SWITCHED INTEGRATOR CONTROL SCHEMES FOR INTEGRATING PLANTS

K. Lau, R. H. Middleton

Centre for Integrated Dynamics and Control
The University of Newcastle
Callaghan NSW 2308 AUSTRALIA
fax: +61 2 49601712
e-mail: {eekl,rick}@ee.newcastle.edu.au

Keywords: switched linear systems, integral control, disturbance rejection, stability, overshoot

Abstract

In this paper, we compare and analyse three switched integrator control schemes for integrating plants. The schemes attempt to circumvent a fundamental limitation of linear time invariant (LTI) control, namely, that the closed loop response to a step change in the reference must overshoot if both the controller and the plant contain an integrator. We show that, for a simple example, the switched schemes all have significantly less overshoot than their LTI counterpart. The stability properties of the schemes are also studied. It is first shown that bounded input bounded state stability is guaranteed. Sufficient frequency domain conditions for asymptotic stability are then derived using passivity analysis.

1 Introduction

Consider the standard feedback system shown in Figure 1. If either the plant $G(s)$ or the controller $C(s)$ contains an integrator, then the steady state error in response to a constant reference signal will be zero. In the case where constant input disturbances are present, integral action is required in the controller. In this case, if $G(s)$ also contains an integrator, then e must satisfy the following integral constraint [11]:

$$\int_0^\infty e(t)dt = 0. \tag{1}$$

This constraint implies that there will always be overshoot in the step response of such a system.

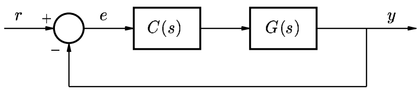

Figure 1: LTI feedback control loop.

It has been observed in [3], that whilst some constraints hold for any internally stabilising controller, others, such as the example given above, are dependent on the linear time invariance of the controller. For such constraints, which apply only to LTI control, a natural question to ask is to what extent can nonlinear or time-varying control be used to circumvent these constraints. In this paper, we consider the use of switched linear control, a particular type of nonlinear control, to circumvent constraint (1).

The control problem to be considered in this paper can be summarised as follows: The plant contains at least one integrator and can be represented by a strictly proper transfer function $G(s)$. It follows that

$$\lim_{s\to 0} G(s) = \infty.$$

The input to the plant is disturbed by $d(t)$, where

$$|d(t)| \le d_{max} \ \ \forall\, t, \tag{2}$$

and d_{max} is known *a priori*. Large changes are permitted in the reference signal $r(t)$, and the control aim can be loosely described as maintaining good error performance with 'reasonable' control energy. This should be achieved in the presence of either reference or disturbance changes.

We analyse and compare three switched control schemes of the form shown in Figure 2. The corresponding switched integrators shall be referred to as the *resetting, saturating,* and *holding integrators*. The motivation for all of the schemes is that integral action in the controller is required only for rejecting step disturbances, not for reference tracking. Hence, constraint (1) can be ameliorated by switching the integrator off when the feedback system is responding to a change in the reference. The schemes use the size(s) of the error and/or the integrator output to decide when to switch off the integrator.[1]

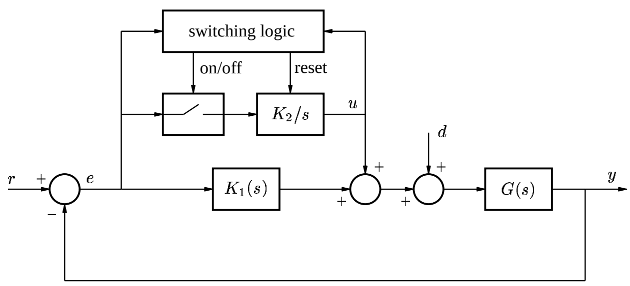

Figure 2: Block diagram of the switched integrator schemes.

We note that the use of resetting to improve the performance of linear controllers is not new. Previous examples include the

[1] We assume that we have no direct knowledge of whether a change in the reference or the disturbance has occurred.

Clegg integrator [2] and the first order reset element (e.g., [1]). The resetting integrator scheme which is studied in this paper is a slight generalisation of the switched linear controller proposed in [3], and was also studied in [8]. In [3], it was shown that, for the particular case of $K_1(s) \equiv K_1$ and $G(s) = 1/s$, the switched scheme ameliorates constraint (1).

Saturating integrators have also been used in control applications in the the past. In particular, they have been used in anti-windup control to reduce the effects of actuator saturation. Here, the saturation serves a different purpose. Hence, the saturation limit may be much smaller than the available actuation.

We also introduce a holding integrator scheme which combines the switching conditions of the resetting and saturating integrators. This scheme attempts to combine the performance of the resetting scheme with the stability properties of the saturating scheme.

An outline of the paper is as follows: We first describe the three schemes in detail and introduce the state space models which are used throughout the paper. Then, in Section 3, we briefly address the technical issue of existence of solutions to the differential equations governing the closed loop systems. In Section 4, we analyse and compare the performance of the schemes for the case in which $K_1(s) \equiv K_1$ and $G(s) = 1/s$. Time responses for a particular choice of K_1 and K_2 are given in Section 5, and Section 6 is concerned with stability properties of the schemes. Section 7 concludes the paper.

2 Switching Details and System Descriptions

In this section, we discuss the resetting, saturating and holding schemes in detail. The schemes can be described by the block diagram in Figure 2 with the following switching and resetting conditions:

Reset. integ.:	sw. closed iff $\lvert e\rvert < \varepsilon_r$ integ. reset when sw. open	(R)
Sat. integ.:	sw. closed iff $\lvert u\rvert < \bar{u}$ or $ue < 0$	(S)
Hold. integ.:	sw. closed iff $\lvert e\rvert < \varepsilon_h$ and ($\lvert u\rvert < \bar{u}$ or $ue < 0$)	(H)

Note that for the holding and saturating schemes, it is assumed that $|u(0)| \leq \bar{u}$, and that we refer to the three switching conditions as switching conditions (R), (S) and (H), respectively.

In each of the three schemes, $K_1(s)$ and K_2 are chosen so that the two frozen linear systems (the systems which are obtained by 'freezing' the switch in the open or the closed position) are asymptotically stable. The parameters ε_r, ε_h and $\bar{u}$ should also satisfy certain constraints (lower bounds). These are discussed in detail in Section 2.2.

We now describe the state space models which will be used throughout this paper. Remove the switched integrator from the scheme in Figure 2. Let x be the state of the remaining subsystem consisting of the feedback arrangement of $K_1(s)$ and $G(s)$ and let $w = [\,d,\ r\,]^T$. The (full) switched integrator scheme can be modelled as follows:

$$\dot{x} = Ax + B_1 u + Bw \tag{3}$$

$$\dot{u} = \begin{cases} K_2(r - Cx), & \text{sw. closed,} \\ 0, & \text{sw. open,} \end{cases} \tag{4}$$

$$u(t_k^+) = 0 \quad \text{for the resetting scheme only,} \tag{5}$$

$$y = Cx, \quad e = r - Cx. \tag{6}$$

Here, $\{t_k\}$ denotes the sequence of times at which the switch opens and B_i is the ith column of B. We let $z = [x^T, u]^T$,

$$\bar{A} = \begin{bmatrix} A & B_1 \\ -K_2C & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B_1 & B_2 \\ 0 & K_2 \end{bmatrix} \text{ and } \bar{C} = [\,C\,0\,].$$

We note that z is the system state, w is the system input, and A and $\bar{A}$ are Hurwitz matrices. It should also be noted that the state space realisations above may, in general, be non-minimal. However, they are detectable and stabilisable.

2.1 Notation

Consider the linear feedback system obtained by keeping the switch closed at all times. $e_d(t)$ denotes the response at e to a unit step disturbance. $e_r(t)$, $u_d(t)$ and $u_r(t)$ are similarly defined signals. We let $e_d^* = \sup\{|e_d(t)|\}$. u_d^* and u_r^* are defined similarly. $e_r^* = \sup\{-e_r(t)\}$, and corresponds to the maximum overshoot of the step response with the switch closed ($\forall\, t$). $S_{on}(s)$ is the sensitivity transfer function of the system, i.e., the transfer function from r to e with the switch closed. $\|\cdot\|_{i_{L_\infty}}$ denotes the induced L_∞ norm of a transfer function. The upper case characters are frequently used to denote the Laplace transform of a signal, for example, $E(s)$, $U(s)$.

2.2 Constraints on the Switching Thresholds

The switching thresholds ε_r, ε_h and $\bar{u}$ should be chosen so that at least the following condition is satisfied:

Steady State (SS) Condition
For step input disturbances bounded by d_{max}, the steady state error is zero.

We can also impose further conditions such as the following:

Bounded Step Disturbance (Bdd. Step Dist.) Condition
For $u(0) = 0$ and $r(t) = 0$ the switch remains closed for step input disturbances bounded by d_{max}.

Bounded Disturbance (Bdd. Dist.) Condition
For $u(0) = 0$ and $r(t) = 0$ the switch remains closed for any input disturbance bounded by d_{max}.

For the resetting scheme, we note that when the switch is open, $u(t) = 0$. It follows that the maximum steady state error with the switch in the open position is $d_{max}/K_1(0)$, and hence that the SS condition is met iff

$$\varepsilon_r > \frac{d_{max}}{K_1(0)}. \tag{7}$$

This inequality is strict because $e(t)$ approaches its steady state value asymptotically.

Similar inequalities can be derived [7] for ε_h and $\bar{u}$, and for each of the other conditions. These inequalities are summarised in Table 1. We note that, for the holding scheme, the constraints on ε_h and $\bar{u}$ both have to be satisfied for the corresponding condition to be met.

	SS	BSD	BD
$\frac{\varepsilon_r}{d_{max}} >$	$1/K_1(0)$	e_d^*	$\|G(s)S_{on}(s)\|_{iL_\infty}$
$\frac{\bar{u}}{d_{max}} \geq$	1	u_d^*	$\left\|\frac{K_2 G(s) S_{on}(s)}{s}\right\|_{iL_\infty}$
$\frac{\varepsilon_h}{d_{max}} >$	$\frac{(\bar{u}+d_{max})}{d_{max}K_1(0)}$	e_d^*	$\|G(s)S_{on}(s)\|_{iL_\infty}$

Table 1: Lower bounds on ε_r, ε_h and $\bar{u}$ for the Steady State (SS), Bdd. Step Dist. (BSD) and Bdd Dist. (BD) conditions.

3 Existence of Solutions for the Schemes (Well-posedness)

Before we analyse the three switched schemes it is important to consider whether they are well-posed, i.e., whether a solution to the differential equations governing the closed loop systems exist. The question of existence is discussed in some detail in [7]. This section contains a brief summary of the main conclusions in [7].

By using arguments similar to those in [5], it may be shown that the saturating scheme has solutions in the sense of Carathéodory provided that r and d are piecewise continuous and bounded.

It is not clear whether solutions exist for the resetting and holding schemes (as described in Section 2). However, in each of these cases, existence may be guaranteed by adding hysteresis to the switch. We let $h > 0$ and define q_ε by

$$q_\varepsilon(t,\varepsilon) = \begin{cases} 1, & |e(t)| \leq \varepsilon, \\ 0, & |e(t)| \geq \varepsilon + h, \\ q_\varepsilon(t^-,\varepsilon), & \text{otherwise.} \end{cases}$$

We assume that $q_\varepsilon(0^-,\varepsilon) = 0$. Hysteresis can be added to the switch in the resetting integrator scheme by replacing switching condition (R) by the following:

$$\text{sw. closed iff } q_\varepsilon(t,\varepsilon_r) = 1. \quad \text{(RH)}$$

Similarly, we can add hysteresis to switching condition (H) as follows:

$$\text{sw. closed iff } q_\varepsilon(t,\varepsilon_h) = 1 \text{ and } (|u| < \bar{u} \text{ or } ue < 0). \quad \text{(HH)}$$

When these switching conditions are used, existence is guaranteed if d and r are bounded, and r is piecewise right continuous with a bounded derivative and a minimum piece size. Since the hysteresis level h can be arbitrarily small, we assume that $h = 0$ in Sections 4 and 5 to simplify the analysis.

4 Example Performance Comparison

In this section, we compare the performance of the three schemes, when $G(s) = 1/s$ and $K_1(s) \equiv K_1 > 0$. We note that this example was studied in [3], and that, in this case, the controller switches between proportional (P) and proportional integral (PI) control.

Following [3], we assume that with the PI controller, the closed loop system has distinct real poles. Let the poles be at $-\lambda_1$ and $-\lambda_2$, where $\lambda_1 > \lambda_2 > 0$, and let $\rho = \lambda_2/\lambda_1$. In this case, the sensitivity function with the switch closed is given by

$$S_{on}(s) = \frac{s^2}{s^2 + K_1 s + K_2} = \frac{s^2}{(s+\lambda_1)(s+\lambda_2)}.$$

We note that $K_1 = \lambda_1 + \lambda_2$ and $K_2 = \lambda_1\lambda_2$.

The following lemma summarises some useful properties of the PI controlled system responses. The reader should refer to Section 2.1 for definitions of u_d, u_d^* etc.

Lemma 4.1 *For the example described above,*

(a) $u_d^* = 1$ *and* $u_d(t)$ *decreases monotonically from* 0 *to* -1

(b) $e_r^* = \rho^{\frac{1+\rho}{1-\rho}}$ *(c)* $e_d^* = \frac{1}{\lambda_1}\rho^{\frac{\rho}{1-\rho}}$

(d) $u_r^* = \lambda_2 \rho^{\frac{\rho}{1-\rho}}$ *(e)* $\|G(s)S_{on}(s)\|_{iL_\infty} = 2e_d^*$

(f) $\left\|\frac{K_2 G(s) S_{on}(s)}{s}\right\|_{iL_\infty} = 1$

Proof See [7]. □

We let $\bar{u} = d_{max}$, $\varepsilon_r = d_{max}/K_1$ and $\varepsilon_h = 2d_{max}/K_1$. These values are the lower bounds given in the SS column of Table 1, and hence give the 'limiting performance'. It can be shown [7] that ε_r satisfies the bdd. step dist. condition but not the bdd. dist condition. ε_h satisfies the bdd. dist. condition.

We note that, if $r(t) = \bar{r}$, $d(t) = \bar{d}$, and the PI closed loop has initial conditions $e(t_1)$ and $u(t_1)$, then for $t_2 > 0$,

$$e(t_1 + t_2) = e(t_1)e_r(t_2) + \left(\bar{d} + u(t_1)\right) e_d(t_2) \quad \text{and} \quad (8)$$

$$u(t_1 + t_2) = e(t_1)u_r(t_2) + u(t_1) + \left(\bar{d} + u(t_1)\right) u_d(t_2). \quad (9)$$

We compare the overshoot (in y) of the three schemes when $d(t) = 0$ and $r(t)$ is a step of height $\bar{r}$. Without loss of generality, we assume that $\bar{r} > 0$. For a given pair of eigenvalues ($-\lambda_1$ and $-\lambda_2$), we define $y_r^*(\bar{r},\lambda_1,\lambda_2)$ as the overshoot for the resetting integrator scheme. The functions y_s^* and y_h^* are defined in a similar manner for the saturating and holding schemes, respectively.

When $\bar{r}$ is 'small', the overshoot for all three schemes is the same as the overshoot with PI control, i.e., $\bar{r}e_r^*$. Expressions for the overshoot when $\bar{r}$ is 'large' follow:

The overshoot in the resetting integrator scheme is given by [3],

$$y_r^*(\bar{r},\lambda_1,\lambda_2) = \varepsilon_r e_r^* = \frac{d_{max}}{\lambda_1} f_r\left(\frac{\lambda_2}{\lambda_1}\right) \quad \text{for } \bar{r} \geq \varepsilon_r,$$

where $f_r(\rho) = (1+\rho)^{-1}\rho^{\frac{1+\rho}{1-\rho}}$. Graphs of the relative overshoot $y_r^*/\bar{r}$ against $\bar{r}$ are given in Figure 3 for $\lambda_2 = 1$ and $\lambda_1 = 1.1,\ 2,\ 5,$ and 10. We observe that as λ_1 increases, the overshoot with PI control decreases. Also, the resetting integrator behaves as a linear (normal) integrator over a smaller range of values for $\bar{r}$ (i.e., the length of the flat section of the graph decreases). We note that, in the nonlinear region, y_r^* decreases hyperbolically.

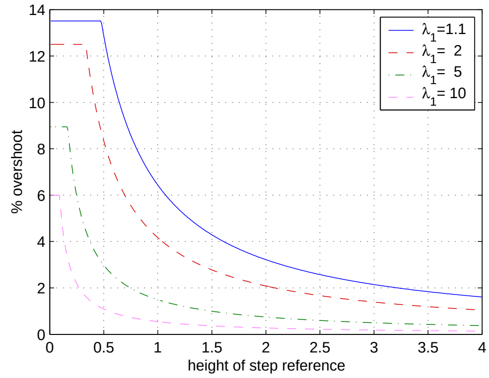

Figure 3: Resetting scheme - overshoot vs. reference step size for $\lambda_2 = 1$ and several values of λ_1.

For the saturating scheme, saturation will not occur for $\bar{r} < \bar{u}/u_r^*$. If $\bar{r} \geq \bar{u}/u_r^*$, the switch will open and close exactly once. This can be seen from the following argument:

Let t_1 be the (first) time at which the integrator saturates and the switch opens. Since $\bar{u}$ satisfies the SS condition, the switch will close at some time $t_2 > t_1$. $e(t_2) = 0$ because $e(t)$ is continuous and changes sign at $t = t_2$. From Equations (8) and (9) we get

$$\begin{aligned} e(t) &= \bar{u} e_d(t - t_2) \\ u(t) &= \bar{u} u_d(t - t_2) + \bar{u}, \end{aligned} \tag{10}$$

for $t > t_2$. Since the expression for $u(t)$ decreases monotonically (Lemma 4.1) from $\bar{u}$ to 0 the switch will remain closed for $t > t_2$.

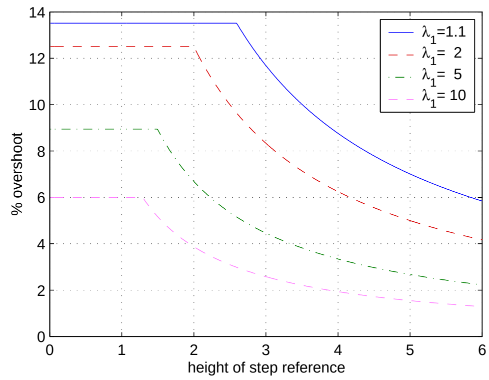

Figure 4: Saturating scheme - overshoot vs. reference step size for $\lambda_2 = 1$ and several values of λ_1.

From Equation (10), it follows that

$$y_s^*(\bar{r}, \lambda_1, \lambda_2) = \bar{u} e_d^* \;=\; \frac{d_{max}}{\lambda_1} f_s\left(\frac{\lambda_2}{\lambda_1}\right) \quad \text{for } \bar{r} \geq \frac{\bar{u}}{u_r^*},$$

where $f_s(\rho) = \rho^{\frac{\rho}{1-\rho}}$. Plots of $y_s^*/\bar{r}$ against $\bar{r}$ are given in Figure 4 for $\lambda_2 = 1$ and several values of λ_1.

We now consider the holding scheme. It can be shown that $\varepsilon_h u_r^* < e^{-1} d_{max} < \bar{u}$. This implies that the integrator will not saturate (when r is a step and $d(t) = 0$), and so the holding scheme is similar to the resetting scheme with a switching threshold of ε_h. It follows that

$$y_h^*(\bar{r}, \lambda_1, \lambda_2) = \varepsilon_h e_r^* = \frac{2 d_{max}}{\lambda_1} f_r\left(\frac{\lambda_2}{\lambda_1}\right), \quad \text{for } \bar{r} \geq \varepsilon_h,$$

and so the overshoot is twice that of the resetting scheme. We note, however, that ε_r satisfies the bdd. step dist. condition whilst ε_h satisfies only the bdd. dist. condition. If ε_r is doubled so that it is equal to ε_h (and satisfies the bdd. dist. condition), then the overshoot of the two schemes will be the same. Graphs of $y_h^*/\bar{r}$ as a function of $\bar{r}$ are given in Figure 5.

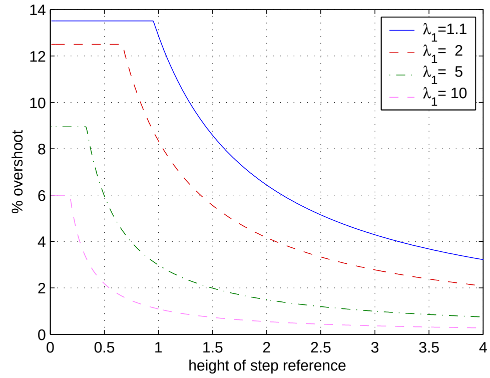

Figure 5: Holding scheme - overshoot vs. reference step size for $\lambda_2 = 1$ and several values of λ_1.

For the case of $\lambda_1 = 3$ and $\lambda_2 = 2$ ($K_1 = 5$, $K_2 = 6$), the relative overshoot for each scheme is plotted against $\bar{r}$ in Figure 6. The overshoot with (pure) PI control is also shown. We observe that the saturating and holding integrators operate in their linear region for a greater range of values of $\bar{r}$, and hence do not perform as well as the resetting scheme for small step references. However, for large values of $\bar{r}$, all three switched schemes offer a significant improvement over PI control.

5 Example Time Responses

We simulate the three schemes with $K_1 = 5$, $K_2 = 6$, $G(s) = 1/s$, $d_{max} = 1$, $\varepsilon_r = 0.21$, $\varepsilon_h = 0.41$, and $\bar{u} = 1$. The reference signal is a step of height $\bar{r} = 3$ arriving at $t = 5$s, and the input disturbance is set to zero. Figure 7 shows $e(t)$ and $u(t)$. The responses for the P and PI controllers are also shown. From the plots it is clear that all of the switched integrators are acting in their nonlinear region, and have less overshoot than pure PI control. In this case, the resetting scheme is the best, with an overshoot of approximately 1 percent. We note that a plot of the percentage overshoot for the three schemes as a function of $\bar{r}$ was given in Figure 6.

6 Stability

In this section, some stability results for the switched schemes are presented. We first consider bounded input bounded state

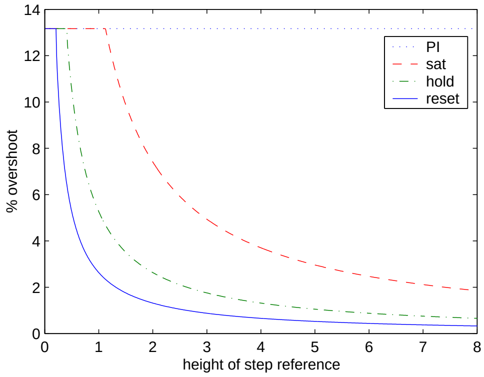

Figure 6: Percentage overshoot for the switched integrator schemes with $\lambda_1 = 3$ and $\lambda_2 = 2$.

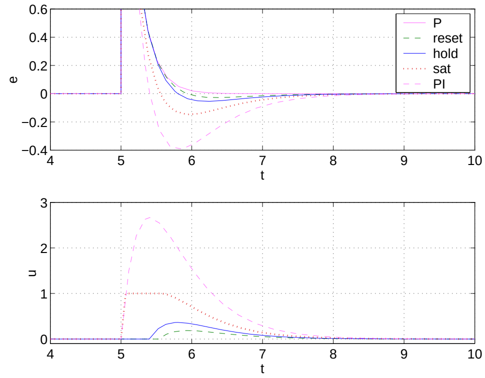

Figure 7: $e(t)$ and $u(t)$ when $\bar{r} = 3$ and $\bar{d} = 0$.

(BIBS) stability (defined in [7]) and then asymptotic stability. For the resetting and holding schemes, the stability of the modified schemes (with hysteresis) is analysed.

The following theorems state that the switched schemes are BIBS stable:

Theorem 6.1 *If A and $\bar{A}$ are Hurwitz, then the resetting integrator system described by Equations* (3) *to* (6) *with switching condition (RH) is BIBS stable.*

Proof The proof of this theorem is given in [8] for the case in which the hysteresis level $h = 0$. A more detailed version, is given in [6, Sect. 2.3]. The proof for the case in which $h > 0$ is almost identical. The only difference is that, when the switch is closed, $|e(t)| < \varepsilon_r + h$ instead of $|e(t)| < \varepsilon_r$. This affects only the definition of a_7 in [8] or c_7 in [6]. $\square$

Theorem 6.2 *If A is Hurwitz, then the switched linear system described by Equations* (3), (4) *and* (6) *with switching condition (S) or (HH) is BIBS stable.*

Proof Suppose that d and r are bounded. The design of the switching conditions, (S) and (HH), ensures that $|u(t)| \le \bar{u}\ \forall\, t$. Since A is Hurwitz, system (3) is BIBS stable. It follows that x is bounded because w and u are bounded. $\square$

When the exogenous inputs, r and d, are set to zero the switched integrator scheme of Figure 2 is equivalent to that shown in Figure 8. We shall refer to this system as the *unforced* switched system. $H(s)$ is the closed loop transfer function from u to y (with the integrator removed) and is given by

$$H(s) = \frac{G(s)}{1 + G(s)K_1(s)} = C(sI - A)^{-1}B_1. \qquad (11)$$

Remark 1 $\bar{A}$ is Hurwitz implies that $K_2H(0) \neq 0$ and that $H(0)$ and K_2 have the same sign. Hence, we may assume, without loss of generality, that $H(0) > 0$.

It is possible to find examples of the switched schemes for which the unforced system contains a limit cycle [6, 8]. So although the schemes are always BIBS stable, they are not, in general, asymptotically stable. In the following, passivity analysis is used to derive some sufficient conditions for asymptotic stability.

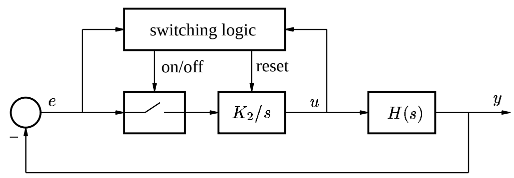

Figure 8: Block diagram of the unforced switched integrator schemes.

We first consider the unforced resetting integrator scheme. It is well known that the feedback connection of two output strictly passive and zero state observable systems is asymptotically stable [4, Lemma 6.7]. The next theorem states that, although the resetting integrator is only passive (not strictly passive), the switching conditions are such that output strict passivity of $H(s)$ implies asymptotic stability of the closed loop.

Theorem 6.3 *Consider the unforced resetting integrator system described by Equations* (3) *to* (6) *with switching condition (RH). Suppose that A and $\bar{A}$ are Hurwitz and that $\exists\, \delta > 0$ s.t.*

$$\mathrm{Re}\,[H(j\omega)] - \delta |H(j\omega)|^2 \ge 0. \qquad (12)$$

Then $z_0 = 0$ is globally asymptotically stable.

Proof The following is an outline of the proof in [7]. We assume that y is controllable from u to simplify the analysis.

Since A is Hurwitz, condition (12) implies that $H(s)$ is output strictly passive [9, Thm. 2.3]. It follows that $\exists$ a matrix $P \ge 0$ s.t. $V_2(x) = x^T P x$ satisfies the following inequality[2] [10, 13]:

$$V_2(x(T)) - V_2(x(0)) \le \int_0^T u(t)y(t) - \delta y^2(t)\, dt. \qquad (13)$$

We now consider the resetting integrator. We note that $K_2 > 0$ (Remark 1), and let $V_1(u) = u^2/(2K_2)$. When the switch is closed $\dot{V}_1(u(t)) = u(t)e(t)$. This is also true when the switch is open because then $u = 0$. $V_1(u(t))$ is continuous when the switch closes but it may decrease instantaneously when the integrator is reset. From this, it can be deduced that

$$V_1(u(T)) - V_1(u(0)) \le \int_0^T -u(t)y(t)\, dt \qquad (14)$$

[2] If the realisation is also observable, then $\exists$ a matrix $P > 0$.

$V_2(x(T))$ and $V_1(u(T))$ are positive, and hence, they may be removed from Inequalities (13) and (14). The inequalities can then be added to yield

$$-V_1(u(0)) - V_2(x(0)) \leq \int_0^T -\delta y^2(t)\, dt.$$

It follows that $y \in L_2[0, T]$ uniformly in T, and therefore that $y \in L_2$.

From Theorem 6.1, y and $\dot{y}$ are bounded. Thus $\frac{d}{dt}(y^2)$ is bounded, and by Barbalat's Lemma, $y^2 \to 0$. So $\exists\, t_\varepsilon > 0$ s.t., for $t > t_\varepsilon$, $|y(t)| < \varepsilon_r$ and the switch is closed. Asymptotic stability follows from the stability of the linear system with the switch closed ($\bar{A}$ is Hurwitz). □

We note that Theorem 6.3 includes the asymptotic stability results of [3] since, in this case, condition (12) is satisfied with $\delta = 1/K_1$.

In the case of the saturating scheme and the holding scheme (with switching condition (HH)), the unforced switched system can be written as a feedback connection of the linear system $K_2H(s)/s$ and a (bounded) time varying gain $k(t) \in \{0, 1\}$. Although absolute stability results, such as the circle criterion [12, 4], apply to this class of systems, they cannot be used here because the system is not asymptotically stable when $k(t) = 0$. The following theorem resembles Popov's criterion:

Theorem 6.4 *Consider the unforced switched system described by Equations* (3), (4) *and* (6) *with switching condition (S) or (HH). Suppose that A and $\bar{A}$ are Hurwitz. If $\exists\, \epsilon > 0$ and $\eta \geq 0$ with $-\frac{1}{\eta}$ not a pole of $H(s)$ s.t.*

$$\begin{aligned} & Z(s) = 1 + (1 + \eta s)\frac{K_2H(s)}{s} \\ \text{and} \quad & \mathrm{Re}\,[Z(j\omega)] - \epsilon \geq 0\; \forall\, \omega \in \mathbf{R}, \end{aligned} \tag{15}$$

then $z_0 = 0$ is asymptotically stable for all $z(0)$ satisfying $|u(0)| \leq \bar{u}$.

Proof See [7]. □

Let $\alpha(\omega) = \mathrm{Re}\,[K_2H(j\omega)]$ and $\beta(\omega) = \mathrm{Im}\,[K_2H(j\omega)]$. The above theorem implies that the saturating and holding schemes are asymptotically stable if

$$\frac{\beta(\omega)}{\omega} - j\alpha(\omega) \in \{\zeta \in \mathbf{C} : \eta\mathrm{Im}\,[\zeta] \leq (\mathrm{Re}\,[\zeta] + 1 - \epsilon)\}\,.$$

If $\eta > 0$ an equivalent condition is that the plot of $\beta(\omega)/\omega - j\alpha(\omega)$, which is the Popov plot of $K_2H(j\omega)/j\omega$, lies below a straight line of slope $1/\eta$ passing through the real axis at $-1+\epsilon$.

We have shown that the switched schemes are always BIBS stable and we have given sufficient frequency domain conditions for asymptotic stability. In [7] we also show that Theorem 6.4 extends to the case of $d(t) = d_0$ ($|d_0| \leq d_{max}$) and $r(t) = r_0$, but that this is not true for Theorem 6.3. The following lemma suggests that the saturating and holding schemes are, in some sense, more stable than the resetting scheme.

Lemma 6.5 *Suppose that A and $\bar{A}$ are Hurwitz and that $\exists\, \delta > 0$ s.t. condition* (12) *holds. Then $\exists\, \epsilon > 0$ and $\eta \geq 0$ s.t. condition* (15) *holds.*

Proof See [7]. □

7 Conclusion

In this paper, the resetting, saturating and holding schemes were studied. For the case in which the plant is an integrator and $K_1(s) \equiv K_1$, the closed loop responses to a step change in the reference (with no input disturbance) were compared. It was found that, in terms of the size of the overshoot in the response, the resetting scheme was slightly better than the other two schemes. However, all of the switched integrators offered a significant improvement over a normal integrator.

It was shown that all the schemes are BIBS stable provided that the two frozen linear systems are asymptotically stable. Sufficient frequency domain conditions for asymptotic stability were also found using passivity analysis. These conditions suggest that the saturating and holding schemes are in some sense 'more stable' than the resetting one.

References

[1] O. Beker, C. V. Hollot, and Y. Chait. Plant with integrator: An example of reset control overcoming limitations of linear feedback. *IEEE Trans. on Automatic Control*, 46(11):1797–9, 2001.

[2] J. C. Clegg. A nonlinear integrator for servomechanisms. *Trans. of the A.I.E.E. Part II*, 77:41–2, 1958.

[3] A. Feuer, G.C. Goodwin, and M. Salgado. Potential benefits of hybrid control for linear time invariant plants. In *Proc. 1997 American Control Conference*, 1997. Session FA06-2.

[4] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd ed., 2002.

[5] S.-J. Kim and I.-J. Ha. On the existence of Carathéodory solutions in mechanical systems with friction. *IEEE Trans. on Automatic Control*, 44(11):2086–9, 1999.

[6] K. Lau. *Time Domain Feedback Performance Limitations - Extensions to Nonlinear, Switched, and Sampled Systems*. PhD thesis, School of Electrical Engineering and Computer Science, The University of Newcastle, 2308, Australia, 2002.

[7] K. Lau and R. H. Middleton. Switched integrator control schemes for integrating plants. Technical Report EE02072, The University of Newcastle, Australia, `http:/www.eecs.newcastle.edu.au/`, 2002. *submitted for publication in the European Journal of Control*

[8] K. Lau and R.H. Middleton. On the use of switching control for systems with bounded disturbances. In *Proc. 39th IEEE Conference on Decision and Control*, Sydney, Australia, Dec. 2000.

[9] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke. *Dissipative Systems Analysis and Control, Theory and Applications*. Communications and Control Engineering. Springer-Verlag, London, 2000.

[10] P. J. Moylan. On a frequency-domain condition in linear optimal control theory. *IEEE Trans. on Automatic Control*, AC-20:806, December 1975.

[11] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin. *Fundamental Limitations in Filtering and Control*. Springer-Verlag, 1997.

[12] J.E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, 1991.

[13] J. C. Willems. Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):353–93, 1972.