NONLINEARITY QUANTIFICATION FOR THE OPTIMAL STATE FEEDBACK CONTROLLER

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Abstract

It is an important question whether nonlinear controller design is needed for a particular application or whether linear controllers are sufficient to achieve the desired control goals. The present paper introduces the closed-loop optimal control law (OCL) nonlinearity measure, a new tool that provides a scheme to quantify the degree of nonlinearity of an optimal state feedback controller for a given control problem. The method builds upon the Optimal Control Structure (OCS) method introduced by Stack and Doyle III [9]. The new closed-loop OCL method is compared to the OCS approach and numerical examples illustrate the theoretical results.

1 Introduction

Continuously increasing economic demands force engineers to optimize industrial processes. The process control strategy has to be included into the optimization effort in order to improve performance and reliability to its best. But in the economic context, not only a mediocre performance of the process causes loss in productivity, but high research and development costs have to be taken into account as well. For linear systems, many suitable and easy-to-implement controller designs are available like PID control, internal model control, H_2- and H_∞-synthesis or model predictive control (MPC). Accordingly very satisfying results can be achieved with little effort. Controller design for nonlinear systems is much more involved. As most real processes show nonlinear behaviour, an important question to ask is: in which cases is nonlinear controller design necessary or of advantage?

There exist many approaches to define a quantitative measure for the plant nonlinearity. But most measures rely on the open-loop behaviour of the plant. Many measures can be interpreted as some kind of prediction error of the best linear approximation [7, 1]. Other nonlinearity measures can be found in [4, 8, 3]. However, the question raised above rather concerns the structure of a suitable controller than that of the plant. In view of this, Stack and Doyle III [9] suggest to measure the nonlinearity of a controller instead of the nonlinearity of the plant. This method is termed control-relevant nonlinearity quantification. In the present work, controller nonlinearity and control problem nonlinearity will be used as synonyms.

A method to describe a benchmark controller for a general class of nonlinear systems is given by the classical optimal control theory. The controller structure is not restricted in advance but only the optimization criterion must be specified. In order to circumvent the derivation of the exact solution for the optimal state feedback controller, Stack and Doyle III define the so-called Optimal Control Structure (OCS) [9]. By this means, interesting questions can be examined like the dependence of control-relevant nonlinearity on the set-point and region of operation [8], the relation between nonlinearity and controller aggressiveness [9, 6] and the severity of certain classes of nonlinear behaviour [6]. Extensions to measurement feedback have also been made [8, 6].

This paper introduces a new nonlinearity measure and aims at clarifying the relation between the nonlinearity of the optimal control law (OCL) and the corresponding Optimal Control Structure (OCS). The present work will consider the state feedback problem. Thus, the nonlinearity that is captured by the described method concerns the input-to-state nonlinearity, as the information available for feedback is assumed to be the full state information. Output nonlinearities thus do not affect the value of the nonlinearity measure. The basics of control-relevant nonlinearity quantification will be reviewed in Section 2. In Section 3, the new closed-loop OCL nonlinearity measure is introduced that captures the nonlinearity of an optimal state feedback controller in a closed-loop setup. The differences between the OCS approach and the OCL nonlinearity are discussed. In Section 4, numerical results illustrate the theoretical insights of precedent sections with the help of two example systems. Section 5 summarizes the key points and draws the conclusions.

2 Control-relevant nonlinearity quantification

2.1 A nonlinearity measure

Nonlinearity tests can classify systems to be linear or nonlinear. But there is a vast variety of different nonlinear behaviours, and many nonlinearities are weak and do not affect the controller design process. This fact shows the need to be able to not only

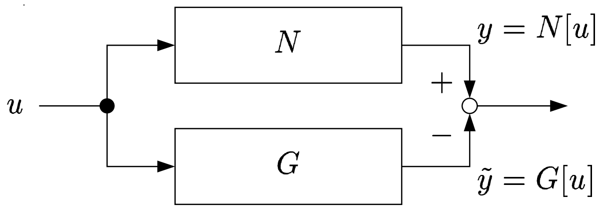

Figure 1: Setup for comparison of a nonlinear system N with a linear approximation G.

qualify a system to be nonlinear, but to quantify the degree of nonlinearity inherent to the system. For this purpose, we will use the following nonlinearity measure as introduced in Refs. [1, 5].

Definition (nonlinearity measure) The nonlinearity measure $\phi_N^{\mathcal{U}}$ is defined as

$$\phi_N^{\mathcal{U}} := \inf_{G\in\mathcal{G}} \sup_{\boldsymbol{u}(\cdot)\in\mathcal{U}} \frac{||G[\boldsymbol{u}] - N[\boldsymbol{u}]||_{L_2}}{||N[\boldsymbol{u}]||_{L_2}} \tag{1}$$

where $N: \mathcal{U} \to \mathcal{Y}$ is the nonlinear operator that describes the input-output behaviour of the plant. $G: \mathcal{U} \to \mathcal{Y}$ is an approximation of N belonging to the class of linear operators $\mathcal{G}$. $\mathcal{U} \subset L_2$ is the space of admissible input signals.

In the further analysis, the L_2-norm

$$||\boldsymbol{f}||_{L_2} = \left(\int_0^\infty |\boldsymbol{f}(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}$$

defined on the space $\mathcal{Y} \subset L_2$ of output signals is used. Thus, the measure is only well defined, if the system N is L_2-stable. In general, the usage of different norms is possible.

This nonlinearity measure can be interpreted as the relative prediction error of the best linear approximation. The given measure corresponds to a *relative* error, because the error term $||\boldsymbol{e}]|| = ||G[\boldsymbol{u}] - N[\boldsymbol{u}]||$ is normalized by the norm of the corresponding output signal of the nonlinear system $||N[\boldsymbol{u}]||$. Note that the description "best approximation" refers to the approximation, that minimizes the *worst case* error and not the average or some other kind of weighted error. The measure tales on a value between zero (if N is linear) and one (if N is highly nonlinear) [1]. An illustration of the setup for the comparison of the nonlinear system N with a linear approximation G is shown in Fig. 1. $\phi_N^{\mathcal{U}}$ can be calculated by different computational schemes. For the results of this paper, we used the convex optimization method to get the best possible results. See Refs. [1, 5] for details on the computational methods.

By choosing the considered input signals such that the relevant amplitude range of the input or state signals are captured, the nonlinearity can be computed for different regions of operation of the process.

2.2 Aim of control-relevant nonlinearity assessment

An important question to the control engineer is whether a nonlinear controller is needed to achieve adequate performance or whether a linear controller does the job. As already mentioned, a nonlinear plant may or may not require a nonlinear controller. Knowledge about the degree of nonlinearity of the plant is therefore only of limited use. The consequent idea of control-relevant nonlinearity as introduced by Stack and Doyle III is to directly examine the nonlinearity of suitable controllers for a given plant instead of just analyzing the plant's open-loop behaviour [9]. The controller nonlinearity is determined by the three factors

1. plant dynamics,
2. region of operation and
3. performance criterion.

Open-loop process nonlinearity measurement takes the first two points into account. The third point is new in control-relevant nonlinearity quantification. In a more general context, not only the performance criterion has to be considered but the controller design method has to be mentioned as well. Following the idea of Ref. [9], optimal control theory with an integral performance criterion is used here as it represents a benchmark for the achievable performance.

2.3 Open-loop optimal control law nonlinearity

For the derivation of an optimal controller, a model of the plant is assumed to be given in state-space form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{2}$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector and $\boldsymbol{u}(t) \in R^p$ is the control input. The controller is sought, which is optimal with respect to the integral cost criterion

$$J(\boldsymbol{x_0})[\boldsymbol{u}] = \int_{t_0}^{T} F(\boldsymbol{x}, \boldsymbol{u})\mathrm{d}t. \tag{3}$$

where the trajectory of $\boldsymbol{x}(t)$ has to satisfy the plant dynamics Eq. (2) and the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$.

It is common knowledge that for a time-invariant system together with a time-invariant cost criterion and an infinite horizon $T = \infty$ the resulting optimal control can be formulated as a static state feedback control law $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ [10], i.e. the optimal control depends only on the current state vector of the plant.

The goal of control-relevant nonlinearity quantification is to measure the nonlinearity of this control law $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ to decide whether linear control is sufficient or not. To this end, it is in principle possible to apply the previously defined nonlinearity measure to the system

$$N_{OCL}:\ \boldsymbol{x} \mapsto \boldsymbol{u},\ \boldsymbol{u}(t) = \boldsymbol{k}\left(\boldsymbol{x}(t)\right)\ \forall t \tag{4}$$

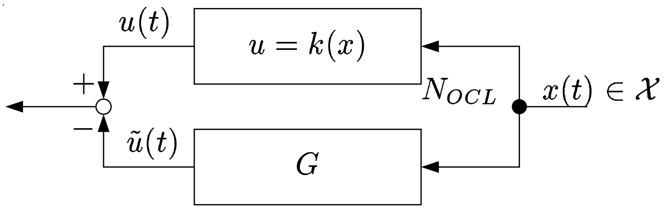

Figure 2: Setup for (open-loop) comparison of the optimal control law $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ with a linear dynamic approximation G.

describing the optimal control law (OCL) where the state vector $\boldsymbol{x}$ is the input to the controller and the control signal $\boldsymbol{u}$ is its output, leading to the following definition.

Definition (open-loop OCL nonlinearity measure) The open-loop optimal control law (OCL) nonlinearity for the control problem at hand is defined as

$$\phi_{N_{OCL}}^{\mathcal{X}(\mathcal{B})} := \inf_{G \in \mathcal{G}} \sup_{\boldsymbol{x} \in \mathcal{X}(\mathcal{B})} \frac{\|N_{OCL}[\boldsymbol{x}] - G[\boldsymbol{x}]\|_{L_2}}{\|N_{OCL}[\boldsymbol{x}]\|_{L_2}} \qquad (5)$$

where the set $X(\mathcal{B})$ describes all input functions $\boldsymbol{x}(\cdot) \in L_2$ that only take values in the region of operation $\mathcal{B} \subset R^n$.

The setup of open-loop nonlinearity quantification for the optimal control law is illustrated in Fig. 2 in analogy to the general framework in Fig. 1. The approach is termed *open-loop* nonlinearity because the considered input signals $\boldsymbol{x}(\cdot) \in \mathcal{X}(\mathcal{B})$ do not need to be possible state trajectories of the plant or of the closed-loop system. This fact will become clearer in Sec. 3.

In accordance with what has been said in Sec. 2.2, the controller nonlinearity $\phi_{N_{OCL}}^{\mathcal{X}(\mathcal{B})}$ is influenced by (i) the plant dynamics of Eq. (2), (ii) the region of operation characterized by the set $\mathcal{B}$ and (iii) the performance criterion given by Eq. (3).

In practice the optimal control law $\boldsymbol{k}(\boldsymbol{x})$ only be computed rarely and if so with significant effort, rendering the presented measure impractical. The advantage of a nonlinearity measure should be that it assesses *prior to* controller design whether nonlinear control is necessary. The following sections will therefore deal with approaches that try to quantify the nonlinearity of the OCL without explicitly deriving it.

2.4 The Optimal Control Structure (OCS) approach

The OCS approach, established by Stack and Doyle III in [9], uses the following results from optimal control theory to approximately quantify the nonlinearity of the optimal control law $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ without explicitly deriving it.

If $\boldsymbol{x}^*$ is an optimal state trajectory and $\boldsymbol{u}^*$ the corresponding input history, then there exists a co-state trajectory $\boldsymbol{\lambda}^*$, such that the equations

$$\dot{\boldsymbol{x}}^*(t) = \boldsymbol{f}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t)) \qquad (6)$$

$$\dot{\boldsymbol{\lambda}}^{*T}(t) = -\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t)) \qquad (7)$$

$$0 = -\frac{\partial H}{\partial \boldsymbol{u}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t)) \qquad (8)$$

with the Hamiltonian defined as

$$H(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}) = F(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{\lambda}^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \qquad (9)$$

are satisfied for all times $0 < t < T$ and the boundary conditions

$$\boldsymbol{x}^*(t_0) = \boldsymbol{x}_0 \qquad (10)$$
$$\boldsymbol{\lambda}^{*T}(T) = 0$$

hold. If it is known, e.g. from physical reasoning, that exactly one optimal control exists, then the solution of this boundary value problem with split boundary conditions determines the optimal control signal.

Eq. (6) represents the plant dynamics. In Ref. [9], Eqs. (7, 8) are viewed as a dynamical system that represent the controller behaviour under certain circumstances and these equations are called the Optimal Control Structure (OCS). Within the OCS approach, the boundary conditions are neglected and only the OCS equations are examined. If the OCS equations are linear, then the resulting optimal controller is linear as well. This fact allows for example to examine analytically whether a controller qualitatively tends to linearity if a parameter of the plant or cost criterion tends to a limit value [9].

To get quantitative results, the nonlinearity of the OCS system can be assessed by different nonlinearity measures. In the original work [9], coherence analysis is used which can cause problems in the case of higher-order systems. In the present work as in Ref. [6], the nonlinearity measure of Eq. (1) is used, providing the advantage that it can treat the general case $x \in R^n$.

Definition (OCS nonlinearity measure) The OCS nonlinearity measure is defined as

$$\phi_{N_{OCS}}^{\mathcal{X}(\mathcal{B})} := \inf_{G \in \mathcal{G}} \sup_{\boldsymbol{x} \in \mathcal{X}(\mathcal{B})} \frac{\|N_{OCS}[\boldsymbol{x}] - G[\boldsymbol{x}]\|_{L_2}}{\|N_{OCS}[\boldsymbol{x}]\|_{L_2}} \qquad (11)$$

with the transfer operator N_{OCS} given by Fig. 3. For the system N_{OCS}, $\boldsymbol{x}$ represents the input, $\boldsymbol{u}$ is the output and the adjunct variable $\boldsymbol{\lambda}$ is an internal state. The set $\mathcal{X}(\mathcal{B})$ describes all input functions $\boldsymbol{x}(\cdot) \in L_2$ that take values in the region of operation $\mathcal{B} \subset R^n$.

Note that the time in the state equations in Fig. 3 is reversed as compared to Eq. (7) by $\tilde{t} = T - t$, i.e. the sign of the right-hand side is changed to its opposite. This is done for two reasons:

1. In most cases, the plant and the OCS locally have complementary stability properties, i.e. the linearization of the

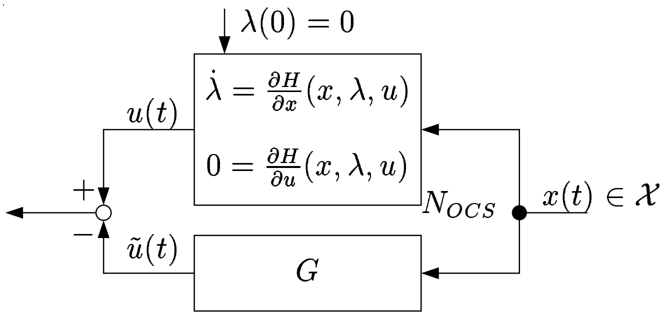

Figure 3: Setup for (open-loop) comparison of the Operator N_{OCS} with a linear dynamic approximation G in the Optimal Control Structure (OCS) approach.

OCS Eqs. (7, 8) has the same poles as the linearization of the plant Eq. (6) but mirrored at the imaginary axis. Thus, if the plant is stable, the OCS system is anti-stable. If the time is reversed, the OCS becomes stable again and the nonlinearity measure can be computed.

2. For a finite horizon, the end condition in (10) is only equivalent to the zero initial condition of the OCS system in Fig. 3 if the time is reversed.

However, there are some drawbacks to the OCS approach. First of all, the OCS is a dynamical system whereas the optimal control law is a static relationship. Thus the question remains if the OCS structural nonlinearity reflects properly the structural nonlinearity of the optimal control law. Moreover, dynamical systems can show much richer behaviour and the OCS behaviour strictly corresponds to the optimal controller only if the state trajectory is an *optimal* trajectory (marked by a star in Eqs. (6-8)). This could lead to the conjecture that the OCS nonlinearity could be an upper bound on the optimal control law nonlinearity. In Sec. 4 an example will illustrate that this is in general not the case.

Summarizing, the OCS method is an approach to control-relevant nonlinearity assessment that does not require the computation of the optimal controller, but utilizes first-order necessary conditions for optimality to examine the structural nonlinearity. A big advantage of the OCS analysis is that for some pairs of plant structure and cost criterion the OCS equations can be studied analytically [9]. For most practical applications, a numerical evaluation can be done for the defined measure. But the characterized problems show that there are serious concerns whether the OCS nonlinearity measure correctly reflects the nonlinearity of the optimal control law.

3 Closed-loop optimal control law nonlinearity

In Section 2.3 we presented an ideal but infeasible method for control-relevant nonlinearity assessment that consisted of the application of the nonlinearity measure from Sec. 2.1 to the optimal control law. We then presented the OCS approach for practical derivation of results. While this approach proves to be very useful for analytical examinations, the numerical values have to be interpreted carefully due to the heuristics inherent to that approach. In this section we introduce a novel method for control-relevant nonlinearity assessment based on the previously presented nonlinearity measure and inspired by the OCS approach. We will see that its definition stays closer to the ideal method described in Sec. 2.3 and that it is even more appropriate with respect to the philosophy of optimal control theory.

Definition (closed-loop OCL nonlinearity measure) The closed-loop optimal control law (OCL) nonlinearity for a control problem is defined as

$$\tilde{\phi}^{\mathcal{B}}_{N_{OCL}} := \inf_{\boldsymbol{k} \in R^n} \sup_{\boldsymbol{x}_0 \in \mathcal{B}} \frac{\left\| N_{OCL}[\boldsymbol{x}^*_{x_0}] - \boldsymbol{k}^T \boldsymbol{x}^*_{x_0} \right\|_{L_2}}{\left\| N_{OCL}[\boldsymbol{x}^*_{x_0}] \right\|_{L_2}} \quad (12)$$

with $N_{OCL}[\boldsymbol{x}^*_{x_0}] := \boldsymbol{u}^*_{x_0}$ and $\boldsymbol{x}^*_{x_0}$ being the solution to the infinite horizon control problem for the initial condition x_0. The region $\mathcal{B} \subset R^n$ of initial conditions replaces the set of considered input signals in comparison to the previous definitions. To be consistent, the set $\mathcal{B} \subset R^n$ should be positive invariant for the closed-loop system, i.e. any trajectory that starts from a point in $\mathcal{B}$ remains in $\mathcal{B}$ for all times. If it is not positive invariant, then trajectories may temporarily leave the specified region of operation and the "real" region of operation is bigger than the specified one.

This definition is the application of the nonlinearity measure from Sec. 2.1 to the optimal control law with respect to the *closed-loop trajectories*. I.e. the worst case input signal in the nonlinearity measure calculation is taken from the set of optimal trajectories instead of all L_2-signals with restricted amplitude. This is justified by the fact, that the demanded control task is to optimally regulate the system for a given initial condition. In the case of further disturbances, the described controller loses its optimality property. Thus, considering optimal trajectories only amounts to taking account of the conditions, under which the optimal control law is derived. Fig. 4 illustrates that the optimally controlled closed-loop "generates" the trajectories that are used in the nonlinearity quantification. As there is a one-to-one relationship between initial conditions and trajectories, regarding optimal trajectories is equivalent to considering initial conditions. Therefore the supremum in Eq. (3) is taken over initial conditions.

There is one more difference to the previously defined nonlinearity measures. Contrary to the OCS method, only *static* approximations of the optimal nonlinear controller are taken into account. This is adequate, as it is known that the optimal control law is a static state feedback. As stability is no issue in the case of static relations, a further big advantage of the closed-loop OCL nonlinearity quantification is that it can be applied regardless of the stability of the plant or of the OCS equations. Only a solution to the optimal control problem needs to exist.

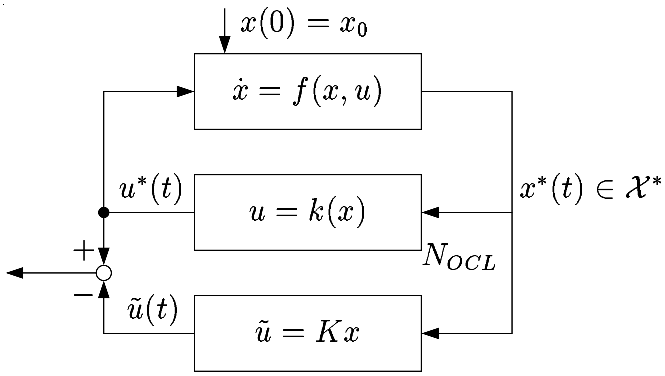

Figure 4: Setup for closed-loop comparison of the Operator N_{OCL} with a linear static approximation K.

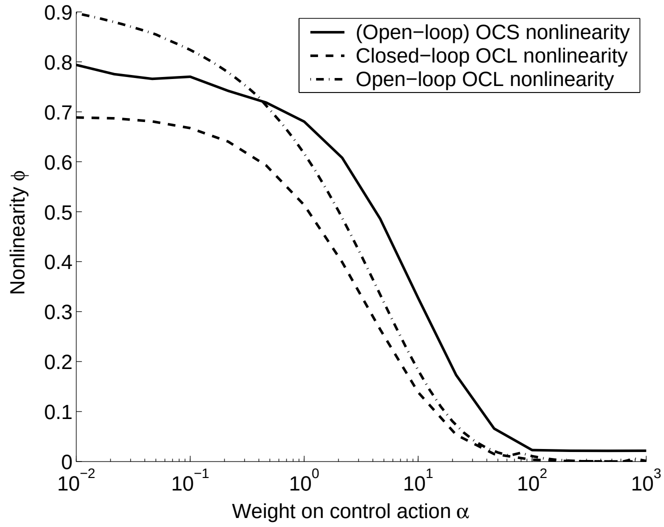

Figure 5: Variation of different nonlinearity measures with penalty weight on control action α for the Hammerstein example system. Simulation parameters: Maximal amplitude of inputs or initial conditions respectively $|x| \leq 20$.

The computational scheme is again based on convex optimization. The optimal control problem is solved for a finite number of points in the set $\mathcal{B}$ and for a finite but large horizon T. For the solution of these problems, any numerical method can be used [2, 10]. Thus, even input or state constraints can be included. Having computed the optimal trajectories, the remaining problem is reformulated as a constrained minimum search over the static controller gain matrix K. A sufficiently large horizon can be found by increasing T until $\tilde{\phi}^{\mathcal{B}}_{N_{OCL}}$ does not change any more. Even though the computations are not trivial, the problem considered is still much simpler than the problem to explicitly compute the nonlinear optimal *feedback law*, as only optimal *open-loop trajectories* need to be computed here.

In this section, the optimal control law (OCL) nonlinearity measure has been introduced. The definition follows that in Section 2, but stays closer to the original problem of measuring optimal control law nonlinearity. Important definitions of control-relevant nonlinearity have been adapted from the OCS approach: the three-fold problem structure with plant dynamics, region of operation and cost criterion as well as the concept of using optimal control theory. The important extension here is to consider exact solutions for the optimal control problem. By this means, a certain degree of approximation inherent to the OCS method is released and a further step is made toward a real closed-loop measure. Moreover, the methodology here is applicable to a broader class of systems.

4 Examples

In this section, the nonlinearity measures for two simple scalar example systems are compared, an input-affine system and a Hammerstein system. Both systems belong to classes that are commonly met in the modeling of processes. The cost criterion for both systems is taken to be

$$J(x_0)[u] = \int_{t_0}^{T} x(t)^2 + \alpha u(t)^2 \mathrm{d}t \qquad (13)$$

where x and u are scalars and the weight on the control action $\alpha > 0$ is the parameter that influences the controller aggressiveness. Often the dependence of nonlinearity on a control problem parameter is more of interest than a single value. In the following, the dependence of the nonlinearity measures on the controller aggressiveness will be explored.

The first system to be examined is of Hammerstein type with a saturating input nonlinearity

$$\dot{x} = -x + \arctan(u). \qquad (14)$$

As this is a scalar system, the optimal control law can be calculated and it is possible to evaluate the open-loop OCL nonlinearity measure. Fig. 5 shows the OCS, the open-loop OCL and the closed-loop OCL nonlinearity measures as functions of the parameter α. All three measure give similar qualitative behaviours. For small values of α, corresponding to an aggressive control policy, the controller nonlinearity is high. Whereas for large α, the controller is only taking little action and the control signal stays in the almost linear part of the input nonlinearity and the controller structure needs not to be nonlinear.

Fig. 5 shows two more facts. Firstly, the results show that the OCS nonlinearity is not an upper bound on the open-loop OCL nonlinearity. Secondly, the closed-loop OCL nonlinearity in this case is a lower bound on the open-loop OCL nonlinearity. It can be expected that in most cases, the restriction of considered input signals will be more important than the smaller class of linear approximations (i.e. static instead of dynamic systems) resulting in the closed-loop OCL nonlinearity being lower than the open-loop OCL nonlinearity. But recall that the closed-loop OCL nonlinearity better takes account of the optimal control setup and is thus favorable as indicator for the controller nonlinearity.

The second example system is a CSTR model taken from [6]

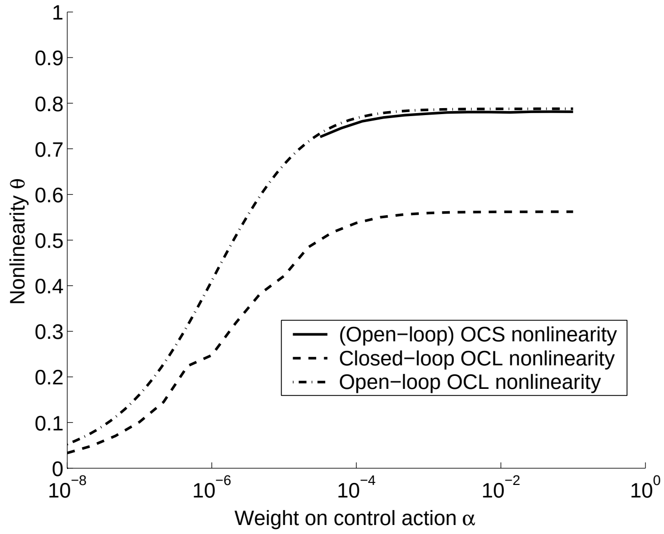

Figure 6: Variation of different nonlinearity measures with penalty weight on control action α for the input-affine CSTR example system. Simulation parameters: Maximal amplitude of inputs or initial conditions respectively $|x| \leq 0.25$.

and given by the equation

$$\dot{x} = -50(0.7 + x)^2 - \frac{0.5(x - 1.8)(x - 0.3)}{x - 0.55}(49.9074 + u) \tag{15}$$

where x is a concentration and u is a flow rate. Fig. 6 shows that the OCS and the open-loop OCL nonlinearity are very close. But in this case, for smaller values than $\alpha = 10^{-5}$ the OCS structure is not stable. The OCS nonlinearity can thus only be calculated for values of α larger than 10^{-5}, whereas the OCL gives results for the whole range.

The examples illustrate that the OCS is neither an upper nor a lower bound on open-loop OCL nonlinearity, even if for the presented example, the qualitative results are very similar. The closed-loop OCL nonlinearity measure on the other hand better reflects the controller nonlinearity and is, as seen, applicable to a more general class of systems. Moreover, the closed-loop OCL nonlinearity measure was the fastest to be computed, as no simulation of the linear dynamic approximations is necessary.

5 Conclusion

An important question is whether nonlinear controller design is necessary. This leads to the idea of quantifying the degree of nonlinearity of a suitable controller for a given control problem instead of only analyzing the plant dynamics. Based on ideas of the Optimal Control Structure (OCS) approach, the closed-loop optimal control law (OCL) nonlinearity measure has been introduced that gives a true assessment of the controller nonlinearity. This novel measure has several advantages. Firstly, it reflects more exactly the desired information about the optimal controller. Secondly, it is applicable to a broader range of control problems, as stability of the plant or governing equations plays no role. Moreover, in principle input and state constraints can be included. Finally, the OCL nonlinearity proved to be computationally more efficient for the example systems, because the only time consuming calculation is the calculation of the optimal trajectories and the simulation of the linear dynamic approximations is not necessary any more.

In summary, the closed-loop OCL nonlinearity measure represents an important step ahead, but it will probably not be the last. Future work will concern output feedback problems and the relation between controller nonlinearity and closed-loop performance.

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