SPEED-GRADIENT APPROACH TO MODELING DYNAMICS OF PHYSICAL SYSTEMS

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Abstract

The Speed-gradient approach to building dynamical models of physical systems is described. It is based on the Speed-gradient principle: *Among all possible motions only those are realized for which the input variables change proportionally to the speed gradient of appropriate goal functional*. Application of the principle is demonstrated by examples: motion of a particle in the potential field; wave, diffusion and heat transfer equations; viscous flow equation. Based on the Speed-gradient principle the proof of the Onzagger principle from thermodynamics and its extension to a class of systems far from equilibrium are given.

1 Introduction

During decades the interest of the physics community in control theory was not substantial. The situation changed dramatically in the 90s after it was discovered that even small feedback introduced into a chaotic system can change its behavior significantly, e.g., turn chaotic motion into periodic one [1]. The seminal paper [1] gave rise to a variety of publications demonstrating metamorphoses of numerous systems - both simple and complicated - under the action of feedback. However, the potential of modern nonlinear control theory still was not seriously appreciated although the key role of nonlinearity definitely was. On the other hand, new problems seem not traditional for control theorists: the desired position or the desired trajectory of the system is not specified whilst the "small feedback" requirement is imposed instead. It took some time to realize that such kind of problems is typical for control of more general oscillatory behavior and to work out a unified approach to nonlinear control of oscillations and chaos [2]. It needs only one more effort to make the next step and to undertake a systematic study of the properties of physical (as well as chemical, biological, etc.) systems by means of feedback actions.

A number of new applications of control theory to physics were presented in [3, 4]. Particularly, the phenomenon of feedback resonance was studied and its possible applications to nonlinear spectroscopy, escape from a potential well, and stabilization of unstable modes were demonstrated. In [5] the subject and methodology of *cybernetical physics* were outlined.

This paper is aimed at demonstration of some applications of the "feedback" approach to physical problems. Namely, the Speed-gradient approach to modeling of physical systems dynamics is described in Section 2. The proposed approach is demonstrated by examples: motion of a particle in the potential field; wave, diffusion and heat transfer equations; viscous flow equation. Other possible applications of the approach are illustrated in Section 3 by the proof of the Onzagger principle from thermodynamics.

2 Speed-gradient Principle and Laws of Dynamics

In this section we will study the links between control laws in technical systems and laws of dynamics in physical systems. It will be demonstrated that the methods of control system design can be applied to explanation of evolution principles for natural systems. To be more specific, a number of dynamical models of physical systems will be interpreted as speed-gradient algorithms for properly chosen goal functionals.

Consider a class of physical systems described by systems of differential equations

$$\dot{x} = f(x, u, t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, u is the vector of free (input) variables, $t \geq 0$. The problem of system evolution modeling can be posed as the

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search of a law of changing $u(t)$ meeting some criterion of "natural", or "reasonable" behavior of the system.

The most common criteria describing both physical (natural) and technical (artificial) systems are formulated as variational principles (e.g., Hamilton's principle of least action). They are based on specification of a functional (usually, integral functional) and determination of real system motions $\{x(t), u(t)\}$ as points in an appropriate functional space providing extrema of the specified functional. In order to explicitly describe either a control law or system dynamics the powerful machinery of calculus of variations and optimal control is used.

In addition to integral principles, differential (local) ones were proposed: Gauss principle of least constraint, principle of minimum energy dissipation, etc. It has been pointed out by M. Planck [9] that the local principles have some preference with respect to integral ones because they do not fix dependence of the current states and motions of the system on its later states and motions. One more local evolution principle is motivated by the Speed-gradient method [2, 6, 7, 8]. It can be formulated as follows.

Speed-gradient Principle: *Among all possible motions only those are realized for which the input variables change proportionally to the speed gradient of an appropriate goal functional.*

In the next section using the Speed-gradient principle will be illustrated by examples.

3 Examples: Speed-gradient Laws of Dynamics

Suppose that the model (1) has a simple form

$$\dot{x} = u. \tag{2}$$

The relation (2) means just that we are seeking for law of change of the state velocities. According to the Speed-gradient principle, first we need to introduce the goal function $Q(x)$. The choice of $Q(x)$ should reflect the tendency of natural behavior to decrease the current value $Q(x(t))$.

Example 1. Motion of a particle in the potential field. In this case the vector $x = \mathrm{col}\,(x_1, x_2, x_3)$ consists of coordinates x_1, x_2, x_3 of a particle. Choose smooth $Q(x)$ as the potential energy of a particle and derive the Speed-gradient law in the differential form. To this end, calculate the speed gradient

$$\dot{Q} = \left[\nabla_x Q(x)\right]^T u, \qquad \nabla_u \dot{Q} = \nabla_x Q(x).$$

Then, choosing the diagonal positive definite gain matrix $\Gamma = m^{-1} I_3$, where $m > 0$ is a parameter, I_3 is the 3×3 identity matrix, we arrive at familiar Newton's law $\dot{u} = -m^{-1} \nabla_x Q(x)$ or

$$m\ddot{x} = -\nabla_x Q(x). \tag{3}$$

Note that the speed-gradient laws with nondiagonal gain matrix Γ can be incorporated if a non-Euclidean metrics in the space of inputs is adopted induced by the matrix Γ^{-1}. Moreover, admitting dependence of the metric matrix Γ on x, we can obtain evolution laws for complex mechanical systems described by Lagrangian and Hamiltonian equations.

The Speed-gradient principle applies not only to finite dimensional systems, but also to infinite dimensional (distributed) ones. Particularly, x may be a vector of a Hilbert space $\mathcal{X}$ and $f(x, u, t)$ may be a nonlinear operator defined in a dense set $D_F \subset \mathcal{X}$ (in such a case the solutions of (1) should be understood as generalized ones).

Example 2. Wave, diffusion and heat transfer equations. Let $x = x(r)$, $r = \mathrm{col}\,(r_1, r_2, r_3) \in \Omega$ be the temperature field or the concentration of a substance field defined in the domain $\Omega \subset \mathbb{R}^3$. Choose the goal functional as the following nonuniformity measure of the field

$$Q_t(x) = \frac{1}{2} \int\limits_{\Omega} \left|\nabla_r x(r,t)\right|^2 \, dr, \tag{4}$$

where $\nabla_r x(r,t)$ is the spatial gradient of the field. Assuming zero boundary conditions for simplicity, we have

$$\dot{Q}_t = -\int\limits_{\Omega} \Delta x(r,t) u(r,t) \, dr, \quad \nabla_u \dot{Q}_t = -\Delta x(r,t),$$

where $\Delta = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial r_i^2}$ is the Laplace operator. Therefore the Speed-gradient evolution law in differential form is

$$\frac{\partial^2}{\partial t^2} x(r,t) = -\gamma \Delta x(r,t), \tag{5}$$

which corresponds to the D'Alembert wave equation, while its finite form is

$$\frac{\partial x}{\partial t}(t) = -\gamma \Delta x(r,t) \tag{6}$$

and coincides with the diffusion or heat transfer equation.

Note that the differential form of Speed-gradient laws corresponds to reversible processes while the finite form generates irreversible ones.

Example 3. Viscous flow equation. Let $v(r,t) \in \mathbb{R}^3$ be the velocity field of fluid, $p(r,t)$ be the pressure field, i.e., $x = \mathrm{col}\,(v(r,t), p(r,t))$. Introduce the goal functional as follows

$$Q_t = \int\limits_{\Omega} p(r,t)\,dr + \nu \int\limits_{\Omega} |\nabla_r v(r,t)|^2\,dr, \qquad (7)$$

where $\nu > 0$. Calculation of the Speed-gradient with respect to (2) yields $\nabla_u \dot{Q}_t = \nabla_r p - \nu\Delta v$. Then, the differential form of speed gradient is just the Navier-Stokes equation for viscous fluid motion

$$\rho\frac{\partial v}{\partial t}(r,t) = -\nabla_r p(r,t) + \nu\Delta v(r,t), \qquad (8)$$

where $\nu > 0$ is the viscosity coefficient, $\rho = \gamma^{-1}$ is density.

Other examples of reproducing dynamical equations for mechanical, electrical and thermodynamic systems can be found in [7]. The Speed-gradient principle applies to a broad class of physical systems subjected to potential and/or dissipative forces. On the other hand, it seems that the systems with vortex motion (e.g., mechanical systems affected by gyroscopic forces) do not belong to it.

4 Onzagger Equations

The Speed-gradient approach provides the new insight for various physical facts and phenomena. For example, we will give evidence for an extended version of the symmetry principle for kinetic coefficients (Onzagger principle) in thermodynamics [11] (it is also called the Maxwell-Betti theorem in elasticity theory). Consider an isolated physical system whose state is characterized by a set of variables (thermodynamic parameters) $\xi_1, \xi_2, \ldots, \xi_n$. Let $x_i = \xi_i - \xi_i^*$ be deviations of the variables from their equilibrium values $\xi_1^*, \xi_2^*, \ldots, \xi_n^*$. Let the dynamics of the vector $x_1, x_2, \ldots, x_n$ be described by the differential equations

$$\dot{x}_i = u_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \qquad (9)$$

Linearize equations (9) near equilibrium

$$\dot{x}_i = -\sum_{k=1}^{n} \lambda_{ik} x_k, \quad i = 1, 2, \ldots, n. \qquad (10)$$

The *Onzagger principle* [11] claims that the values λ_{ik} (so called kinetic coefficients) satisfy the equations

$$\lambda_{ik} = \lambda_{ki}, \quad i, k = 1, 2, \ldots, n. \qquad (11)$$

In general, the Onzagger principle is not valid for all systems or far from equilibrium. Its existing proofs (see, e.g., [10]) require additional postulates. Below the new proof is given showing that it is valid for irreversible Speed-gradient systems without exceptions.

First of all, the classical formulation of Onzagger principle (11) should be extended to nonlinear systems. A natural extension is the following set of identities:

$$\frac{\partial u_i}{\partial x_k}(x_1, x_2, \ldots, x_n) = \frac{\partial u_k}{\partial x_i}(x_1, x_2, \ldots, x_n). \qquad (12)$$

Obviously, for the case when the system equations (9) have linear form (10) the identities (12) coincide with (11). However, since linearization is not used in the formulation (12) there is a hope that the extended version of the Onzagger law holds for some nonlinear systems far from equilibrium. The following theorem specifies a class of systems for which this hope comes true.

Theorem 1. *Assume that there exists a smooth function $Q(x)$ such that equations (9) represent the Speed-gradient law in finite form for the goal function $Q(x)$.*

Then, the identities (12) hold for all $x_1, x_2, \ldots, x_n$.

Proof of Theorem 1. The proof is very simple. Since (9) is the Speed-gradient law for $Q(x)$, its right-hand sides can be represented in the form

$$u_i = -\gamma\frac{\partial \dot{Q}}{\partial Q_i}, \quad i = 1, 2, \ldots, n.$$

Therefore $u_i = -\gamma(\partial Q/\partial x_i)$ (in view of $\dot{Q} = (\nabla_x Q)^T u$). Hence

$$\frac{\partial u_i}{\partial x_k} = -\gamma\frac{\partial^2 Q}{\partial x_i \partial x_k} = \frac{\partial u_k}{\partial x_i},$$

and identities (12) are valid. ■

Thus, for Speed-gradient systems the extended form of Onzagger equations (12) hold without linearization, i.e., they are valid not only near the equilibrium state. It is worth mentioning that the above derivation is valid only under assumption that all the derivatives exist, i.e. all the involved functions are smooth. It excludes a number of nonsmooth physical problems, like description of shock waves, etc.

5 Discussion

We have shown that nonlinear control design methods developed in control theory (cybernetics) may provide new interpretations and new insights for dynamical models of physical systems. Moreover, using nonlinear control methods allows one to investigate new phenomena like feedback resonance. Existence of such strong analogies between dynamics of physical systems and control systems is not very surprising because both are generated by similar variational principles. Systematic usage of the above analogy to study physical systems constitutes a new research area in physics that can be called *cybernetical physics*.

The subject of cybernetical physics is investigation of natural systems depending on (weak) feedback interactions with the environment. Its methodology heavily relies on the design methods developed in cybernetics. However, the approach of cybernetical physics differs from the conventional use of feedback in control applications (e.g., robotics, mechatronics, see [12]) aimed mainly at driving a system to a pre-specified position or a given trajectory. The cybernetical methodology may also gain new insights into chemistry, biology and environmental studies. Perhaps the oncoming years will provide new important contributions in this exciting field.

It is interesting to link the speed-gradient approach with the results of H.Rosenbrock [13, 14] who demonstrated how to obtain Schrödinger's equation and some other results from the elementary theory of quantum mechanics by means of optimality principle of dynamic programming. Although using extremal principles is by no means a new approach, most of previous applications belong to the engineering area where optimality is a goal of creating an artificial engineering system. In the contrast, the goal-seeking in physics was many times criticized as a way of scientific description of nature. For example, let us quote A.Einstein [15], following [14]:

“For the scientist there is only being, but no wishing, no valuing, no good, no evil; no goal”.

Rosenbrock characterizes such an opinion as outdated, arguing that the goal-seeking is natural for much broader class of systems than just living organisms.

The above Speed-gradient principle as a local (differential) extremal principle relies upon goal-seeking idea even more heavily than integral variational principles and it may add arguments into the discussion. In the cases where obtaining physical results is easier from extremal principle than from system equations (see [14]), using simple Speed-gradient formulation may further facilitate analysis of a physical system.

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