OPERATOR THEORETIC SOLUTION TO THE MIMO EXTENSION OF ORDAP

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Abstract

In this paper we consider the MIMO form of the optimal robust disturbance attenuation problem (ORDAP) posed by Zames and Owen. In particular, an operator theoretic solution is developed solving the problem exactly. The solution is given in terms of an operator defined on particular Banach space versions of matrix valued H^2 spaces. The results obtained here generalize similar results obtained for SISO systems and a particular version of the MIMO two-disc problem.

Notation

$\mathbb{R}$, $\mathbb{C}$ stand for the field of real and complex numbers respectively . $< \cdot\,, \cdot >$ denotes either the inner or duality product depending on the context. I denotes the identity map. If B is a Banach space then $B^\star$ denotes its dual space. For an n-vector $\zeta \in \mathbb{C}_n$, where $\mathbb{C}_n$ denotes the n-dimensional complex space, $|\zeta|$ is the Euclidean norm. $\mathbb{C}_{n\times n}$ is the space of $n \times n$ matrices A, where $|A|$ is the largest singular value of A. $\mathbb{C}^{\max}_{2n}$, $\mathbb{C}^1_{2n}$ and $\mathbb{C}_{2n}$ denote the complex Banach space of $2n$-vectors ζ, $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$; ζ_1, $\zeta_2 \in \mathbb{C}_n$ with respectively, the norms

$$\begin{aligned} |\zeta|_{\max} &= \max(|\zeta_1|, |\zeta_2|), \;\; |\zeta|_1 = |\zeta_1| + |\zeta_2|, \\ \text{and} \qquad & |\zeta| = \sqrt{|\zeta_1|^2 + |\zeta_2|^2} \end{aligned} \tag{1}$$

Clearly, $\mathbb{C}^1_{2n}$ is the dual space of $\mathbb{C}^{\max}_{2n}$ and vise-versa.
$\tilde{\mathbb{C}}_{2n\times n}$, and $\hat{\mathbb{C}}_{2n\times n}$ denote the complex Banach space of $2n \times n$ matrices A, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, A_1, $A_2 \in \mathbb{C}_{n\times n}$, with respectively the following norms

$$\|A\|_\sim \;:=\; \max_{\substack{|\zeta| \leq 1 \\ \zeta \in \mathbb{C}_n}} (|A_1\zeta| + |A_2\zeta|) \tag{2}$$

$$\begin{aligned} \|A\|_\wedge \;&:=\; \inf \sum_k |\zeta_k|\, |\xi_k|_{\max},\; \zeta_k \in \mathbb{C}_n,\; \xi_n \in \mathbb{C}^{\max}_{2n} \\ &: \;\; A = \sum_k \xi_k\, \zeta_k^T \end{aligned} \tag{3}$$

The symbol $\mathfrak{D}$ denotes the unit disc of the complex plane, $\mathfrak{D} = \{z \in \mathbb{C} : \; |z| < 1\}$. $\partial\mathfrak{D}$ denotes the boundary of $\mathfrak{D}$, $\partial\mathfrak{D} = \{z \in \mathbb{C} : \; |z| = 1\}$. If E is a subset of $\partial\mathfrak{D}$, then E^c denotes the complement of E in $\partial\mathfrak{D}$. m denotes the normalized Lebesgue measure on the unit circle $\partial\mathfrak{D}$, $m(\partial\mathfrak{D}) = 1$. m a.e. is the label used for "Lebesgue almost everywhere". For a matrix or vector-valued function F on the unit circle, $|F|$ is the real-valued function defined on the unit circle by $|F|(e^{i\theta}) = |F(e^{i\theta})|$, $\theta \in [0, 2\pi)$. If X denotes a finite dimensional complex Banach space, $L^p(X)$, $1 \leq p \leq \infty$, stands for the Lebesgue-Bochner space of p-th power absolutely integrable X-valued functions on $\partial\mathfrak{D}$ under the norm

$$\|f\|^p_{L^p(X)} \;:=\; \int_{[0,2\pi)} \|f(e^{i\theta})\|^p_X dm, \;\text{ for }\; 1 \leq p < \infty \tag{4}$$

$$\|f\|_{L^\infty(X)} \;:=\; \operatorname*{ess\,sup}_{\theta\in[0,2\pi)} \|f(e^{i\theta})\|_X, \;\text{ for }\; p = \infty \tag{5}$$

where $f \in L^p(X)$, and $\|\cdot\|_X$ denotes the norm on X [17]. If $f \in L^p(X)$, $1 \leq p \leq \infty$, the k-th Fourier coefficient is defined by $\hat{f}_k \,\Delta\, \int_{\partial\mathfrak{D}} f(z) z^{-k} dm$, which define the well known Fourier series representation of f. $H^p(X)$, $1 \leq p \leq \infty$, is the Hardy space of X-valued analytic functions on the unit disc $\mathfrak{D}$, viewed as a closed subspace of $L^p(X)$. In fact these spaces can be realized as

$$H^p(X) = \{f \in L^p(X) : \hat{f}_k = 0 \;\text{ if }\; k < 0\} \tag{6}$$

The space $H^1_o(X)$ is defined as $\{f \in H^1(X),$ such that $\int_0^1 f(e^{i\theta})dm = 0\}$. Finally, $\mathcal{C}(X)$ denotes the space of continuous X-valued functions defined on $\partial\mathfrak{D}$. $\Re(A)$ denote the real part of A.

1 Introduction

A basic problem of feedback synthesis is the selection of a feedback control law to maximally suppress the effect of a class of output disturbances on the output of an uncertain plant. This objective may be captured in the form of an optimization known as the optimal robust disturbance attenuation (ORDAP), which was posed by Zames in [6], and later considered by Francis [2, 3], Owen and Zames [10, 5], and the author [7, 11, 9]. In the MIMO case the problem statement can be formulated as follows:
A stable LTI system P is assumed to belong to set of plants described by a weighted sphere in $H^\infty(\mathbb{C}_{n\times n})$ defined by

$$\begin{aligned} \mathfrak{B}(P_o, V) = \{(I + VX)P_o :\; X \in H^\infty(\mathbb{C}_{n\times n}), \\ \|X\|_\infty < 1, P_o \in H^\infty(\mathbb{C}_{n\times n}),\; V^{\pm 1} \in H^\infty\} \end{aligned} \tag{7}$$

The objective is to synthesize a robustly stabilizing feedback law C for the set $\mathfrak{B}(P_o, V)$ which minimizes the W weighted sensitivity norm $\|W(I+PC)^{-1}\|_\infty$ uniformly over all plants $P \in \mathfrak{B}(P_o, V)$. This amounts to solving the optimization given by

$$\mu = \inf_{C \text{ rob. stab.}} \sup_{P \in \mathfrak{B}(P_o,V)} \|W(I+PC)^{-1}\|_\infty \tag{8}$$

In [10] it was shown that the optimal robust disturbance attenuation μ was equal to the smallest fixed point of a function $\chi : [0, \infty) \rightarrow [0, \infty)$ defined in terms of the MIMO version of the two-disc problem,

$$\chi(r) = \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \operatorname{ess} \sup_{\theta \in [0,2\pi)} \max_{\substack{|\zeta| \leq 1 \\ \zeta \in \mathbb{C}_n}} \big(|W_1(I-P_oQ)W(e^{i\theta})\zeta| + r|P_oQV(e^{i\theta})\zeta|\big) \tag{9}$$

Motivated in part by this result and in part by an output disturbance and sensor noise disturbance rejection problem the subject of [10, 8, 11, 12] was another form for of the MIMO two-disk problem given by

$$\inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \|\, |W(I-P_oQ)| + |VP_oQ| \,\|_\infty \tag{10}$$

In [10] the optimization defined by $\chi(r)$ was approximated by a parametrized version of (10). That is $\chi(r)$ was approximated by

$$\chi(r)' = \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \|\, |W(I-P_oQ)| + r|VP_oQ| \,\|_\infty \tag{11}$$

where the approximation is exact in the SISO case. For the general MIMO case however (11) serves only as an upper bound for (9). This restrictions means that the results of [10, 11, 12] apply only to an approximate form of the ORDAP in the MIMO case.

In this paper the duality and operator theoretic methods developed in [10, 11, 8, 12] are applied directly to the two-disc problem (9) generated by the MIMO form of the ORDAP. In a similar vein to [8, 12] we characterize the optimal solutions to (9) using operator theory. In particular, certain vector valued H^2 spaces of matrix-valued functions are introduced, on which the norm of (9) is induced. The optimal μ is then shown to be equal exactly to the induced norm of a certain operator, analogous to the Sarason operator [20] and [8, 12], however the operator here is strictly defined on Banach space versions of matrix-valued H^2 Hardy spaces. The key is that these spaces are isomorphic to the Hilbert space version of an H^2 space of matrix-valued functions involving the Frobenius matrix norm. It is also demonstrated the existence of "maximal vectors" under certain conditions which lead to the computation of the optimal controller. The operator theoretic solution obtained here allows the numerical algorithm developed in [12], modulo simple modifications, to be applied to the MIMO form of the ORDAP. In contrast to [10, 8, 12], these methods can achieve numerically arbitrary accurate solution to the MIMO form of ORDAP. The operator solution developed here relies on an interplay between the Banach space duality description of the optimization problem (9) and analytic function theory. The duality theory has been worked out in [9, 13] and is briefly reviewed in the following section.

2 Duality Theory

Denote by $A^\star$ the dual space of any Banach space A. If M is a subspace of A then $M^\perp$ is the subspace of $A^\star$ which annihilates M, that is

$$M^\perp := \{f \in A^\star \; : < f \, , \, m \; > = \, 0, \; \forall m \in M\}$$

Isometric isomorphism between Banach spaces is denoted by $\simeq$.

$A_\star$ is said to be the predual space of A if $(A_\star)^\star \simeq A$, and a subspace ${}^\perp M$ of $A_\star$ is a preannihilator of a subspace M of A if, $({}^\perp M)^\perp \simeq M$. We shall use the following standard result of Banach space duality theory asserts that when a predual and preannihilator exist, then for any $K \in A$ [15]

$$\min_{m \in M} \|K - m\|_A = \sup_{f \in {}^\perp M, \, \|f\|_{A_\star} \leq 1} | < K, f > |$$

Observe that the optimization problem (9) is equivalent to finding the shortest distance from a vector function to a Banach subspace, defined as follows. Let the Banach space $H^\infty(\tilde{\mathbb{C}}_{2n\times n})$ equipped with the following norm:

$$\|K\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} := \sup_{z \in \mathfrak{D}} \max_{\substack{|\zeta| \leq 1 \\ \zeta \in \mathbb{C}_n}} (|K_1(z)\zeta| + |K_2(z)\zeta|)$$

$$K^T \; = \; (K_1^T, \; K_2^T)^T \in H^\infty(\tilde{\mathbb{C}}_{2n\times n}) \tag{12}$$

Since K is analytic in $\mathfrak{D}$, then $\max_{\substack{|\zeta| \leq 1 \\ \zeta \in \mathbb{C}_n}} |K_1(z)\zeta| + |K_2(z)\zeta|$ is subharmonic and satisfies the maximum principle [17]. Therefore:

$$\|K\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} = \operatorname{ess} \sup_{\theta \in [0,2\pi)} \max_{\substack{|\zeta| \leq 1 \\ \zeta \in \mathbb{C}_n}} \big(|K_1(e^{i\theta})\zeta| + |K_2(e^{i\theta})\zeta|\big)$$

Inner outer factorization of VP_o, and 'absorption' of the outer factor of P_o into the free parameter Q imply that (9) can be written as [10]:

$$\begin{aligned} \chi(r) &= \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} W \\ V \end{pmatrix} P_oQ \right\|_\infty \\ &= \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} U\tilde{W} \\ \tilde{V} \end{pmatrix} Q \right\|_\infty \end{aligned} \tag{13}$$

where $U \in H^\infty(\mathbb{C}_{n\times n})$ is the inner part of the plant P_o, $\tilde{W}$ and $\tilde{V}$ are outer in $H^\infty(\tilde{\mathbb{C}}_{2n\times n})$ (see [10]). The optimization (13) is the distance from $\begin{pmatrix} W \\ 0 \end{pmatrix}$ to the subspace

$S := \begin{pmatrix} U\tilde{W} \\ \tilde{V} \end{pmatrix}$ of $H^\infty(\tilde{\mathbb{C}}_{2n\times n})$ [10, 5].

Assuming: **(A1)** $\tilde{W}^\star\tilde{W}(e^{i\theta}) + \tilde{V}^\star\tilde{V}(e^{i\theta}) > 0$, $\forall\theta \in [0, 2\pi)$, then S is closed in $H^\infty(\tilde{\mathbb{C}}_{2n\times n})$, and there exists an outer spectral factor Λ of $\tilde{W}^\star\tilde{W} + \tilde{V}^\star\tilde{V}$, such that $\Lambda^\star\Lambda = \tilde{W}^\star\tilde{W} + \tilde{V}^\star\tilde{V}$, and $S = RH^\infty(\mathbb{C}_{n\times n})$, where $R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$, $R_1 = U\tilde{W}\Lambda^{-1}$, $R_2 = \tilde{V}\Lambda^{-1}$, so that R is inner, i.e., $R^\star R = I$ [10, 5].

Thus $\chi(r)$, by "absorbing" r into $\tilde{W}$ and $\tilde{V}$, can be expressed in the minimal distance form:

$$\chi(r) = \inf_{Q\in H^\infty(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} Q \right\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \quad (14)$$

Introduce the space $L^1(\hat{\mathbb{C}}_{2n\times n})$ the Lebesgue space of absolutely integrable functions with the norm:

$$\|G\| := \int_0^1 \|G(e^{i\theta})\|_\wedge dm, \;\; G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \in \quad (15)$$

Recall that the Banach space $L^\infty(\tilde{\mathbb{C}}_{2n\times n})$ is the space of essentially bounded $2n \times n$-matrix valued functions on the unit circle, which is equipped with the same norm as $H^\infty(\tilde{\mathbb{C}}_{2n\times n})$. The dual space of $L^1(\hat{\mathbb{C}}_{2n\times n})$ has been characterized in [9, 13] as

$$L^\infty(\tilde{\mathbb{C}}_{2n\times n}) \simeq \left(L^1(\hat{\mathbb{C}}_{2n\times n})\right)^\star$$

The preannihilator of S is given by [9, 13, 12].

$$\hat{S} = \left((I - RR^\star) \oplus R\overline{H}^1_o(\mathbb{C}_{n\times n})\right)/\hat{X} \quad (16)$$

where

$$\hat{X} = \left((I - RR^\star) \oplus R\overline{H}^1_o(\mathbb{C}_{n\times n})\right) \cap \overline{H}^1_o(\hat{\mathbb{C}}_{2n\times n})$$

The quotient norm in $\hat{S}$ is denoted by $\|\cdot\|_{\hat{S}}$ and is given by $[F]\|_{\hat{S}} = \inf_{h\in\hat{X}} \|F + h\|$, $\forall\ [F] \in \hat{S}$. We assume henceforth

(A2) W is continuous on the unit circle, and $\mu_o > \mu_{oo}$, where

$$\mu_{oo} := \inf_{Q\in\mathcal{C}(\mathbb{C}_{n\times n})} \operatorname{ess}\sup_{\theta\in[0,2\pi)} \max_{\substack{|\zeta|\le 1 \\ \zeta\in\mathbb{C}_n}} \left(|(W - R_1Q)(e^{i\theta})\zeta| + |R_2Q(e^{i\theta})\zeta|\right) \quad (17)$$

i.e. when the open unit disc analyticity constraint on Q is removed.

The following Theorem which appeared in [7, 9] asserts the existence of an optimal Youla parameter $Q \in H^\infty(\mathbb{C}_{n\times n})$ achieving μ_o, and consequently an optimal feedback controller for the optimization problem (9).

Theorem 1 *[9, 13, 7]*
Under assumptions (A1) and (A2), there exists at least one $Q_o \in H^\infty(\mathbb{C}_{n\times n})$ such that

$$\begin{aligned}\mu_o &= \inf_{Q\in H^\infty(\mathbb{C}_{n\times n})} \operatorname{ess}\sup_{\theta\in[0,2\pi)} \max_{\substack{|\zeta|\le 1 \\ \zeta\in\mathbb{C}_n}} \left(|(W - R_1Q)(e^{i\theta})\zeta| + |R_2Q(e^{i\theta})\zeta|\right) && (18)\\ &= \operatorname{ess}\sup_{\theta\in[0,2\pi)} \max_{\substack{|\zeta|\le 1 \\ \zeta\in\mathbb{C}_n}} \left(|(W - R_1Q_o)\zeta| + |R_2Q_o\zeta|\right) \\ &= \max_{\substack{\|[F]\|_{\hat{\mathbb{S}}}\le 1 \\ [F]\in\hat{\mathbb{S}}}} \left|\int_0^1 Tr\{(W^\star\ ,\ 0)F(e^{i\theta})\}dm\right| && (19)\end{aligned}$$

Under assumption (A1) and (A2) it has been shown in [7, 9] that the optimal solution is in fact flat or allpass, that is

$$\max_{\substack{|\zeta|\le 1 \\ \zeta\in\mathbb{C}_n}} \left(|(W - R_1Q_o)(e^{i\theta})\zeta| + |R_2Q_o(e^{i\theta})\zeta|\right) = \mu \qquad \forall\theta \in [0, 2\pi) \quad (20)$$

Theorem 1 not only shows the dual and predual description of the problem under assumptions (A1) and (A2), showing existence of at least one optimal control law, but also plays an important role in developing the operator theoretic solution to the optimization (9) in the sequel.

3 Operator Theoretic Approach to the MIMO Extension of ORDAP

3.1 A Particular Multiplication Operator

Let $L^2(\mathbb{C}_{n\times n})$ $(H^2(\mathbb{C}_{n\times n}))$ denote the Banach space of Lebesgue square integrable (and analytic) $\mathbb{C}_{n\times n}$-valued functions in the unit disc under the norm

$$\|f\|^2_{L^2(\mathbb{C}_{n\times n})} = \int_0^1 |f(e^{i\theta})|^2 dm, \quad f \in L^2(\mathbb{C}_{n\times n}) \quad (21)$$

where $|\cdot|$ denotes the largest singular value.

Likewise define $L^2(\tilde{\mathbb{C}}_{2n\times n})$ $(H^2(\tilde{\mathbb{C}}_{2n\times n}))$ to be the Banach space of $\tilde{\mathbb{C}}_{2n\times n}$-valued and Lebesgue square integrable (and analytic) functions on the unit disc endowed with the norm, for $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in L^2(\tilde{\mathbb{C}}_{2n\times n})$:

$$\|F\|^2_{L^2(\tilde{\mathbb{C}}_{2n\times n})} = \int_0^1 \max_{\substack{|\zeta|\le 1 \\ \zeta\in\mathbb{C}_n}} \left(|F_1(e^{i\theta})\zeta| + |F_2(e^{i\theta})\zeta|\right)^2 dm \quad (22)$$

Let $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in H^\infty(\tilde{\mathbb{C}}_{2n\times n})$, then Φ may be viewed as a multiplication operator M_Φ acting from $L^2(\mathbb{C}_{n\times n})$ (or $H^2(\mathbb{C}_{n\times n})$) into $L^2(\tilde{\mathbb{C}}_{2n\times n})$ (or $H^2(\tilde{\mathbb{C}}_{2n\times n})$), more precisely

$$M_\Phi f = \Phi f\ , \quad \forall f \in \mathrm{L}^2(\mathbb{C}_{n\times n}) \quad (23)$$

Clearly, M_Φ is a bounded linear operator. We show in the next Proposition that the operator induced norm is equal to $\|\Phi\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})}$.

Proposition 1 *Let Φ and M_Φ be defined as above. Then*

$$1)\ \|M_\Phi\| = \sup_{\substack{\|g\|_{L^2(\mathbb{C}_{n\times n})} \leq 1 \\ g\in L^2(\mathbb{C}_{n\times n})}} \|M_\Phi g\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} = \|\Phi\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \quad (24)$$

$$2)\ \|M_\Phi\| = \sup_{\substack{\|g\|_{L^2(\mathbb{C}_{n\times n})} \leq 1 \\ g\in H^2(\mathbb{C}_{n\times n})}} \|M_\Phi g\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \quad (25)$$

The dual space of $L^2(\tilde{\mathbb{C}}_{2n\times n})$ is given by $L^2(\hat{\mathbb{C}}_{2n\times n})$ and vise-versa, hence $L^2(\tilde{\mathbb{C}}_{2n\times n})$ is reflexive [16]. In the next Proposition we characterize the dual space of $H^2(\tilde{\mathbb{C}}_{2n\times n})$.

Proposition 2 *Let $H^2(\tilde{\mathbb{C}}_{2n\times n})$ and $H^2(\hat{\mathbb{C}}_{2n\times n})$ defined as above. Then*

$$1)\ H^2(\tilde{\mathbb{C}}_{2n\times n}) \simeq \left(H^2(\hat{\mathbb{C}}_{2n\times n})\right)^\star \quad (26)$$

$$2)\ H^2(\hat{\mathbb{C}}_{2n\times n}) \simeq \left(H^2(\tilde{\mathbb{C}}_{2n\times n})\right)^\star \quad (27)$$

Hence $H^2(\tilde{\mathbb{C}}_{2n\times n})$ and $H^2(\hat{\mathbb{C}}_{2n\times n})$ are reflexive Banach spaces.

3.2 Exact Operator Theoretic Solution for the ORDAP of MIMO Systems

The vector valued Hardy spaces $H^2(\tilde{\mathbb{C}}_{2n\times n})$ and $H^2(\mathbb{C}_{n\times n})$ are isomorphic to particular vector valued H^2 spaces. To see this define first the space $H^2(\mathbb{C}_{n\times n})$ as the Hardy space of $\mathbb{C}_{n\times n}$-valued functions under the following norm

$$\|f\|_2 := \int_0^1 \|f(e^{i\theta})\|_F dm, \quad f\in H^2(\mathbb{C}_{n\times n}) \quad (28)$$

where $\|f(e^{i\theta})\|_F$ is the matrix Frobenius norm, i.e., $\|f(e^{i\theta})\|_F = \sqrt{Tr(f^\star f)(e^{i\theta})}$. Clearly, $H^2(\mathbb{C}_{n\times n})$ is a Hilbert space with inner product

$$< f,g > := \int_0^1 Tr(g^\star f)(e^{i\theta})dm, \quad f,g\in H^2(\mathbb{C}_{n\times n}) \quad (29)$$

Since $\mathbb{C}_{n\times n}$ is finite dimensional all norms on $\mathbb{C}_{n\times n}$ are equivalent, in particular the matrix norms $|\cdot|$ and $\|\cdot\|_F$ are equivalent, i.e., there exist positive scalars α and β such that

$$\alpha\|A\|_F \leq |A| \leq \beta\|A\|_F, \quad \forall A\in\mathbb{C}_{n\times n}$$

It is then clear that the identity map I_1 from $H^2(\mathbb{C}_{n\times n})$ onto $H^2(\mathbb{C}_{n\times n})$ is isomorphic.

In a similar fashion, define the space $H^2(\mathbb{C}_{2n\times n})$ as the Hardy space of $\mathbb{C}_{2n\times n}$-valued functions under the following norm

$$\|f\|_2 := \int_0^1 \|f(e^{i\theta})\|_F dm, \quad f\in H^2(\mathbb{C}_{2n\times n}) \quad (30)$$

where $\|f(e^{i\theta})\|_F$ is the matrix Frobenius norm for $2n\times n$ matrices. It is clear that the identity map I_2 from $H^2(\mathbb{C}_{2n\times n})$ onto the Hilbert space $H^2(\tilde{\mathbb{C}}_{2n\times n})$ is isomorphic.

A Theorem of Lindenstrauss and Tzafriri [19] asserts that if X is a Banach space isomorphic to a Hilbert space, then every closed subspace Y of X is complemented. That is, there exists a bounded linear projection from X onto Y. In the sequel we shall define a projection operator onto the subspace $H^2(\tilde{\mathbb{C}}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$.

Let Π be the orthogonal projection on the closed subspace $H^2(\mathbb{C}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$ of $H^2(\mathbb{C}_{2n\times n})$, where $H^2(\mathbb{C}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$ is the orthogonal complement of $RH^2(\mathbb{C}_{n\times n})$, $\left(RH^2(\mathbb{C}_{n\times n})\right)^\perp$. Here orthogonality is understood to be with respect to the inner product of matrices, i.e., A, $B\in H^2(\tilde{\mathbb{C}}_{2n\times n})$ are orthogonal if and only if

$$< B,A > = \int_0^1 Tr\{A^\star B\}(e^{i\theta})dm = 0 \quad (31)$$

Next, define the bounded linear projection Π' as follows

$$\begin{aligned}\Pi' \ :\ \ & H^2(\tilde{\mathbb{C}}_{2n\times n}) \xrightarrow{I_1} H^2(\mathbb{C}_{2n\times n}) \xrightarrow{\Pi} H^2(\mathbb{C}_{2n\times n}) \\ & \ominus RH^2(\mathbb{C}_{n\times n}) \xrightarrow{I_2} H^2(\tilde{\mathbb{C}}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n}) \\ & \Pi' := I_2\circ\Pi\circ I_1 \end{aligned} \quad (32)$$

where $\circ$ is the usual composition, and I_2 is the previous identity restricted to the subspace $H^2(\mathbb{C}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$.

It is clear from the definition of Π', and the properties of the orthogonal projection Π that every vector function $f\in H^2(\tilde{\mathbb{C}}_{2n\times n})$ can be decomposed *uniquely* as the sum of two functions as follows

$$f = g + h, \text{ where } \begin{aligned} g &\in H^2(\tilde{\mathbb{C}}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n}) \\ h &\in RH^2(\mathbb{C}_{n\times n}) \end{aligned} \quad (33)$$

and g, h are orthogonal in the sense that $< g,h > = 0$. Also note that

$$g = \Pi' f \quad (34)$$

Since the orthogonal projection Π gives the best approximation of elements of $H^2(\mathbb{C}_{2n\times n})$ by functions in the subspace $H^2(\mathbb{C}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$ in the $\|\cdot\|_2$-norm. The projection operator Π' plays a similar role as Π but for approximations in the $\|\cdot\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})}$-norm by elements of $H^2(\tilde{\mathbb{C}}_{2n\times n})\ominus RH^2(\mathbb{C}_{n\times n})$.

Next, define the following operator which is analogous to the Sarason operator for the standard optimal H^∞ problem [20]

$$\begin{array}{ll}\Xi' & : H^2(\mathbb{C}_{n\times n}) \longrightarrow H^2(\tilde{\mathbb{C}}_{2n\times n}) \ominus RH^2(\mathbb{C}_{n\times n}) \\ \text{by} & \Xi' = \Pi' M \begin{pmatrix} W \\ 0 \end{pmatrix} \end{array} \tag{35}$$

where $M\begin{pmatrix} W \\ 0 \end{pmatrix}$ is the multiplication operator associated to $\begin{pmatrix} W \\ 0 \end{pmatrix}$. The following Theorem quantifies optimal performance in terms of Ξ'.

Theorem 2 *Under assumptions (A1), (A2) and* $\mu'_o > \mu'_{oo}$*, the following hold:*

i) μ'_o *is equal to the operator induced norm of* Ξ'*, namely*

$$\mu'_o = \|\Xi'\| \tag{36}$$

ii) There exists a maximal vector for Ξ'*, i.e.,* $f \in H^2(\mathbb{C}_{n\times n})$*,* $\|f\|_{L^2(\mathbb{C}_{n\times n})} = 1$ *such that*

$$\|\Xi'\| = \|\Xi' f\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \tag{37}$$

Proof

1. First note that $\forall Q \in H^\infty(\mathbb{C}_{n\times n})$, and $\forall G \in H^2(\mathbb{C}_{n\times n})$, we have $QG \in H^2(\mathbb{C}_{n\times n})$, i.e., $QH^2(\mathbb{C}_{n\times n}) \subset H^2(\mathbb{C}_{n\times n})$, hence $\Pi' RQG = 0$. Let $F \in H^2(\mathbb{C}_{n\times n})$, with norm $\|F\|_{L^2(\mathbb{C}_{n\times n})} \le 1$, we have

$$\begin{aligned} &\min_{G\in H^2(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} F - RQ \right\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \\ &\le \min_{G\in H^2(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} F - RQG \right\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \\ &\le \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} F - RQF \right\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \\ &\le \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ \right\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \|F\|_{L^2(\mathbb{C}_{n\times n})} \\ &\le \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ \right\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \end{aligned}$$

Consequently

$$\begin{aligned} \|\Xi'\| &= \sup_{\substack{\|F\|_{L^2(\mathbb{C}_{n\times n})} \le 1 \\ F \in H^2(\mathbb{C}_{n\times n})}} \min_{G\in H^2(\mathbb{C}_{n\times n})} \\ & \qquad \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} F - RG \right\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \qquad (38) \\ &\le \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ \right\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \qquad (39) \end{aligned}$$

For the reverse inequality, note that

$$\left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ \right\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} = \int_0^1 Tr\{(W^\star, 0)F_o(e^{i\theta})\} dm$$

and flatness implies $\left\| \begin{pmatrix} F_{o1}(e^{i\theta}) \\ F_{o2}(e^{i\theta}) \end{pmatrix} \right\|_\wedge = 1$, m a.e.

Then there exists a function $h \in H^2$ such that

$$|h(e^{i\theta})|^2 = \left\| \begin{pmatrix} F_{o1}(e^{i\theta}) \\ F_{o2}(e^{i\theta}) \end{pmatrix} \right\|_\wedge, \quad m \text{ a.e.} \tag{40}$$

and $\|h\|_{L^2} = 1$. Note that $F_o h \in (RH^2(\mathbb{C}_{n\times n}))^\perp$, $\|F_o h\|_{L^2(\hat{\mathbb{C}}_{2n\times n})} = 1$.

$$\begin{aligned} &\left| \int_0^1 Tr\{(W^\star, 0)F_o(e^{i\theta})\} dm \right| = \\ &\left| \int_0^1 Tr\{\overline{h} I_n (W^\star, 0) F_o h\}(e^{i\theta}) dm \right| = \qquad (41) \\ &\left| \int_0^1 Tr\left\{ \left(\Pi' \begin{pmatrix} W \\ 0 \end{pmatrix} h I_n \right)^\star F_o h \right\}(e^{i\theta}) dm \right| \\ &(\text{since } F_o h \in (RH^2(\mathbb{C}_{n\times n}))^\perp) \\ &\le \sup_{\substack{\|F\|_{L^2(\hat{\mathbb{C}}_{2n\times n})} \le 1 \\ F \in (RH^2(\mathbb{C}_{n\times n}))^\perp}} \left| < \Pi' \begin{pmatrix} W \\ 0 \end{pmatrix} h I_n, F > \right| \\ &\le \left\| \Pi' \begin{pmatrix} W \\ 0 \end{pmatrix} h I_n \right\|_{L^2(\tilde{\mathbb{C}}_{2n\times n})} \le \left\| \Pi' \begin{pmatrix} W \\ 0 \end{pmatrix} \right\| \\ &= \|\Xi'\| \qquad (42) \end{aligned}$$

Inequalities (39) and (42) imply that $\|\Xi'\| = \mu$.

2. Follows from (39) and (42).

As in [7, 11, 12] the projection operator Π' can be shown to be given by

$$\Pi' = I - RP_+ R^\star \tag{43}$$

where this time I is the identity map on $H^2(\tilde{\mathbb{C}}_{2n\times n})$, R is viewed as a multiplication acting from $H^2(\mathbb{C}_{n\times n})$, and P_+ is the positive Riesz projection from $L^2(\tilde{\mathbb{C}}_{2n\times n})$ into $H^2(\tilde{\mathbb{C}}_{2n\times n})$. The optimal performance μ is then equal to

$$\mu = \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RP_+ R_1^\star W \right\| \tag{44}$$

The results of [7, 11, 12] carry over, and the norm of Ξ' can approximated as close as needed by special norms of a sequence of matrices along the lines of the numerical algorithm developed in [7, 12]. However, we need n independent maximal vectors to determine completely the optimal performance (and thus the optimal controller) $\begin{pmatrix} W \\ 0 \end{pmatrix} - RQ_o$, which in this case is also in general highly non-unique.

It is worth noting that, in the MIMO form of the ORDAP corresponds the norm of the operator $\|\Xi'\|$ gives the best uncertainty reduction achieved by a single feedback control law as defined in [6].

4 Conclusion

In this paper we have given an operator theoretic solution to the MIMO form of the ORDAP generalizing previous similar results obtained for the SISO case and the

MIMO form of the two-disk problem (10). By restricting the operator Ξ' to certain finite dimensional subspaces of $H^2(\mathbb{C}_{n\times n})$ it can be shown that the norm $\|\Xi'\|$ can be approximated arbitrary by norms of certain matrices, allowing the development of finite dimensional numerical algorithms, along the lines of [7, 12].

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