\cdot)$ be an admissible feedback* *for Σ, and Σ^Δ be the feedback system of Theorem 3.2. Then*

$$\begin{aligned}\mathbb{F}_s^\Delta = (I - \mathbb{F}_s\Delta(\cdot))^{-1}\mathbb{F}_s &= \mathbb{F}_s(I - \Delta(\cdot)\mathbb{F}_s)^{-1}\\ &= C(\cdot)\Phi^\Delta(\cdot, s),\\ \Sigma^\Delta(t,s) - \Sigma(t,s) &= \Sigma(t,s)\begin{pmatrix}0 & 0\\ 0 & \Delta(\cdot)\end{pmatrix}\Sigma^\Delta(t,s)\\ &= \Sigma^\Delta(t,s)\begin{pmatrix}0 & 0\\ 0 & \Delta(\cdot)\end{pmatrix}\Sigma(t,s).\end{aligned}$$

A part of this proposition was already established in the proof of Theorem 3.2, the rest can be verified by algebraic manipulations.

Definition 3.4 *(a) A non–autonomous control system (T, Φ) is called* exactly (approximately) controllable on $[s,t]$ *if $\Phi(t,s)$ is surjective (has dense range) and it is called* exactly (approximately) null controllable on $[s,t]$ *if $T(t,s)X$ is contained in the (closure of) $\Phi(t,s)L^p([s,t],U)$.*
(b) A non–autonomous observation system (T, Ψ) is called (continuously) initially observable on $[s,t]$ *if $\Psi(t,s)$ is injective (bounded from below) and* (continuously) finally observable on $[s,t]$ *if* $\ker\Psi(t,s) \subset \ker T(t,s)$ *(if $\|T(t,s)x\| \leq c\,\|\Psi(t,s)x\|_p$ for a constant $c > 0$ and $x \in X$).*

Proposition 3.3 implies that these control theoretic properties remain unchanged under feedback. It also guarantees that repeated feedbacks behave nicely.

Proposition 3.5 *Let Σ be an absolutely regular non–autonomous system, $p \in (1,\infty)$, $\Delta(\cdot)$ be an admissible feedback for Σ, and Σ^Δ be the corresponding feedback system. Then Σ possesses one of the properties in Definition 3.4 if and only if Σ^Δ has the same property.*

Proposition 3.6 *Let Σ be an absolutely regular non–autonomous system with $p \in (1,\infty)$, $\Delta(\cdot)$ be an admissible feedback for Σ, Σ^Δ be the corresponding feedback system, and $\tilde{\Delta}(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$. Then $\tilde{\Delta}(\cdot)$ is admissible for Σ^Δ if and only if $\Delta(\cdot) + \tilde{\Delta}(\cdot)$ is admissible for Σ. If this is the case, then $\Sigma^{\Delta+\tilde{\Delta}} = (\Sigma^\Delta)^{\tilde{\Delta}}$.*

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