CONTINUOUS-TIME $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL WITH NON-COMMON LYAPUNOV VARIABLES VIA CONVERGENT ITERATIONS

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Abstract

This paper presents a multi-objective output-feedback $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework with non-common Lyapunov variables (NCLV's) for continuous-time systems, and clearly summarized the numerical algorithms for controller design. This $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework is less conservative than the traditional one with common Lyapunov variables (CLV's). Although the computation of this new method is more time-consuming, the controller obtained remains the same order as the one designed by the traditional method. Furthermore, this framework is ready to be extended to a more general multi-objective framework with slight changes. The advantage of this framework is illustrated by a numerical example.

1 Introduction

In the past decade, $\mathcal{H}_2/\mathcal{H}_\infty$ control has been studied extensively. Early work [2, 15, 8, 30, 28] focused on designing sub-optimal controllers by solving algebraic Riccati equations (ARE's). Later on, the linear-matrix-inequality (LMI) technique [3], a numerically attractive alternative, was applied to $\mathcal{H}_\infty$ control [11, 14]. It was convenient to combine different specifications in terms of LMI's and design a multi-objective controller [19, 16]. Unfortunately, in order to linearize bilinear variables, one had to equalize all the Lyapunov variables, i.e. a CLV was used, which resulted in a conservative design.

Recently, there has been some progress in designing less-conservative $\mathcal{H}_2/\mathcal{H}_\infty$ controllers. Roughly speaking, there exist three methods for less conservative design: Youla parameterization, convergent synthesis iterations and dilated LMI's.

[20, 13, 21, 22, 7] utilized a Youla parameterization technique to compute sub-optimal controllers. The objective value converged to the true optimum at a cost of controller dimension increase.

[23, 24, 10] presented an $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework via successive iterations. [23, 24] substituted non-positive quadratic terms with their upper bounds and proposed a convergent (strictly speaking, non-divergent) iterative algorithm. Because the iterative algorithm could not guarantee the global optimization, the more substitutions one used, the more conservative the design would be.

[5, 6, 1, 10] decoupled the Lyapunov and controller variables by "dilating" LMI's and introducing a new common variable. This novel idea initially came from [5] and was then applied to a discrete-time $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework. [1] presented the so-called reciprocal projection lemma to split certain Lyapunov terms, thus extending results in [6] to continuous-time systems. Although some work [1, 9] has been carried out to design $\mathcal{H}_2$ and eigenstructure assignment or D-stability controllers for continuous-time systems, it still remains open and challenging to incorporate $\mathcal{H}_\infty$ control into the existing framework [1]. This difficulty comes from the fact that certain terms are always in the same row and column of the LMI for $\mathcal{H}_\infty$ control. In this paper, we present an $\mathcal{H}_2/\mathcal{H}_\infty$ framework for continuous-time system via LMI dilation and give an iterative algorithm.

The paper is organized as follows. Section 2 states system models and gives a preliminary lemma. Section 3 derives the new characterizations for $\mathcal{H}_\infty$ analysis and lists the compatible LMI's for $\mathcal{H}_2$ analysis, while Section 4 addresses the $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem. Section 5 demonstrates the advantage of this framework by an example and Section 6 makes concluding remarks and maps out future research directions. The notation is standard. $A_1 \oplus A_2$ is the direct sum of matrices A_1 and A_2, i.e. $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. $G \star K$ denotes the Redheffer star-product of G and K.

2 Preliminaries

The system models used throughout this paper are continuous-time multi-input, multi-output (MIMO) and linear-time-invariant (LTI). The plant G is given by the state-space equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + B_w w + Bu \\ z & = & C_z x + D_{zw} w + D_z u \\ y & = & Cx + D_w w \end{array} \right. \quad (1)$$

where w, u, z, and y are the vectors of exogenous inputs, control inputs, regulated outputs, and measured outputs respectively.

The plant G can be partitioned into two channels G_1 and G_2 by a linear transformation

$$\begin{array}{rcl} G_j & \triangleq & (L_j \oplus I) G (R_j \oplus I) \\ & \triangleq & \left(\begin{array}{c|cc} A & B_j & B \\ \hline C_j & D_j & E_j \\ C & F_j & 0 \end{array} \right) \end{array} \quad (2)$$

where $j \in \{1, 2\}$, and the matrices L_j and R_j select the appropriate input and output channels respectively.

The dynamical controller is $K = C_K(sI - A_K)^{-1}B_K + D_K$. Then, the closed-loop system $T = G \star K$ has the following realization

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc|c} A + BD_KC & BC_K & B_w + BD_KD_w \\ B_KC & A_K & B_KD_w \\ \hline C_z + D_zD_KC & D_zC_K & D_{zw} + D_zD_KD_w \end{array}\right) \tag{3}$$

and the closed-loop subsystems $T_j = G_j \star K = L_jTR_j$ for different channels are given by

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array}\right) \triangleq \left(\begin{array}{cc|c} A + BD_KC & BC_K & B_j + BD_KF_j \\ B_KC & A_K & B_KF_j \\ \hline C_j + E_jD_KC & E_jC_K & D_j + E_jD_KF_j \end{array}\right) \tag{4}$$

To facilitate subsequent derivations, we state the following lemma [1] without proof.

Lemma 1 (Reciprocal Projection Lemma) *For any $P > 0$, the following statements are equivalent:*

(i): $\Psi + S + S^T < 0$

(ii): $\begin{pmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{pmatrix} < 0$ *is feasible with respect to W.*

3 New Characterizations for $\mathcal{H}_2/\mathcal{H}_\infty$ Analysis

3.1 $\mathcal{H}_\infty$ Analysis

Theorem 1 ($\mathcal{H}_\infty$ Analysis) *The following statements, involving a symmetric variable X_1 and general variables $\bar{L}_1$ and V, are equivalent.*

(i) : *$\mathcal{A}$ is stable and $\|T_1\|_\infty < \gamma$.*

(ii): $\exists X_1 > 0$ *such that*

$$\begin{pmatrix} \mathcal{A}^TX_1 + X_1\mathcal{A} & X_1\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^TX_1 & -I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \tag{5}$$

(iii): $\exists X_1 > 0$, $\bar{L}_1$, *and V such that*

$$\begin{pmatrix} -(V^T + V) & V^T\mathcal{A} + X_1 & V^T\mathcal{B}_1 & 0 & V^T \\ \mathcal{A}V + X_1 & -X_1 & 0 & 0 & 0 \\ \mathcal{B}_1^TV & 0 & -I & \mathcal{D}_1^T & 0 \\ 0 & 0 & \mathcal{D}_1 & \Pi & \mathcal{C}_1 \\ V & 0 & 0 & \mathcal{C}_1^T & -X_1 \end{pmatrix} < 0 \tag{6}$$

where $\Pi = -\gamma^2I + \Phi(X_1, \mathcal{C}_1, \bar{L}_1)$ *and* $\Phi(X_1, \mathcal{C}_1, \bar{L}_1) \triangleq \bar{L}_1^TX_1\bar{L}_1 - \bar{L}_1^T\mathcal{C}_1^T - \mathcal{C}_1\bar{L}_1$.

Proof: The equivalence between (i) and (ii) is a standard result [11, 18, 29]. In the sequel, we will prove the equivalence between (ii) and (iii).

Define $Y_1 \triangleq X_1^{-1}$. A congruence transformation $Y_1 \oplus I \oplus I$ to (5) yields

$$\begin{pmatrix} Y_1\mathcal{A}^T + \mathcal{A}Y_1 & \mathcal{B}_1 & Y_1\mathcal{C}_1^T \\ \mathcal{B}_1^T & -I & \mathcal{D}_1^T \\ \mathcal{C}_1Y_1 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \tag{7}$$

By the reciprocal projection lemma, (7) is equivalent to

$$\begin{pmatrix} Y_1 - (W + W^T) & \mathcal{A}Y_1 + W^T & \mathcal{B}_1 & Y_1\mathcal{C}_1^T \\ Y_1\mathcal{A}^T + W & -Y_1 & 0 & 0 \\ \mathcal{B}_1^T & 0 & -I & \mathcal{D}_1^T \\ \mathcal{C}_1Y_1 & 0 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \tag{8}$$

Define $V \triangleq W^{-1}$ (W is assumed to be invertible, which is always possible by perturbation if necessary [1]). A congruence transformation $V \oplus X_1 \oplus I \oplus I$ to (8) yields

$$\begin{pmatrix} V^TX_1^{-1}V - (V^T + V) & V^T\mathcal{A} + X_1 & V^T\mathcal{B}_1 & V^TX_1^{-1}\mathcal{C}_1^T \\ \mathcal{A}^TV + X_1 & -X_1 & 0 & 0 \\ \mathcal{B}_1^TV & 0 & -I & \mathcal{D}_1^T \\ \mathcal{C}_1X_1^{-1}V & 0 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \tag{9}$$

Let

$$\Psi \triangleq \begin{pmatrix} -(V^T + V) & V^T\mathcal{A} + X_1 & V^T\mathcal{B}_1 & 0 \\ \mathcal{A}^TV + X_1 & -X_1 & 0 & 0 \\ \mathcal{B}_1^TV & 0 & -I & \mathcal{D}_1^T \\ 0 & 0 & \mathcal{D}_1 & -\gamma^2I - \mathcal{C}_1X_1^{-1}\mathcal{C}_1^T \end{pmatrix}$$

Then, (9) can be written as

$$\Psi + \begin{pmatrix} V^T \\ 0 \\ 0 \\ \mathcal{C}_1 \end{pmatrix} X_1^{-1} \begin{pmatrix} V & 0 & 0 & \mathcal{C}_1^T \end{pmatrix} < 0 \tag{10}$$

By the Schur lemma [3], (10) is equivalent to

$$\begin{pmatrix} -(V^T + V) & V^T\mathcal{A} + X_1 & V^T\mathcal{B}_1 & 0 & V^T \\ \mathcal{A}^TV + X_1 & -X_1 & 0 & 0 & 0 \\ \mathcal{B}_1^TV & 0 & -I & \mathcal{D}_1^T & 0 \\ 0 & 0 & \mathcal{D}_1 & -\gamma^2I - \mathcal{C}_1X_1^{-1}\mathcal{C}_1^T & \mathcal{C}_1 \\ V & 0 & 0 & \mathcal{C}_1^T & -X_1 \end{pmatrix} < 0 \tag{11}$$

We will show in the following that (11) is equivalent to (6).

Motivated by the idea in [24], we construct the following inequality

$$(\bar{L}_1^T - \mathcal{C}_1X_1^{-1})X_1(\bar{L}_1 - X_1^{-1}\mathcal{C}_1^T) \geq 0 \tag{12}$$

which is always satisfied since X_1 is positive. Then, we obtain

$$-\mathcal{C}_1X_1^{-1}\mathcal{C}_1^T \leq \bar{L}_1^TX_1\bar{L}_1 - \bar{L}_1^T\mathcal{C}_1^T - \mathcal{C}_1\bar{L}_1 = \Phi(X_1, \mathcal{C}_1, \bar{L}_1) \tag{13}$$

It is clear that (6) is a sufficient condition for (11). Also, by setting $\bar{L}_1 = X_1^{-1}\mathcal{C}_1^T$, (11) becomes (6), i.e. (11) is a sufficient condition for (6). Hence (11) is equivalent to (6).

Therefore, the statements (i), (ii), and (iii) are equivalent. The proof is completed. □

Remark 1 *The item $\Phi(X_1, \mathcal{C}_1, \bar{L}_1)$ in* (6) *is not linear. However, if $\bar{L}_1$ is set to a constant matrix, $\bar{L}_1^TX_1\bar{L}_1$ will be linear and* (6) *will become a sufficient condition for* (5). *Under this condition the computed objective value is an upper bound on the true optimum.*

3.2 $\mathcal{H}_2$ Analysis [1]

For completeness, we also list the LMI characterizations for $\mathcal{H}_2$ control. Readers can refer to [1] for the details.

Theorem 2 ($\mathcal{H}_2$ Analysis) *For a given system T_2 with $\mathcal{D}_2 = 0$, the following statements, involving symmetric variables X_2, Z_2 and a general variable V, are equivalent.*

(i): *$\mathcal{A}$ is stable and $\|T_2\|_2^2 < \alpha$.*

(ii): *$\exists X_2 > 0$, $Z_2 > 0$ such that*

$$\begin{pmatrix} \mathcal{A}^T X_2 + X_2\mathcal{A} & X_2\mathcal{B}_2 \\ \mathcal{B}_2^T X_2 & -I \end{pmatrix} < 0 \quad (14)$$

$$\begin{pmatrix} X_2 & \mathcal{C}_2^T \\ \mathcal{C}_2 & Z_2 \end{pmatrix} > 0, \qquad \text{Trace } Z_2 < \alpha \quad (15)$$

(iii): *$\exists X_2 > 0$, $Z_2 > 0$ and V such that*

$$\begin{pmatrix} -(V^T + V) & V^T\mathcal{A} + X_2 & V^T\mathcal{B}_2 & V^T \\ \mathcal{A}^T V + X_2 & -X_2 & 0 & 0 \\ \mathcal{B}_2^T V & 0 & -I & 0 \\ V & 0 & 0 & -X_2 \end{pmatrix} < 0 \quad (16)$$

$$\begin{pmatrix} X_2 & \mathcal{C}_2^T \\ \mathcal{C}_2 & Z_2 \end{pmatrix} > 0, \qquad \text{Trace } Z_2 < \alpha \quad (17)$$

4 Multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis

In this section, we adopt two techniques to linearize the matrix inequalities in Section 3 and present an $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework.

- The term such as $V^T\mathcal{A}$, $V^T\mathcal{B}_1$, and $V^T\mathcal{B}_2$ in Theorem 1 and Theorem 2 will become bilinear when the synthesis problem is concerned. We follow [11, 19, 1] and linearize these bilinear matrix inequalities (BMI's) via variable changes.

- The term $\Phi(X_1, \mathcal{C}_1, \bar{L}_1)$ in (6) is not linear and can not be linearized by simple algebraic operations. In this paper, motivated by [24], we use successive iterations to solve this problem. Specifically, we set the variable $\bar{L}_1$ to an initial value and solve the LMI's with respect to variables X_1, X_2 and V. The objective value is an upper bound on the true optimum. Once we obtain a controller and its associated variables, we reset $\bar{L}_1$ and design a new controller. The new objective value is always no more than the old one, i.e., the system performance converges to a sub-optimal value.

Partition the variable V and V^{-1} as follows:

$$V \triangleq \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & \star \end{pmatrix}, \quad W = V^{-1} \triangleq \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & \star \end{pmatrix} \quad (18)$$

where $V_{11}, V_{12}, V_{21}, W_{11}, W_{12}, W_{21} \in \mathbb{R}^{n\times n}$. Let $\Pi_V \triangleq \begin{pmatrix} V_{11} & I \\ V_{21} & 0 \end{pmatrix}$ and $\Pi_W \triangleq \begin{pmatrix} I & W_{11} \\ 0 & W_{21} \end{pmatrix}$, then $W\Pi_V = \Pi_W$ and $V\Pi_W = \Pi_V$.

Define

$$\begin{cases} \hat{\mathbf{A}} & \triangleq V_{21}^T A_K W_{21} + V_{21}^T B_K C W_{11} + V_{11}^T B C_K W_{21} \\ & \quad + V_{11}^T (A + B D_K C) W_{11} \\ \hat{\mathbf{B}} & \triangleq V_{21}^T B_K + V_{11}^T B D_K \\ \hat{\mathbf{C}} & \triangleq C_K W_{21} + D_K C W_{11} \\ \hat{\mathbf{D}} & \triangleq D_K \\ U & \triangleq V_{11}^T W_{11} + V_{21}^T W_{21} \end{cases} \quad (19)$$

Then, we obtain

$$\begin{cases} \Pi_W^T V^T \mathcal{A} \Pi_W \triangleq \overrightarrow{V^T A} = \begin{pmatrix} V_{11}^T A + \hat{\mathbf{B}} C & \hat{\mathbf{A}} \\ A + B\hat{\mathbf{D}} C & A W_{11} + B\hat{\mathbf{C}} \end{pmatrix} \\ \Pi_W^T V^T \mathcal{B}_j \triangleq \overrightarrow{V^T B_j} = \begin{pmatrix} V_{11}^T B_j + \hat{\mathbf{B}} F_j \\ B_j + B\hat{\mathbf{D}} F_j \end{pmatrix} \\ \mathcal{C}_j \Pi_W \triangleq \overrightarrow{C_j} = \begin{pmatrix} C_j + E_j \hat{\mathbf{D}} C & C_j W_{11} + E_j \hat{\mathbf{C}} \end{pmatrix} \\ \Pi_W^T V \Pi_W \triangleq \overrightarrow{V} = \begin{pmatrix} V_{11} & I \\ U^T & W_{11}^T \end{pmatrix} \\ \Pi_W^T X_j \Pi_W \triangleq \overrightarrow{X_j} \end{cases} \quad (20)$$

To be compatible with the variable changes, $\Phi(X_1, \mathcal{C}_1, \bar{L}_1)$ is re-defined as

$$\begin{aligned} \Phi(\overrightarrow{X_1}, \overrightarrow{\mathcal{C}_1}, L_1) &\triangleq L_1^T \Pi_W^T X_1 \Pi_W L_1 - L_1^T \Pi_W^T \mathcal{C}_1^T - \mathcal{C}_1 \Pi_W L_1 \\ &= L_1^T \overrightarrow{X_1} L_1 - L_1^T \overrightarrow{\mathcal{C}_1}^T - \overrightarrow{\mathcal{C}_1} L_1 \end{aligned}$$

where $L_1 = \Pi_W^{-1} X_1^{-1} \mathcal{C}_1^T = (\overrightarrow{X_1})^{-1} (\overrightarrow{\mathcal{C}_1})^T$. When L_1 is set to a constant matrix, we denote the above expression by $\Phi(\overrightarrow{X_1}, \overrightarrow{\mathcal{C}_1})$.

For a given system T in (3) with T_1 measured by the $\mathcal{H}_\infty$ norm and T_2 measure by the $\mathcal{H}_2$ norm, there are three typical multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problems:

Problem (i): Minimize $\|T_2\|_2$ subject to $\|T_1\|_\infty < \gamma$;

Problem (ii): Minimize $\|T_1\|_\infty$ subject to $\|T_2\|_2^2 < \alpha$;

Problem (iii): Minimize $k\|T_1\|_\infty + (1-k)\|T_2\|_2^2$ where $k \in [0, 1]$.

In this paper, due to limited space we only consider Problem (i), which can be interpreted as achieving optimal $\mathcal{H}_2$ performance while guaranteeing a certain level of robust stability.

Now, we are in a position to present a sufficient $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework for continuous-time systems.

Theorem 3 (Sufficient $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis) *For a system G in* (1) *with a given γ, there exists a controller K for Problem (i) if one can minimize $J \triangleq \sqrt{\alpha} = \sqrt{\text{Trace } Z_2}$ while the following LMI's, involving variables $\overrightarrow{X_1}$, $\overrightarrow{X_2}$, U, V_{11}, W_{11}, $\hat{\mathbf{A}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{D}}$, are feasible.*

$$\begin{pmatrix} -(\overrightarrow{V}^T + \overrightarrow{V}) & \overrightarrow{V^T A} + \overrightarrow{X_1} & \overrightarrow{V^T B_1} & 0 & \overrightarrow{V}^T \\ (\cdot)^T & -\overrightarrow{X_1} & 0 & 0 & 0 \\ (\cdot)^T & (\cdot)^T & -I & \mathcal{D}_1^T & 0 \\ (\cdot)^T & (\cdot)^T & (\cdot)^T & \overrightarrow{\Pi} & \overrightarrow{C_1} \\ (\cdot)^T & (\cdot)^T & (\cdot)^T & (\cdot)^T & -\overrightarrow{X_1} \end{pmatrix} < 0 \quad (21)$$

$$\begin{pmatrix} -(\overrightarrow{V}^T+\overrightarrow{V}) & \overrightarrow{V^TA}+\overrightarrow{X_2} & \overrightarrow{V^TB_2} & \overrightarrow{V}^T \\ (\cdot)^T & -\overrightarrow{X_2} & 0 & 0 \\ (\cdot)^T & (\cdot)^T & -I & 0 \\ (\cdot)^T & (\cdot)^T & (\cdot)^T & -\overrightarrow{X_2} \end{pmatrix} < 0 \quad (22)$$

$$\begin{pmatrix} \overrightarrow{X_2} & \overrightarrow{C_2}^T \\ (\cdot)^T & Z_2 \end{pmatrix} > 0 \quad (23)$$

where $\overrightarrow{\Pi} = -\gamma^2 I + \Phi(\overrightarrow{X_1}, \overrightarrow{\mathcal{C}_1})$, $\Phi(\overrightarrow{X_1}, \overrightarrow{\mathcal{C}_1}) = L_1^T \overrightarrow{X_1} L_1 - L_1^T \overrightarrow{\mathcal{C}_1}^T - \overrightarrow{\mathcal{C}_1} L_1$ *and* L_1 *is set to a constant value.*

Proof: The LMI's (21) $\sim$ (23) can be obtained by simple congruence transformations on (6), (16) and (17) respectively. The details are omitted here. $\square$

Remark 2 *When a solution to the LMI's* (21) $\sim$ (23) *exists, the controller can be derived from the following scheme [1]:*

- *Compute the non-singular matrices* V_{21} *and* W_{21} *such that* $V_{21}^T W_{21} = U - V_{11}^T W_{11}$.
- *Compute the controller* A_K, B_K, C_K, *and* D_K *by reversing the formulas in* (19).

Theorem 3 gives a sufficient $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework and the objective value J is an upper bound on the optimum. To decrease the conservatism, we need the following algorithm to decrease the upper bound.

Algorithm 1 (Synthesis Iterations) *For a given system* G *with the control objective of minimizing* $\|T_2\|_2$ *subject to* $\|T_1\|_\infty < \gamma$, *one can carry on the following non-divergent iterations to compute the controller.*

1. *If the conventional* $\mathcal{H}_2/\mathcal{H}_\infty$ *synthesis framework in [19] with CLV's is feasible for the system* G, *one can compute the LMI's to obtain the Lyapunov variable* X *and the associated closed-loop subsystem* $T_1 = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \end{array}\right)$; *Otherwise one can not continue the iteration to design a less conservative* $\mathcal{H}_2/\mathcal{H}_\infty$ *controller.*
2. *Let* $W = -X^{-1}\mathcal{A}^T + X^{-1} \triangleq \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, *then* $\Pi_W = \begin{pmatrix} I & W_{11} \\ 0 & W_{21} \end{pmatrix}$. *Set* $L_1 = \Pi_W^{-1} X^{-1} \mathcal{C}_1^T$ *and compute a solution to the LMI's* (21) $\sim$ (23) *with objective value* $J = \sqrt{\text{Trace } Z_2}$. *From* (19) *and* (20), *the controller* K *and* $\overrightarrow{X_1}$, $\overrightarrow{\mathcal{C}_1}$ *can be obtained.*
3. *Initialize* $i \leftarrow 1$ *and let* $J^{(0)} = J$, $K^{(0)} = K$, $\overrightarrow{X_1}^{(0)} = \overrightarrow{X_1}$ *and* $\overrightarrow{\mathcal{C}_1}^{(0)} = \overrightarrow{\mathcal{C}_1}$.
4. *Set* $L_1 = \left\{\overrightarrow{X_1}^{(i-1)}\right\}^{-1} \left\{\overrightarrow{\mathcal{C}_1}^{(i-1)}\right\}^T$ *and compute the LMI's* (21) $\sim$ (23) *in Theorem 3.*
5. *The objective value is* $J = \sqrt{Trace(Z_2)}$. *The controller* K *and matrices* $\overrightarrow{X_1}$, $\overrightarrow{\mathcal{C}_1}$ *can be computed by* (19) *and* (20). *Let* $J^{(i)} = J$, $K^{(i)} = K$, $\overrightarrow{X_1}^{(i)} = \overrightarrow{X_1}$ *and* $\overrightarrow{\mathcal{C}_1}^{(i)} = \overrightarrow{\mathcal{C}_1}$.
6. *If* $(J^{(i-1)} - J^{(i)})/J^{(i-1)} < \epsilon$ *for some* $\epsilon > 0$ *then stop.*
7. *Set* $i \leftarrow i + 1$ *and return to Step 4.*

Theorem 4 *For a given system* G, *if the conventional* $\mathcal{H}_2/\mathcal{H}_\infty$ *synthesis with CLV's [19] is feasible, the new LMI's characterizations in Theorem 3 are always feasible. Hence, the synthesis iterations in Algorithm 1 can always be carried out. Furthermore, a sequence of controllers* $K^{(i)}$ *is given by Step 5 such that*

$$\|T_1^{(i)}\|_\infty < \gamma$$

and

$$J_{opt} \le \cdots \le J^{(i)} \le \cdots \le J^{(1)} \le J^{(0)}$$

where $T_j^{(i)} = G_j \star K^{(i)}$, $J^{(i)} = \|T_2^{(i)}\|_2$ *and* J_{opt} *is the real optimum.*

Proof: From Theorem 1, we know that if the conventional LMI (5) for $\mathcal{H}_\infty$ analysis is feasible, then the new LMI (6) with $\bar{L}_1 = X_1^{-1}\mathcal{C}_1^T$ is feasible. We will show below that if a solution for (5) is given, one can use it to derive a special solution for (6) without computing the LMI itself.

By setting $X_1 = X$ and $\bar{L}_1 = X^{-1}\mathcal{C}_1^T$, (6) for $\mathcal{H}_\infty$ analysis becomes

$$\begin{pmatrix} -(V^T+V) & V^T\mathcal{A}+X & V^T\mathcal{B}_1 & 0 & V^T \\ \mathcal{A}^TV+X & -X & 0 & 0 & 0 \\ \mathcal{B}_1^TV & 0 & -I & \mathcal{D}_1^T & 0 \\ 0 & 0 & \mathcal{D}_1 & -\gamma^2I-\mathcal{C}_1X_1^{-1}\mathcal{C}_1^T & \mathcal{C}_1 \\ V & 0 & 0 & \mathcal{C}_1^T & -X \end{pmatrix} < 0 \quad (24)$$

Reversing the procedure for proving Theorem 1, we observe that the above LMI is equivalent to

$$\begin{pmatrix} Y-(W+W^T) & \mathcal{A}Y+W^T & \mathcal{B}_1 & Y\mathcal{C}_1^T \\ Y\mathcal{A}^T+W & -Y & 0 & 0 \\ \mathcal{B}_1^T & 0 & -I & \mathcal{D}_1^T \\ \mathcal{C}_1Y & 0 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \quad (25)$$

where $Y = X^{-1}$.

Let $W = -X^{-1}\mathcal{A}^T + X^{-1}$. Using the Schur lemma, we obtain

$$\begin{pmatrix} \mathcal{A}^TX+X\mathcal{A} & X\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^TX & -I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma^2I \end{pmatrix} < 0 \text{ and } X > 0 \quad (26)$$

which are the standard LMI's for $\mathcal{H}_\infty$ control as shown in the statement (ii) of Theorem 1.

Using a similar technique and setting $X_2 = X$ and $V = (I - \mathcal{A}^T)^{-1}X$, we can show that the new LMI's (16) and (17) for $\mathcal{H}_2$ control include the conventional LMI's (14) and (15) as a special case.

Hence, if there exists a feasible solution $X_1 = X_2 = X$ for (5), (14) and (15), one can always find a special solution $X_1 = X_2 = X$ and $V = (I - \mathcal{A}^T)^{-1}X$ satisfying (6), (16) and (17). We can obtain the similar results for $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem using congruence transformation.

The proof of the non-divergence of the objective value is similar to that in [24], hence it is omitted here. $\square$

5 Numerical Example

In this section, we give a numerical example to illustrate the advantage of $\mathcal{H}_2/\mathcal{H}_\infty$ control with NCLV's. Consider an unstable plant [19] with equations:

$$\dot{x} = \begin{pmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} w + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = x_2 + 2w$$

and performance outputs:

$$z_\infty = \begin{pmatrix} x_1 \\ u \end{pmatrix}, \qquad z_2 = \begin{pmatrix} x_2 \\ x_3 \\ u \end{pmatrix}$$

where x, w, u, and y denote the state, disturbance, control and measured signals respectively. We are interested in the $\mathcal{H}_\infty$ performance from w to z_∞ and the $\mathcal{H}_2$ performance from w to z_2. Let the closed-loop system from w to z_∞ is denoted by T_1 and from w to z_2 by T_2.

(a) Design of the optimal $\mathcal{H}_2$ controller: We design an optimal $\mathcal{H}_2$ controller for system T_2 using the function "h2lqg.m" in the *Robust Control Toolbox* [4]. The controller is

$$K_{opt} = \frac{-5.7266(s+5.089)(s-0.2718)}{(s+5.085)(s^2+3.669s+9.931)}$$

and the closed-loop system $\mathcal{H}_2$ performance is $\|T_2\|_2 = 7.7484$. With the controller K_{opt}, the $\mathcal{H}_\infty$ performance is $\|T_1\|_\infty = 23.5873$.

(b) Design of a conventional $\mathcal{H}_2/\mathcal{H}_\infty$ controller: Using the function "hinfmix" in the *LMI Control Toolbox* [12], we design a mixed $\mathcal{H}_\infty/\mathcal{H}_2$ controller. To compare with the optimal $\mathcal{H}_2$ controller, we minimize the $\mathcal{H}_2$ norm $\|T_2\|_2$ subject to $\|T_1\|_\infty \leq 23.5873$ and obtain the controller

$$K_{mix} = \frac{-7.5978(s+5.098)(s-0.05412)}{(s+5.099)(s^2+4.256s+10.26)}$$

The $\mathcal{H}_2$ upper bound is 8.9625 which is 15.67% higher than the optimum, while the actual $\mathcal{H}_2$ performance is $\|T_2\|_2 = 8.0704$.

(c) Design of a $\mathcal{H}_2/\mathcal{H}_\infty$ controller with NCLV's: Now, we design the $\mathcal{H}_\infty/\mathcal{H}_2$ controller with NCLV's. The program for the synthesis framework presented in this paper can be obtained by contacting the authors. To compare with the conventional mixed control, we also minimize the $\mathcal{H}_2$ norm $\|T_2\|_2$ subject to $\|T_1\|_\infty \leq 23.5873$.

When the convergence tolerance is $\epsilon = 1 \times 10^{-4}$, i.e. $(J^{(i-1)} - J^{(i)})/J^{(i-1)} < 1 \times 10^{-4}$, the results of synthesis iterations are shown in Table 5. It is clear that, only through three iterations, the objective value $J^{(i)}$ decreased from 8.9602 to 8.4947, which is 9.63% greater than the optimum. The best controller we can obtain is

$$K_{new} = \frac{-7.2791(s+5.125)(s-0.2081)}{(s+5.127)(s^2+4.123s+9.703)}$$

Iter. i	Upper bound $J^{(i)}$	Relative error $\frac{J^{(i-1)}-J^{(i)}}{J^{(i-1)}}$	Actual $\|T_2\|_2$	Actual $\|T_1\|_\infty$
0	8.9602		8.0711	17.8957
1	8.4957	5.184×10^{-2}	8.0031	18.1886
2	8.4948	1.0593×10^{-4}	8.0027	18.1958
3	8.4947	1.1771×10^{-5}	8.0028	18.1957

Table 1: Synthesis iterations.

with $\|T_2\|_2 = 8.0027$.

Note that, although the new $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework is less conservative than the traditional one [19], the improvement in this example is not as significant as one would expect. The future work is to further decrease the conservatism of the design.

6 Concluding Remarks

This paper presented a continuous-time $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis framework, which was a counterpart of de Oliveria's discrete-time synthesis framework [6]. The main contribution is to incorporate $\mathcal{H}_\infty$ control into Apkarian's continuous-time synthesis framework for eigenstructure assignment and $\mathcal{H}_2$ control [1]. To achieve this objective, we decoupled the controller and Lyapunov variables by introducing a new auxiliary variable, and substituted a non-linear term with a sequence of convergent (strictly speaking, non-divergent) linear upper bounds.

Compared with the Youla parametrization technique, this framework computes a controller with the same order as the generalized plant, which is important to controller implementation. As demonstrated in the example, the synthesis iterations can more quickly converge to a sub-optimum than those presented in [24] since only one non-linear term need to be substituted by a sequence of upper bounds.

The conservatism of this work comes from two factors: (1) A common auxiliary variable was used for different control specifications. (2) A sequence of upper bounds are utilized to substitute the non-linear item. Although the synthesis framework in this paper is less conservative than the traditional one with CLV's, more computation is needed due to increased number of decision variables and synthesis iterations.

The results in this paper are ready to be extended to set up a more general multi-objective synthesis framework, e.g. a mixed generalized $\mathcal{L}_2$ control, generalized $\mathcal{H}_2$ control, $\mathcal{H}_2$ control framework with pole placement. When more control objectives are involved, the advantage of the framework presented in this paper will be much more clear. It will be of great interest to apply this framework to a real system, such as active suspension control system, which is by its very nature a multi-objective optimization problem [27, 26]. A challenging question is whether one can linearize the non-linear term (e.g. $\mathcal{C}_1 X_1^{-1} \mathcal{C}_1^T$) via other less conservative techniques.

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