ON THE COMPUTATION OF INVARIANT SETS FOR CONSTRAINED NONLINEAR SYSTEMS: AN INTERVAL ARITHMETIC APPROACH

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Abstract

This paper shows how interval arithmetic can be applied to the computation of control invariant sets. The paper reviews some concepts in invariant set theory and presents recursive algorithms to compute sequences of control invariant sets. These ideas can be used to design stable controllers subject to state and control constraints. To prove set inclusion and inversion, interval arithmetic is also used.

1 Introduction

The concept of invariant set is a direct derivation of the theory of Lyapunov. A Lyapunov function can be used to show that an equilibrium point of a dynamical system is asymptotically stable. The region of the state space where this Lyapunov function is bounded is an attraction region of the equilibrium point, and constitutes a positive invariant set. Excellent surveys about invariant sets are given in [3, 10].

Set invariance theory has been applied to a lot of fields in control [2, 6, 1, 10]. This theory is very useful for analysing constrained systems and for designing controllers, which guarantee constraint satisfaction.

Interval mathematics is a generalization of real mathematics in which intervals numbers replace real numbers [14]. Interval arithmetic has been applied to bound the solution set of interval linear and nonlinear systems [9] and finding global minimum points [7]. In the control context, interval arithmetic has been used in robust control [4], parameter and state estimation [8] and model predictive control [12].

This paper deals with the application of interval analysis to design algorithms that determine subsets of the state space of a discrete nonlinear system that can be steered to any given target set by a control law, while fulfilling control and state constraints. In section 2 basic definitions of set invariance theory are presented. Section 3 is dedicated to the interval arithmetic. Section 4 contains the one-step set approximation algorithm and an extension to polytopes is considered in section 5. Finally an example is shown in section 6.

2 Definitions and problem statement

Consider a system described by a nonlinear discrete model:

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state and $u_k \in \mathbb{R}^m$ is the control signal at sample time k. The system can be subject to control and state constraints:

$$x_k \in X,\ u_k \in U \tag{2}$$

where X and U are compact sets, both of them containing the origin.

In the sequel, some concepts and results related to invariant sets theory and computation are presented.

Definition 1 *(Control invariant set). The set $\Omega \in \mathbb{R}^n$ is a control invariant set for the system (1) if and only if $\forall x_k \in \Omega$ then $\exists u_k \in U$ such that $f(x_k, u_k) \in \Omega$.*

Definition 2 *(One-step set $Q(\cdot)$). Consider a target set Ω, then the one-step set to Ω, denoted as $Q(\Omega)$ is the set of states in $\mathbb{R}^n$ for which an admissible control signal exits, which will guarantee that the system will be driven to Ω in one step, i.e. $Q(\cdot) = \{x \in \mathbb{R}^n | \exists u \in U : f(x,u) \in \Omega\}$.*

The computation of this set is a geometric problem and it allows us to establish a geometric condition for control invariance [3, 10].

Theorem 1 *(Geometric condition for invariance). A set Ω is a control invariant set if and only if $\Omega \subseteq Q(\Omega)$.*

The determination of the stabilizable set is quite useful in the design of controllers. For instance in [2, 13] these sets are used to design robust time-optimal controllers, in [10, 5] a robust control invariant set is used to guarantee robust feasibility of a predictive controller and in [11] a sequence of control invariant sets is used to enlarge the domain of attraction of a predictive controller. The computation of the stabilizable sets is based on the calculation of the one-step set. There exists well established techniques to compute them when the system is linear, the constraints are polyhedra and the target set is a polytope. In this case the stabilizable sets are polyhedra [3, 10].

If the system is nonlinear, then the computation of the one-step set is very difficult. This fact converted the invariant sets into a theoretical tool to design controllers for nonlinear systems.

This paper shows that this drawback may be overcome by using approximate procedures to compute the one-step set. These procedures are much simpler to compute. Furthermore, an algorithm based on interval arithmetic is given in the paper. The proposed algorithm is based on the following lemma.

Lemma 1 *Let Ω be a (control) invariant set and let Φ be any set contained in $Q(\Omega)$, then the set $\Gamma = \Phi \cup \Omega$ is a control invariant set.*

Proof:

$\Gamma = \Phi \cup \Omega$ is a control invariant set if and only if $\Gamma \subseteq Q(\Gamma)$. Since $\Omega \subseteq \Gamma$ then $Q(\Omega) \subseteq Q(\Gamma)$. Given that $\Phi \subseteq Q(\Omega)$ and that $\Omega \subseteq Q(\Omega)$, then $\Gamma = \Phi \cup \Omega \subseteq Q(\Omega)$, which yields to $\Gamma \subseteq Q(\Gamma)$.

Note that if $\Omega \subseteq \Phi$, then $\Gamma = \Phi$, and hence Φ is a control invariant set.

■

The main result that can be derived from this lemma is that it is not necessary to compute the exact one-step set to obtain a control invariant, and hence, a inner set can be used. This result is very interesting for nonlinear systems, for which the computation of the one-step set is very difficult.

Then, in order to make tractable the computation of control invariant sets for nonlinear system, an approximate procedure $Q_{ap}(\Omega)$ can be used. This one should provide an inner approach to the one step set, that is, $Q_{ap}(\Omega) \subseteq Q(\Omega)$. By using this idea a sequence of control invariant sets can be derived.

Theorem 2 *Let Ω be an invariant set for the nonlinear system (1). Consider the sequence of sets given by $\Phi_i = Q_{ap}(\Phi_{i-1}) \cap X$ where $\Phi_0 = \Omega$. Then the set given by $\Omega_i = \cup_{j=0}^{i} \Phi_j$ is a control invariant set for all $i \geq 0$.*

Proof:

It is proved by induction. For $j = 1$, $\Omega_1 = \Phi_1 \cup \Omega$. Since $\Phi_1 \subseteq Q(\Omega)$, then from lemma 1 it is inferred that Ω_1 is a control invariant set. Suppose that Ω_{j-1} is a control invariant set. Note that $\Phi_j \subseteq Q(\Phi_{j-1}) \subseteq Q(\Omega_{j-1})$. Since $\Omega_i = \Phi_j \cup \Omega_{j-1}$, from lemma (1) it is deduced that Ω_j is a control invariant set. ■

A method to compute control invariant sets for a general class of nonlinear systems is presented in this paper. The method is based on computing an approximate one-step set using interval arithmetic. The proposed algorithm can approximate the exact set with a given bound on the error. It is clear that lower bounds on the error yield to bigger computational burden, however, the computation is done off-line. A trade-off between computation time and error can be reached. In order to obtain an easy-to-use sequence of control invariant set, a polytopic approximation is presented. It is based on the approximation of the previous control invariant sets by an inner polytope.

3 Interval arithmetic

An interval number $X = [a,b]$ is the set $\{ x : a \leq x \leq b \}$ of real numbers between and including the endpoints a and b. Interval arithmetic is an arithmetic defined on sets of intervals, rather than sets of real numbers. The interval arithmetic is based on operations applied to sets of intervals.

Let $\mathbb{I}$ be the set of real compact intervals $[a,b]$ with $a,b \in \mathbb{R}$. Operations in $\mathbb{I}$ satisfy the expression:

$$A \; op \; B = \{ a \; op \; b \; : \; a \in A, b \in B \} \; for \; A,B \in \mathbb{I} \tag{3}$$

In this way, the four basic interval operations [14] are:

$$[a,b] + [c,d] = [a+c, b+d] \tag{4}$$

$$[a,b] - [c,d] = [a-d, b-c]$$

$$[a,b] * [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$$

$$[a,b]/[c,d] = [a,b] * [1/d, 1/c], \text{ if } 0 \notin [c,d]$$

An extension of the interval arithmetic to include 0 in division can be found in [7]. The interval extension of standard functions $\{sin, cos, tan, arctan, exp, ln, abs, sqr, sqrt\}$ is possible too.

Definition 3 (Box) *A box is an interval vector (vector whose components are intervals). An interval hull of a set $X \subseteq \mathbb{R}^n$, denoted by $\Box X$, is a box that satisfies $X \subseteq \Box X$. Given a box $\Box X = ([a_1,b_1], \ldots, [a_n,b_n])^\top$, $mid(\Box X)$ denotes its center and $diam(\Box X) = (b_1 - a_1, \ldots, b_n - a_n)^\top$.*

Definition 4 (Range) *The range of a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ over a set $X \subset \mathbb{R}^n$ is defined as $f(X) = \{ f(x) | x \in X \}$.*

Definition 5 (Natural interval extension) *If $f : \mathbb{R}^n \to \mathbb{R}$ is a function computable as an expression, algorithm or computer program involving the four elementary arithmetic operations interspersed with evaluations of standard functions then, a natural interval extension of f, denoted $\Box f$, is obtained replacing* *each occurrence of each variable by the corresponding interval variable, by executing all operations according to formulas (4) and by computing ranges of the standard functions.[9]*

Theorem 3 *[Inclusion] A natural interval extension $\Box f$ of a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ over a box $X \subseteq \mathbb{R}^n$ satisfies that $f(X) \subseteq \Box f(X)$. That is the fundamental theorem of the interval arithmetic [14].*

Theorem 4 *[Inclusion Monotonic] A natural interval extension $\Box f$ of a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ over two boxes $Y, X \subseteq \mathbb{R}^n$ satisfies that if $Y \subseteq X$ then $\Box f(Y) \subseteq \Box f(X)$ [9].*

A consequent of Theorem 4 is that if a sequence of boxes X_k converges to a real vector x, then the sequence of interval bounds $\Box f(X_k)$ converges to the real vector $f(x)$. This is an important property of natural interval extension is that [14].

4 One-step set approximation algorithm

Interval algorithms have been used successfully in the resolution of several problems. These algorithms are basically branch and bound [7] algorithms where the ranges of functions are bounded by interval arithmetic.

Let A, B be two sets defined by finite sets of inequalities. Interval branch and bounds algorithms can prove set inclusion $A \subset B$ and solve the set inversion problem $A = f^{-1}(B)$, see [8].

In this paper the application of interval branch and bound algorithm to invariant set theory is considered. Given a target set Ω, the interval algorithm computes a set denoted $B(\Omega)$ with $B(\Omega) \subseteq Q(\Omega)$. By using the result presented in section 2, this approach can be used to compute a sequence of control invariant sets making $Q_{ap}(\Omega) \equiv B(\Omega)$. $B(\Omega)$ is compound by a list of boxes and $B^c(\Omega)$ represents the complement set of $B(\Omega)$ in X. The input parameters of Algorithm 1 are the non linear system (1), the sets X and U that represent the states and inputs allowed, the target set Ω and two tolerances $\varepsilon_{A1}, \varepsilon_{A2}$, that bound the level of division accomplished by the interval branch and bound algorithms. The algorithm returns three lists of boxes:

1. The one-step set approximation $B(\Omega)$.
2. L_n that represents the states that can not be driven to Ω, so $\forall x_k \in L_n \ \nexists u_k \in U \mid f(x_k, u_k) \in \Omega$.
3. L_u that includes the undetermined boxes with $\varepsilon_{A1}, \varepsilon_{A2}$ selected.

The complement of $B(\Omega)$ is the union of L_n and L_u, $B^c(\Omega) = L_u \cup L_n$.

The algorithm gets a box X_i and for that box computes a control action $u_i = select(U)$. The operator $select()$ can be chosen $select(U) = mid(U)$ or the 'best' u_i such that X_i is driven to set Ω. The algorithm checks if the interval extension $\Box f(X_i, select(u))$ belongs to Ω. When the result is positive, X_i is inserted in $B(\Omega)$. If $\Box f(X_i, U) \cap \Omega = \emptyset$ then X_i is inserted in L_n. In the other case, X_i is split to be processed again if the width of X_i is greater than ε_{A1}. If none of the previous conditions is satisfied then Algorithm 2 is invoked. Algorithm 2 produces a similar search, but in U.

Algorithm 1

$(B(\Omega), L_n, L_u)$=*OneStepSetApproximation*$(\Omega, f, X, U, \varepsilon_{A1}, \varepsilon_{A2})$
Alg
    $L = X$
    while $L \neq \emptyset$
        X_i=first(L)
        $X_F = \Box f(X_i, U)$;$X_{mF} = \Box f(X_i, select(U))$
        Select one of:
        (1) **if** $X_F \cap \Omega = \emptyset$ insert X_i in L_n **endif**
        (2) **if** $X_{mF} \subseteq \Omega$ insert X_i in $B(\Omega)$ **endif**
        (3) **if** $diam(X_F) < \varepsilon_{A1}$
            Reachable=*ProcU*$(\Omega, X_i, F, U, \varepsilon_{A2})$
            if *Reachable*='NO' insert X_i in L_n **endif**
            if *Reachable*='YES' insert X_i in $B(\Omega)$ **endif**
            if *Reachable*='UNCERTAIN' insert X_i in L_u
            endif
        endif
        (4) bisect X_i and insert in L
    endwhile
    return $L_n, B(\Omega), L_u$
End

The Algorithm 2 is similar to Algorithm 1. With a fixed state box X_i given, a search in U is achieved. The algorithm returns three possibilities: that a control signal exists that drives the box X_i to Ω; that such a control signal does not exist or it is not possible to assure any answer with the division level ε_{A2}.

Algorithm 2

Reachable=*ProcU*$(\Omega, X_i, f, U, \varepsilon_{A2})$
Alg
    $\tilde{L} = U$
    while $\tilde{L} \neq \emptyset$
        Ua=first($\tilde{L}$)
        $X_U = \Box f(X_i, Ua)$;$mX_U = \Box f(X_i, select(Ua))$
        select one of:
        (1)**if** $mX_U \subseteq \Omega$ Return 'YES' **endif**
        (2)**if** $mX_U \cap \Omega \neq \emptyset$
            if $diam(U_a) > \varepsilon_{A2}$
                bisect U_a and insert in $\tilde{L}$
            else
                insert U_a in $\tilde{L_u}$
            endif
        endif
        (3)Reject U_a
    endwhile
    if $L_u = \emptyset$ **return** 'NO'
    else return 'UNCERTAIN'

endif
End

Theorem 5 . *Let Ω, X, U compact (closed and bounded) sets, $\breve{\Omega}$ the interior of Ω and a system (1) where f is continuous in Ω. Consider the proposed algorithm in which $max\{\varepsilon_{A1}, \varepsilon_{A2}\} < \varepsilon$:*

(i)$\forall x_k \in B(\Omega)$ then $\exists u_k \in U \mid f(x_k, u_k) \in \Omega$
(ii)There is a real positive number $\varepsilon > 0$ such that:
(ii).1 if $\exists x_k \in X \mid \forall u_k \in U \; f(x_k, u_k) \notin \Omega$ then $x_k \notin B(\Omega) \cup L_u$
(ii).2 if $\forall x_k \in X$, $\exists u_k \in U \mid f(x_k, u_k) \in \breve{\Omega}$ then $x_k \in B(\Omega)$

Proof:

(i) The first part is easy to prove. A box X_i is inserted in $B(\Omega)$ only when $\exists u_k \in U$ such $\square f(X_i, u_k) \subseteq \Omega$ so, by Theorem 3 $\forall x_k \in X_i \mid \exists u_k \in U \; f(x_k, u_k) \in \Omega$ then $x_k \in B(\Omega)$.

As a consequence of this result, it follows that $B(\Omega) \subseteq Q(\Omega)$ so the algorithm returns a correct $Q_{ap}(\Omega)$.

(ii) To prove the second part is considered that z_k denotes the vector (x_k, u_k) and $Ball(z_k, r)$ denotes the sphere of radius r and center z_k.

(ii).1 If $\exists x_k \in X \mid \forall u_k \in U \; f(z_k) \notin \Omega$ then $\exists Ball(z_k, \varepsilon)$ such that $f(Ball(z_k, \varepsilon) \cap \Omega = \emptyset$. By interval monotonic inclusion $\exists Z_k \subset Ball(z_k, \varepsilon)$ with $z_k = mid(Z_k)$ such that $\square f(Z_k) \subseteq f(Ball(z_k, \varepsilon))$ where Z_k is a box. So, $\square f(Z_k) \cap \Omega = \emptyset$. Then, any box $\hat{Z}_k \subseteq Z_k$ fulfills $\square f(\hat{Z}_k) \cap \Omega = \emptyset$. So with appropriate ε_{A1} and ε_{A2}, Algorithm 1 inserts x_k in L_n.

(ii).2 If $\forall x_k \in X \mid \exists u_k \in U \; f(z_k) \in \breve{\Omega}$ then $\exists Ball(z_k, \varepsilon)$ such that $f(Ball(z_k, \varepsilon) \subseteq \breve{\Omega}$. By interval monotonic inclusion $\exists Z_k \subset Ball(z_k, \varepsilon)$ with $z_k = mid(Z_k)$ such that $\square f(Z_k) \subseteq f(Ball(z_k, \varepsilon))$ where Z_k is a box. So, $\square f(Z_k) \subseteq \breve{\Omega}$. Then, any box $\hat{Z}_k \subseteq Z_k$ fulfills $\square f(\hat{Z}_k) \subseteq \breve{\Omega}$. So with appropriate ε_{A1} and ε_{A2}, Algorithm 1 inserts x_k in $B(\Omega)$.

■

A first consequence of theorem 5 is that if $\varepsilon_{A1}, \varepsilon_{A2} \longrightarrow 0$ then $B(\Omega) \longrightarrow Q(\Omega)$, so $B(\Omega)$ can be a reliable approximation of the exact set $Q(\Omega)$ and because X, U are the search space of Algorithms 1-4, state and control constraints are fulfilled.

5 One-step set polytopic approximation

An approximate one-step set can be obtained by means of Algorithms 1-2. This set is represented by a list of boxes. In order to obtain a simpler representation, polytopes are proposed. Given a target set Ω and the $B(\Omega)$ set obtained by Algorithms 1-2, Algorithms 3-4 returns a polytope denoted $P(\Omega)$ with $P(\Omega) \subseteq B(\Omega) \subseteq Q(\Omega)$. A new sequence of control invariant sets can be obtained by $Q_{ap}(\Omega) \equiv P(B(\Omega))$.

The input parameters of Algorithm 3 are: $B(\Omega)$, the number of facets of the polyhedral n and a tolerance δ . The algorithm returns a polytope $P(\Omega)$. Each facet of the polyhedral is of the form $b_{min} \leq c^t x \leq b_{max}$ where c fulfills $\|c\| = 1$ and the scalars b_{max} and b_{min} are calculated by the optimization problems:

$$b_{max} = \max_{x \in B(\Omega)} c^t x$$

$$b_{min} = \min_{x \in B(\Omega)} c^t x$$

The solutions of these optimization problems are obtained going through the list of boxes $B(\Omega)$. When b_{max} and b_{min} are known the constraints $c^t x \leq b_{max} - \delta$ and $-c^t x \leq -(b_{mix} + \delta)$ are added to the polytope $P(\Omega)$. This process is repeated n times.

Algorithm 3

$P(\Omega) = InsidePolytope(B(\Omega), n, \delta)$
Alg
$P(\Omega) = \emptyset$
for 1 to n
c = Select a slope vector
b_{max} = Maximun value of $c^t x$ with $x \in B(\Omega)$
b_{min} = Minimum value of $c^t x$ with $x \in B(\Omega)$
Add to $P(\Omega)$ the constraints
$c^t x \leq b_{max} - \delta$ and $-c^t x \leq -(b_{min} + \delta)$
endfor
$P(\Omega)$=Delete unnecessary constraints to $P(\Omega)$
return $P(\Omega)$
Alg

Algorithm 3 provides a polytopic approximation of $B(\Omega)$, but does not guarantee the absence of states that belongs to $B^c(\Omega)$ in $P(\Omega)$. Thus, it is needed to erase the states that belong to the resulting $P(\Omega) \cap B^c(\Omega)$. The Algorithm 4 is proposed to obtain an inner polytope. The input parameters of Algorithm 4 are: $B^c(\Omega)$ and $P(\Omega)$.

Algorithm 4 discards boxes B_i of $B^c(\Omega)$ that does not belong to $P(\Omega)$. If $B_i \cap P(\Omega) \neq \oslash$ the nearest face of the polytope $P(\Omega)$is moved until the box is out of $P(\Omega)$ (Compact $P(\Omega)$).

Algorithm 4

$P(\Omega) = EraseStatesNotInside(P(\Omega), B^c(\Omega))$
Alg
while $B^c(\Omega) \neq \emptyset$
B_i = First element of $B^c(\Omega)$
if $B_i \cap P(\Omega) \neq \oslash$ Compact $P(\Omega)$
else Reject B_i
endif
endwhile
$P(\Omega)$=Delete unnecessary constraints to $P(\Omega)$
return $P(\Omega)$
End

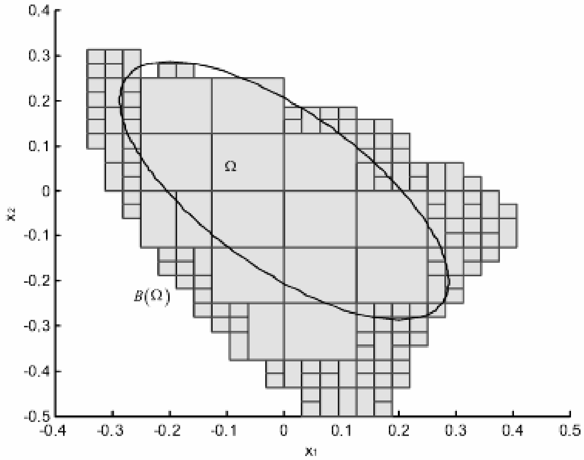

Figure 1.a: $B(\Omega)$ with ε_{A1}=0.03 and ε_{A2}=0.1

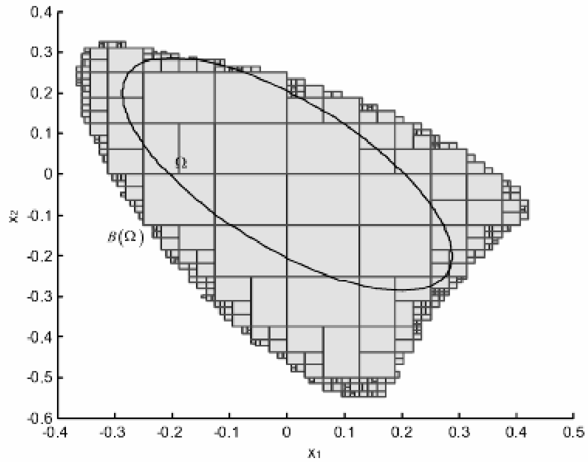

Figure 1.b: $B(\Omega)$ with ε_{A1}=0.01 and ε_{A2}=0.1

Theorem 6 *Let $B(\Omega)$ a set represented by a list of boxes, and $P(\Omega)$ a polytope calculated by Algorithms 3-4 then $P(\Omega) \subseteq B(\Omega)$ and $P(\Omega) \cap B^c(\Omega) = \emptyset$.*

The proof of this theorem follows from the previous discussion and it is omitted because of lack of space.

6 Example

As illustrative example of the proposed controller, the technique is applied to a system used in [5] and described by the following ODEs:

$$\dot{x}_1 = x_2 + u(\mu + (1-\mu)x_1)$$

$$\dot{x}_2 = x_1 + u(\mu + 4(1-\mu)x_2)$$

where the parameter $\mu = 0.5$ and the input constraint is $|u| \leq 2$. The system is discretized with a sampling time of 0.1 time-units using a fourth order RK. A linear locally stabilizing state feedback gain $K = [2.118\ 2.118]$ is used to derive the initial control invariant set $\Omega = \{x \in \mathbb{R}^2 | x^t P x \leq 0.7\}$ where:

$$P = \begin{bmatrix} 16.5926 & 11.5926 \\ 11.5926 & 16.5926 \end{bmatrix}$$

Fig. 1.a shows the one step approximation $B(\Omega)$ and Ω. $B(\Omega)$ has been computed using Algorithm 1 and 2 with $\varepsilon_{A1} = 0.03$ and $\varepsilon_{A2} = 0.1$.

A better approximation to the one step set can be obtained using a smaller error bound as shown in Fig. 1.b where $\varepsilon_{A1} = 0.01$ and $\varepsilon_{A2} = 0.1$ have been choosen.

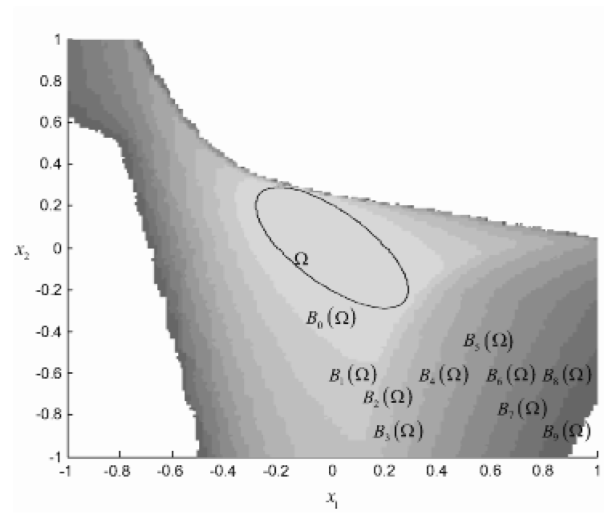

Figure 2: $B_i(\Omega)$ with i=0..9

Fig. 2 shows a sequence $B_i(\Omega)$ with $i = 1..9$ using $\varepsilon_{A1} = 0.01$, $\varepsilon_{A2} = 0.1$ and Algorithms 1-2.

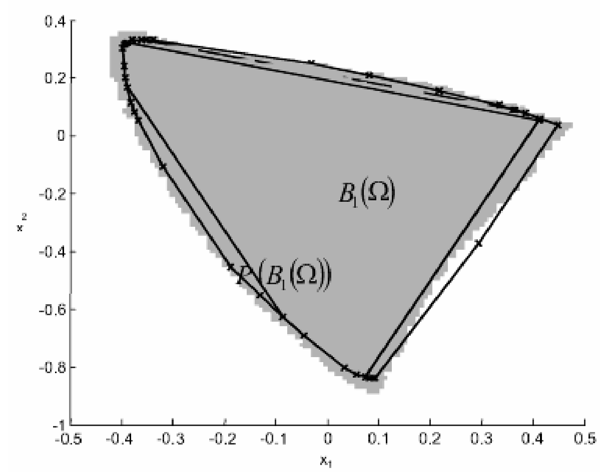

Figure 3: $P(B_1(\Omega))$

The inner polytope $P(B_1(\Omega))$ is shown in Fig. 3. The polytope has been obtained using Algorithms 3-4.

A sequence $P_i(\Omega)$ with $i = 0..9$ obtained by applying Algorithms 1-2-3-4 can be seen in Fig. 4.

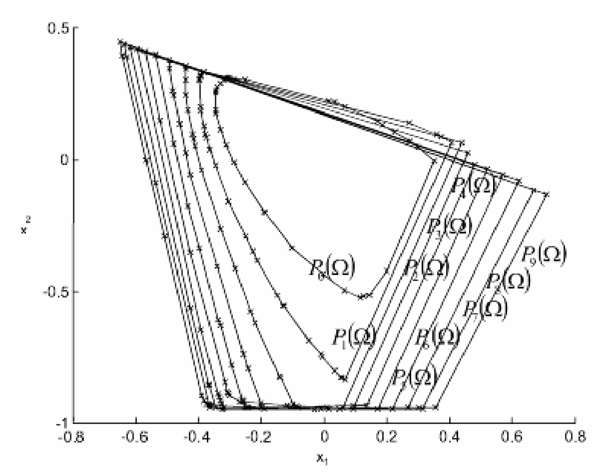

Figure 4: $P_i(\Omega)$ with i=0..9

7 Conclusion

A solution to the problem of determining the reachability of a target set is proposed. The paper shows how given an initial control invariant set, a sequence of control invariant sets can be computed recursively. Two possibilities are considered to define the control invariant sets: boxes and polytopes. A list of boxes provides a reliable approximation to the exact one-step set while polytopes provide a simpler definition but a worse approximation. The sequences of controlled sets can be used to design controllers and to guarantee stability because state and control constrains are fulfilled.

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