REDUCED-ORDER ROBUST $\mathcal{H}_\infty$ FILTERING FOR LINEAR PARAMETER VARYING SYSTEMS

G. Iulia Bara, Mohamed Boutayeb

LSIIT-UMR CNRS-ULP 7005
ENSPS, bd Sébastien Brant
67412 Illkirch, France
Iuliana.Bara@ensps.u-strasbg.fr
Mohamed.Boutayeb@ipst-ulp.u-strasbg.fr
Phone: (+33)3 90 24 48 62
Fax: (+33)3 90 24 44 80

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Abstract

In this paper, we address the robust reduced-order filtering problem for linear parameter-varying (LPV) systems using an $\mathcal{H}_\infty$-setting. The stability and the performance in a $\mathcal{L}_2$-gain sense of the filtering error is based on the existence of an affine parameter-dependent Lyapunov function. Our synthesis method gives sufficient conditions for the filter design which are expressed as new easily tractable LMI feasibility conditions with equality constraints. A numerical example illustrates the applicability of our method.

1 Introduction

There are many motivations to study the linear parameter-varying (LPV) systems : many physical systems can be modelled as a linear system whose state space representation matrices are functions of a set of parameters that can be time-varying. The gain scheduling methodology conceptually involves a linear parameter-dependent plant and thus the implication of parameter-dependent systems theory is obvious.

These reasons justify the efforts in the extension of $\mathcal{H}_\infty$-type control to time-varying or LPV systems. Earlier stability analysis and control synthesis results were based on quadratic stability using a single quadratic Lyapunov function [3, 2] which is usually conservative. Recently, some progress have been reported on the use of parameter-dependent Lyapunov functions [2, 9, 5]. There is an increasing interest in the robust $\mathcal{H}_\infty$ literature, apart from the robust control area, and a number of papers addressed the problem of robust state estimation. Reduced-order (or functional) filters are often desirable to reduce the conservatism of the full-order filter design results as well as the complexity and computational burden of the real-time filtering process. The interested reader can see [8, 13] and the references therein for reduced-order robust filtering problems of linear time invariant systems. In [4], we find a solution to the optimal unbiased reduced-order filtering problem for linear time-varying systems. The case of uncertain systems was considered in [7] for polytopic systems and in [12] for systems with a nonlinear dependence in the uncertain parameter. In the case of LPV systems, the filter synthesis methods are based on the measurability of the parameters and we find only few works on the reduced-order robust parameter-dependent filter design. In [10], a robust full-order filter with reduced-order state output is synthesized using LMI conditions.

In this paper, we address the reduced-order parameter-dependent filter design problem for affine LPV systems. In order to guarantee exponential stability with performance in a $\mathcal{L}_2$-gain sense for the filtering error, we use the parameter-dependent γ-performance notion which is a generalized LPV version of the standard $\mathcal{H}_\infty$ problem. This notion uses a parameter-dependent Lyapunov function and it is expressed as a parameter-dependent version of the bounded real lemma. First, we give new sufficient conditions for the filter synthesis in terms of LMI with equality constraint resolution. Second, the use of affine structures for the unknown design matrices and the application of the multiconvexity concept allow to formulate the filter synthesis problem as a finite-set LMI feasibility problem with equality constraint.

The paper is organized as follows. In Section 2, we present the problem under consideration and in Section 3 we present our robust filter synthesis method. We end this paper by an illustrative example and the conclusions.

Notations: The notations used throughout this paper are standard. Uppercase letters denote matrices of appropriate dimensions and lowercase letters denote vectors or scalars. The notation $A > 0$ ($A \geq 0$) stands for a positive definite (semi-definite) symmetric matrix. In long matrix expressions, we use $(\star)$ for terms that are induced by symmetry. $(\cdot)^+$ denotes the Moore-Penrose generalized inverse of a matrix.

2 Problem statement

The problem addressed in this paper is the following. Consider we are given the class of linear parameter-varying (LPV) systems represented by:

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) + B_w(\rho)w(t) \quad (1a)$$
$$y(t) = C(\rho)x(t) + D(\rho)u(t) + D_w(\rho)w(t) \quad (1b)$$
$$z(t) = Lx(t) \quad (1c)$$

where $x(t) \in I\!R^n$ is the state vector, $u(t) \in I\!R^m$ is the control input vector, $y(t) \in I\!R^p$ is the measured output vector and $w(t) \in I\!R^{m_w}$ is the disturbance vector. The vector $z(t) \in I\!R^q$ is the output to be estimated as a linear combination of state $x(t)$. The matrices $A(\rho)$, $B(\rho)$, $B_w(\rho)$, $C(\rho)$, $D(\rho)$ and $D_w(\rho)$ are parameter-dependent matrices of appropriate dimensions which are affine, that means:

$$\left[\begin{array}{c|cc} A(\rho) & B(\rho) & B_w(\rho) \\ \hline C(\rho) & D(\rho) & D_w(\rho) \end{array}\right] = \left[\begin{array}{c|cc} A_0 & B_0 & B_{w_0} \\ \hline C_0 & D_0 & D_{w_0} \end{array}\right] + \sum_{i=1}^{N} \rho_i \left[\begin{array}{c|cc} A_i & B_i & B_{w_i} \\ \hline C_i & D_i & D_{w_i} \end{array}\right] \quad (2)$$

where A_i, B_i, B_{w_i}, C_i, D_i and D_{w_i} are known matrices. Without loss of generality, we can assume that the matrix L is constant.

The parameter vector $\rho(t) = \begin{bmatrix} \rho_1(t) & \rho_2(t) & \ldots & \rho_N(t) \end{bmatrix}^T$ is assumed to be real, continuous, time-varying and satisfying the following constraints:

1. each parameter $\rho_i(t)$ is real time measurable and it ranges between known extremal values $\rho_i(t) \in [\underline{\rho_i}, \overline{\rho_i}]$
2. the variation rate of each parameter $\dot{\rho}_i(t)$ is limited by known upper and lower bounds, that is: $\dot{\rho}_i(t) \in [\underline{\tau_i}, \overline{\tau_i}]$.

Note that the assumptions 1. and 2. are not restrictive. The first one can always be achieved by a change of variable and it means that the parameter vector $\rho(t)$ is valued in a hyper-rectangle with vertices defined by:

$$\mathcal{V} = \{(\omega_1, \omega_2, \ldots, \omega_N) |\ \omega_i \in \{\underline{\rho_i}, \overline{\rho_i}\}\}.$$

The second assumption defines a hyper-rectangle for the rate of variation of the parameter vector whose vertices are

$$\mathcal{S} = \{(\tau_1, \tau_2, \ldots, \tau_N) |\ \tau_i \in \{\underline{\tau_i}, \overline{\tau_i}\}\}.$$

In the sequel of this paper, the parameter dependence upon time is suppressed for clarity. In fact, ρ or ρ_i stands for $\rho(t)$ or $\rho_i(t)$ respectively.

In this paper, we are concerned with the design for system (1) of a robust parameter-dependent reduced-order filter or estimator in the form

$$\dot{\hat{z}}(t) = H(\rho)\hat{z}(t) + J(\rho)u(t) + K(\rho)y(t) \quad (3)$$

where $\hat{z}(t)$ is an estimate of $z(t)$. The problem of the filter design consists in finding the functional matrices $H(\rho)$, $J(\rho)$ and $K(\rho)$ of appropriate dimensions which provide good robust estimation of the output $z(t)$ *i.e.* which provide an uniformly small estimation error

$$e(t) = z(t) - \hat{z}(t) = Lx(t) - \hat{z}(t) \quad (4)$$

for all $\mathcal{L}_2$-bounded disturbance input vector $w(t)$. In order to solve this synthesis problem, we proceed using an $\mathcal{H}_\infty$-setting as presented in the following section.

3 Robust $\mathcal{H}_\infty$ filtering

The robust $\mathcal{H}_\infty$ filtering problem consists in finding the filter that minimizes the worst case estimation error energy $\| e(t) \|_{\mathcal{L}_2}$ over all bounded energy disturbances $w(t)$. Based on the induced $\mathcal{L}_2$-gain property of the $\mathcal{H}_\infty$ norm, we address this problem by using the parameter-dependent γ-performance notion.

By parameter-dependent γ-performance we mean the existence of a parameter-dependent quadratic Lyapunov function ensuring asymptotic stability of an LPV system and a bound on its $\mathcal{L}_2$ gain. We recall this notion in the following.

Definition 3.1. *Given a linear system of the form*

$$\dot{\eta}(t) = A(\rho)\eta(t) + B(\rho)w(t)$$
$$y(t) = C(\rho)\eta(t) + D(\rho)w(t)$$

and the performance level $\gamma > 0$, the parameter-dependent γ-performance problem is solvable if there exists a positive definite matrix $P(\rho)$ such that the following inequality

$$\begin{bmatrix} A(\rho)^T P(\rho) + P(\rho)A(\rho) + \dot{P}(\rho) & P(\rho)B(\rho) & C^T(\rho) \\ B^T(\rho)P(\rho) & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0$$

holds for all admissible trajectories of the parameter vector ρ.

Remark 3.1. i) The parameter-dependent quadratic Lyapunov function is $V(\eta, \rho) = \eta^T P(\rho)\eta$ and this function ensures that the system is asymptotically and exponentially stable and its $\mathcal{L}_2$ gain does not exceed γ. That means

$$\| \eta(t) \|_{\mathcal{L}_2} < \gamma \| w(t) \|_{\mathcal{L}_2}$$

for all $\mathcal{L}_2$-bounded input $w(t)$. In the following, we apply the parameter dependent γ-performance notion on the filtering error system.

ii) When the Lyapunov matrix is affine parameter-dependent *i.e.*

$$P(\rho) = P_0 + \rho_1 P_1 + \ldots + \rho_N P_N$$

then the Definition 3.1 is identical to the *affine quadratic $\mathcal{H}_\infty$ performance* given in [6]. In this case $\dot{P}(\rho) = P(\dot{\rho}) - P_0$.

We use this $\mathcal{H}_\infty$-setting in order to study the stability and guarantee the performance of the estimation error. Using the definition of the estimation error (4), we find that its dynamic is

$$\begin{bmatrix} P(\rho)LA(\rho)L^+ - W(\rho)C(\rho)L^+ + \mathcal{Q}(\rho)(I-LL^+) + (\star) + \dot{P}(\rho) + I & P(\rho)LB_w(\rho) - W(\rho)D_w(\rho) \\ B_w^T(\rho)L^TP(\rho) - D_w^T(\rho)W^T(\rho) & -\gamma^2 I \end{bmatrix} < 0 \tag{5}$$

given by:

$$\begin{aligned}\dot{e} = H(\rho)e + \big(LA(\rho) - H(\rho)L - K(\rho)C(\rho)\big)x + \\ + \big(LB(\rho) - J(\rho) - K(\rho)D(\rho)\big)u \\ + \big(LB_w(\rho) - K(\rho)D_w(\rho)\big)w.\end{aligned} \tag{6}$$

In order to have an estimation error which is independent from the control input and the system state, we put the following conditions

$$LA(\rho) - H(\rho)L - K(\rho)C(\rho) = 0 \tag{7a}$$
$$J(\rho) = LB(\rho) - K(\rho)D(\rho). \tag{7b}$$

If these relations are satisfied then the estimation error is given by:

$$\dot{e} = H(\rho)e + \big(LB_w(\rho) - K(\rho)D_w(\rho)\big)w. \tag{8}$$

From the general solution of a linear matrix equation [11] it follows that the equation (7a) has a solution if and only if

$$\big(LA(\rho) - K(\rho)C(\rho)\big)\big(I - L^+L\big) = 0 \tag{9}$$

When this condition is satisfied, the solution of the equation (7a) according only to the unknown matrix $H(\rho)$ is given by the following parametrization

$$H(\rho) = \big(LA(\rho) - K(\rho)C(\rho)\big)L^+ + \mathcal{Z}(\rho)\big(I - LL^+\big) \tag{10}$$

where $\mathcal{Z}(\rho)$ is an arbitrary matrix of appropriate dimensions. This matrix intervenes as a supplementary degree of freedom for the robust filter synthesis.

Conditions for the synthesis of robust LPV $\mathcal{H}_\infty$ filter are provided by the following result. This result gives sufficient conditions for the quadratic parameter-dependent γ-performance of the filtering error as defined in Definition 3.1.

Theorem 3.1. *There exists a q^{th}-order robust filter of the form* (3) *for the LPV system* (1) *if there exist a symmetric matrix $P(\rho) > 0$ and matrices $\mathcal{Q}(\rho)$ and $W(\rho)$ of appropriate dimensions satisfying the LMI* (5) *and the following equality constraint*

$$\big(P(\rho)LA(\rho) - W(\rho)C(\rho)\big)\big(I - L^+L\big) = 0 \tag{11}$$

for all admissible parameter trajectories. Then the gain matrix $K(\rho)$ is given by $P^{-1}(\rho)W(\rho)$, $\mathcal{Z}(\rho)$ is given by $P^{-1}(\rho)\mathcal{Q}(\rho)$ and the matrices $H(\rho)$ and $J(\rho)$ are given by (10) *and* (7b) *respectively.*

Proof. According to the Definition 3.1, the filtering error equation (8) satisfies the γ-performance criterion if there exists a matrix $P(\rho) > 0$ such that

$$\begin{bmatrix} P(\rho)H(\rho) + (\star) + \dot{P}(\rho) & (\star) & I \\ \big(LB_w(\rho) - K(\rho)D_w(\rho)\big)^T P(\rho) & -\gamma I & 0 \\ I & 0 & -\gamma I \end{bmatrix} < 0 \tag{12}$$

for all parameter trajectories. Applying the Schur complement, replacing $H(\rho)$ by the equation (10) and using the notations $W(\rho) = P(\rho)K(\rho)$ and $\mathcal{Q}(\rho) = P(\rho)\mathcal{Z}(\rho)$, we find that this inequality is equivalent to the inequality (5). As the matrix $P(\rho)$ is positive definite and consequently it is of full rank, we can multiply to the left the equation (9) by $P(\rho)$. Using the above notations, we find that the equality constraint (11) is equivalent to the condition (9) which guarantees the existence of a solution for the equation (7a). As the matrix $P(\rho)$ is positive definite we can compute the matrix gain $K(\rho)$ as $P^{-1}(\rho)W(\rho)$ and $\mathcal{Z}(\rho)$ as $P^{-1}(\rho)\mathcal{Q}(\rho)$. □

In order to design the robust LPV filter, according to the Theorem 3.1, we must solve the matrix inequality (5) with the equality constraint (11) which are linear according to the unknown matrices. Consequently we must solve a convex feasibility problem. But, this problem involves a infinite number of conditions to satisfy. In order to reduce this infinite number of conditions at a finite number and thus to formulate easily computationally tractable conditions, we use the notion of multiconvexity as described in [6]. To this end, we must make a choice on the structure of the unknown matrices $P(\rho)$, $W(\rho)$ and $\mathcal{Q}(\rho)$. In the following, we suppose that the matrices $P(\rho)$ and $W(\rho)$ are affine. As

$$P(\rho)H(\rho) = P(\rho)LA(\rho)L^+ - W(\rho)C(\rho)L^+ + \mathcal{Q}(\rho)(I - LL^+)$$

we chose for $\mathcal{Q}(\rho)$ a second order structure *i.e.*

$$\mathcal{Q}(\rho) = \mathcal{Q}_{00} + \sum_{\substack{i,j=0 \\ i \le j}}^{N} \mathcal{Q}_{ij}\rho_i\rho_j \quad \text{where } \rho_0 = 1 \text{ by convention.}$$

The next theorem gives a sufficient solution to the robust filter synthesis problem based on the multiconvexity principle and the choice of design matrices structures given above. This solution is expressed as a finite number of LMI with a finite number of linear equality constraints and thus it is easily tractable numerically. Note that the following result is based on the affine quadratic $\mathcal{H}_\infty$ performance criterion of the estimation error.

Theorem 3.2. *Consider the LPV system* (1) *and a given $\gamma > 0$. Suppose that there exist positive scalars ν_i for $i \in \{1,..,N\}$, symmetric matrices P_i for $i \in \{0,..,N\}$ and the matrices $\mathcal{Q}_{ij}$ for $i,j \in \{0,..,N\}$, $i \le j$, W_i for $i \in \{0,..,N\}$ such that the LMIs* (13), (14), (15) *are satisfied with the following equality constraints*

$$\big(P_iLA_i - W_iC_i\big)\big(I - L^+L\big) = 0 \quad \textit{for all} \quad i \in \{0,\dots,N\} \tag{16}$$

$$\big(P_iLA_j + P_jLA_i - W_iC_j - W_jC_i\big)\big(I - L^+L\big) = 0 \quad \textit{for all} \quad i,j \in \{0,\dots,N\} \textit{ and } i \neq j \tag{17}$$

$$P(\omega) > 0 \quad \text{for all} \quad \omega \in \mathcal{V} \tag{13}$$

$$\begin{bmatrix} P(\omega)LA(\omega)L^+ - W(\omega)C(\omega)L^+ + \mathcal{Q}(\omega)(I - LL^+) + (\star) + P(\tau) - P_0 + I & P(\omega)LB_w(\omega) - W(\omega)D_w(\omega) \\ B_w^T(\omega)L^T P(\omega) - D_w^T(\omega)W^T(\omega) & -\gamma^2 I + \sum_{i=1}^{N} \nu_i \omega_i^2 I \end{bmatrix} < 0 \quad \text{for all} \quad (\omega, \tau) \in \mathcal{V} \times \mathcal{W} \tag{14}$$

$$\begin{bmatrix} P_i L A_i L^+ - W_i C_i L^+ + \mathcal{Q}_{ii}(I - LL^+) + (\star) & P_i L B_{w_i} - W_i D_{w_i} \\ B_{w_i}^T L^T P_i - D_{w_i}^T W_i^T & \nu_i I \end{bmatrix} \geq 0 \quad \text{for all} \quad i \in \{1, \ldots, N\} \tag{15}$$

where □

$$P(\omega) = P_0 + \omega_1 P_1 + \ldots + \omega_N P_N \tag{18}$$

$$W(\omega) = W_0 + \omega_1 W_1 + \ldots + \omega_N W_N \tag{19}$$

$$\mathcal{Q}(\omega) = \mathcal{Q}_{00} + \sum_{\substack{i,j=0 \\ i \leq j}}^{N} \mathcal{Q}_{ij}\omega_i\omega_j \quad \textit{and} \ \omega_0 = 1. \tag{20}$$

Then the q^{th}-order filter (3) *where $K(\rho)$ is given by $P^{-1}(\rho)W(\rho)$, $\mathcal{Z}(\rho)$ is given by $P^{-1}(\rho)\mathcal{Q}(\rho)$ and the matrices $H(\rho)$ and $J(\rho)$ are given by* (10) *and* (7b) *respectively, is a robust filter ensuring the affine quadratic $\mathcal{H}_\infty$-performance index γ for the estimation error.*

Proof. The proof is based on Theorem 3.1. Consider the function $f(\rho, \dot{\rho})$ given by (21) where $\mathcal{X} \in I\!R^{q+m_w}$.

Choosing a Lyapunov matrix with affine parameter dependence and according to Theorem 3.1, the filtering error (8) satisfies the parameter-dependent γ-performance conditions given by Definition 3.1 or more precisely the affine quadratic $\mathcal{H}_\infty$ performance conditions if $f(\rho, \dot{\rho}) < 0$ with $P(\rho) > 0$ are satisfied for all the admissible parameter trajectories and for all $\mathcal{X} \neq 0$. We require that the function $f(\rho, \dot{\rho})$ be bounded by a multiconvex function

$$f_{conv}(\rho, \dot{\rho}) = f(\rho, \dot{\rho}) + \mathcal{X}^T \sum_i \nu_i \rho_i^2 \mathcal{X} \geq f(\rho, \dot{\rho}) \text{ where } \nu_i > 0.$$

Hence, the equation (8) satisfies the affine $\mathcal{H}_\infty$ performance conditions if $f_{conv}(\rho, \dot{\rho}) < 0$ with $P(\rho) > 0$ affine for all the admissible parameter trajectories and for all $\mathcal{X} \neq 0$.

The inequalities (13) and (14) imply that $f_{conv}(\rho, \dot{\rho})$ is negative definite and $P(\rho)$ is positive definite at the vertices $\mathcal{V} \times \mathcal{S}$ and respectively $\mathcal{V}$ of the domain containing all the parameter trajectories.

The inequalities (15) are the multiconvexity conditions of the function $f_{conv}(\rho, \dot{\rho})$ according to the Lemma 5.1 (see Appendix) with a partial degree $d_k = 2$ for $k = 1, \ldots, N$ and a total degree $d = 2$. As the matrix $P(\rho)$ is affine on the parameters, to ensure that it is positive definite, it is sufficient to impose the condition (13).

To demonstrate the applicability of our method, we consider a numerical example in the next section.

4 Numerical example

Consider an affine LPV system depending on one parameter described by the set of matrices

$$A_0 = \begin{bmatrix} -1.3 & 1.1 \\ 0.3 & -2.2 \end{bmatrix}, A_1 = \begin{bmatrix} 0.2 & -1.2 \\ -1 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0.6 \\ 1.2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, B_{w_0} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, B_{w_1} = \begin{bmatrix} 0.2 \\ -0.3 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.7 & 0.1 \end{bmatrix}, D_0 = \begin{bmatrix} 0.1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.5 \end{bmatrix}, D_{w_0} = \begin{bmatrix} -0.2 \end{bmatrix},$$
$$D_{w_1} = \begin{bmatrix} 0.1 \end{bmatrix}.$$

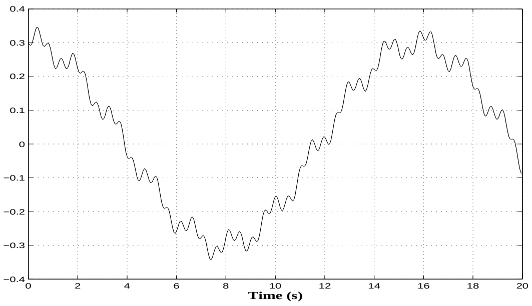
The parameter ρ ranges in $[-0.4, 0.4]$ and its rate $\dot{\rho} \in [-0.5, 0.5]$. We want to estimate $z = x_1$ using the synthesis result given by the Theorem 3.2. The implementation of this theorem allows us to obtain the following matrices: $P(\rho) = 0.6973 + \rho 0.0697$, $W(\rho) = 0.7671 - \rho 0.8368$. As $I - LL^+ = 0$, the matrix $\mathcal{Q}$ does'nt intervene in the filter design.

The smallest performance index achievable with the conditions of Theorem 3.2 is $\gamma = 0.4$. The filter estimation error is indicated in Figure 1(b) and this error corresponds to an evolution of the parameter as shown in Figure 1(a). For this simulation we choose a perturbation signal bounded by $\| w(t) \|_{\mathcal{L}_2} < 1.31$. We can see on Figure 1(b) that the estimation error is exponentially stable and bounded.

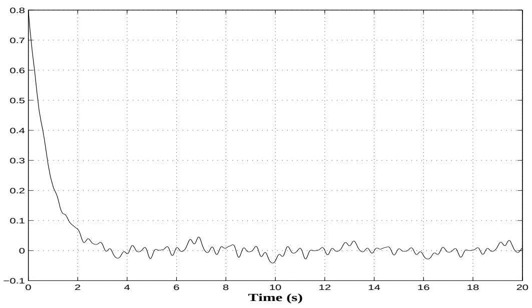
5 Conclusion

In this paper, a systematic framework for robust parameter-dependent reduced-order filter design has been addressed in $\mathcal{H}_\infty$-setting. The main contribution of this paper lies in the proposed method which gives an easily tractable LMI solution with equality constraint to the filtering problem. This solution is based on the existence of a parameter-dependent Lyapunov function and uses a parameter-dependent version of the

$$\mathcal{X}^T \begin{bmatrix} f(\rho,\dot{\rho}) = \rho)LA(\rho)L^+ - W(\rho)C(\rho)L^+ + \mathcal{Q}(\rho)(I - LL^+) + (\star) + P(\dot{\rho}) - P_0 + I & P(\rho)LB_w(\rho) - W(\rho)D_w(\rho) \\ B_w^T(\rho)L^T P(\rho) - D_w^T(\rho)W^T(\rho) & -\gamma^2 I \end{bmatrix} \mathcal{X} \quad (21)$$



(a) The parameter $\rho_1(t)$.



(b) Estimation error

Figure 1: Simulation results

bounded real lemma.

Appendix: Mathematical tools

Our approach relies on the concept of multiconvexity that is convexity along each direction of the parameter space. The concept of multiconvexity for scalar quadratic functions is given in [6] and a similar result for polynomial functions is presented in [1]. We state here the concept of multiconvexity for a polynomial function.

Lemma 5.1 ([1]). *Consider a general polynomial function $f(\rho_1,\dots,\rho_N)$ of arbitrary order. Denote d_k the partial degree with respect to the variable ρ_k, $k = 1,\dots,N$ and d the total degree of the polynomial function. Then $f(\cdot)$ is negative (resp. positive) in the hyper-rectangle whose vertices are given by $\mathcal{V}$ whenever*

$$f(\rho) < 0, \quad (\textit{resp.} > 0) \quad \forall \rho \in \mathcal{V}, \qquad (22)$$

and

$$(-1)^m \frac{\partial^{2m} f(\rho)}{\partial \rho_{l_1}^2 \dots \partial \rho_{l_m}^2} \le 0 \quad (\textit{resp.} \ge 0) \quad \forall \rho \in \mathcal{V}, \qquad (23)$$

where

$$1 \le l_1 \le l_2 \le \cdots \le l_m \le N, \quad 1 \le m \le d/2$$

$$\mathrm{Card}(\{l_j = k, j \in \{1,\dots,m\}\}) \le d_k/2, \quad k = 1,2,\dots,N.$$

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