FINITE FORM REALIZATIONS OF ADAPTIVE CONTROL ALGORITHMS

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Abstract

We suggest a new method to design adaptation algorithms that guarantee improved performance and are applicable for a class of plants with nonconvex parameterization. The main idea of the method is, first, to augment the tuning error (possibly using uncertainty-dependent signals) of the known adaptive schemes in such a way that the desired characteristics of the adaptive system are guaranteed. Then we search for realization of the proposed schemes in an integral-differential form similar to the PI (proportional-integral) rules. Such adaptation schemes in the paper are called adaptive algorithms in *finite forms*. For this new description, neither dependence on state derivatives, nor unknown parameters is required. Sufficient conditions for existence of new finite form realizations of adaptive algorithms are proposed.

1 Introduction

Despite significant progress in adaptive control theory of linear and nonlinear plants [3, 5, 4, 9], plants with relative degree greater than one [16, 14, 15], and systems with nonconvex parameterization [6, 10], there is still a room for further studies, especially when striving for improved performance in the presence of nonconvex parameterization.

Most of the available results in direct adaptive control when considering some different performance measures, for instance LQ performance, deal only with convergence analysis of different adaptive schemes without concern for improving their performance [12, 13]. When they do suggest improvements like those in [11], they do not provide any exact performance measure that can explicitly be computed *a-priori* except probably the bounds on L_2 and L_∞ norms[1] for the tracking errors.

Another unresolved issue of conventional adaptive control theory is nonconvex parameterization of the plant model. Unfortunately, the available approaches encourage the designer to compensate for the nonlinearity (at least, in part) by using an additional damping term, or high-gain feedback [6, 10, 7]. Very recent results on nonparametric adaptation [17] can also be applied to nonlinearly parameterized systems. Nevertheless, all these approaches merely can provide integrability of the squared error.

One impediment to further progress, we believe, is due to the lack of sufficient information in the conventional adaptive schemes to improve the performance and deal with nonconvex parameterization. One way to provide the algorithms with extra information is to augment the tuning errors. Many adaptive control schemes use the error augmentation to make the estimation error be dependent on the controller parameters. This idea is inherent to both Morse's adaptive controllers [14] and those based on Kreisselmeier's observers [16] when dealing with the plants with relative degree grater than one. These augmented errors then are used in conventional gradient schemes

$$\dot{\hat{\theta}} = -\Gamma\tilde{\psi}(\mathbf{x},t)\mathcal{A}(\mathbf{x},\hat{\theta},t), \qquad (1)$$

where $\mathbf{x} \in R^n$ is a state (or output) vector, $\hat{\theta} \in R^d$ is a vector of the controller parameters, $\mathcal{A}(\mathbf{x},\hat{\theta})$ is an operator that depends on particular problem, and gain $\Gamma > 0$.

The existing performance limitations that have been pointed out motivate the following challenging question: is there an augmentation that can create new properties in the system if applied to it (in addition to readily achievable finiteness of L_2 and L_∞ norm bounds). Furthermore, are algorithms with such an augmentation physically realizable, i.e., the controller parameters $\hat{\theta}$ can be computed at any time instant without measuring the unknown signals or parameters?

Instead of searching for the desired adaptive algorithm in the conventional form defined by (1), we suggest to extend this class as follows (as in [1, 17, 18]):

$$\hat{\theta}(\mathbf{x}(t),t) \;\; = \;\; \hat{\theta}_P(\mathbf{x},t) + \hat{\theta}_I(t); \; \dot{\hat{\theta}}_I = \mathcal{A}_2(\mathbf{x},\hat{\theta},t),$$

[1] Function $\nu : R_+ \rightarrow R$ is said to belong to L_2 iff $L_2(\nu) = \int_0^\infty \nu^2(\tau)d\tau < \infty$. The value $\sqrt{L_2(\nu)}$ stands for the L_2 norm of $\nu(t)$. Function $\nu : R_+ \rightarrow R$ belongs to and L_∞ iff $L_\infty(\nu) = \sup_{t\geq 0}\|\nu(t)\| < 0$, where $\|\cdot\|$ is the Euclidean norm. The value of $L_\infty(\nu)$ stands for the L_∞ norm of $\nu(t)$.

$$\hat{\theta}_P(\mathbf{x},t) \;=\; \mathcal{A}_1(\mathbf{x},t). \qquad (2)$$

It is obvious that algorithms (1) belong to the class (2). Furthermore, functions $\hat{\theta}(\mathbf{x},t)$ when written in the differential form (1) may depend on the unknown parameters and unmeasured signal. These simple observations lead to quite unexpected conclusions. Instead of restricting the design procedure to those algorithms that can be realizable in the form of equation (1), one may design the adaptation algorithms in two steps. First, search for the desired augmentation, possibly uncertainty-dependent, to obtain the requested properties of the adaptive control[2]. Second once a suitable tuning error is chosen, find a realization of the algorithm in the form of integral-algebraic equations of the type (2), what is termed by *algorithms in finite form*[3].

The current paper is devoted to solution of the following problem: given the desired augmentation (possibly, derivative-dependent) that guarantees improved performance and ability to deal with nonconvex parameterization for a class of nonlinear systems, find functions $\mathcal{A}_1(\mathbf{x},t)$, $\mathcal{A}_2(\mathbf{x},\hat{\theta},t)$ that guarantee the desired realization. The layout of the paper is as follows. In Section 2 we specify a class of nonlinear dynamical systems under consideration and select the desired augmentation. Section 3 contains the main results of the paper. We show that the realization problem is solvable for a class of nonlinear systems and provide the sufficient conditions which guarantee existence of solutions. Section 4 concludes the paper.

2 Problem Formulation

Let the plant model be given as

$$\begin{aligned} \dot{x}_i &= f_i(\mathbf{x}) + g_i(\mathbf{x})u, \quad i = 1,\dots,m \qquad (3)\\ \dot{x}_j &= f_j(\mathbf{x}) + \nu_{j-m}(\mathbf{x},\theta) + g_j(\mathbf{x})u, \quad j = m+1,\dots,n, \end{aligned}$$

where $\mathbf{x} \in R^n$ is a state vector, $f_i,\ g_i : R^n \to R$, $f_i, g_i \in C^1$, $\theta \in \Omega_\theta \subset R^d$ is a vector of unknown parameters, $\vartheta_i : R^n \times R^d \to R$, $\vartheta_i \in C^1$, u is a control input. For the notational convenience, we will use more compact description of system (3):

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}) + \vartheta(\mathbf{x},\theta) + \mathbf{g}(\mathbf{x})u, \qquad (4)$$

where $\mathbf{x} \in R^n$ is a state vector, $\theta \in \Omega_\theta \subset R^d$ is a vector of unknown parameters, $u \in R$ is a control input, functions $\mathbf{f}(\cdot)$, $\mathbf{g}(\cdot)$, $\vartheta(\cdot,\cdot)$ are specified as follows: $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}),\dots,f_n(\mathbf{x}))^T$, $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}),\dots,g_n(\mathbf{x}))^T$ $\vartheta(\mathbf{x},\theta) = (0,\dots,0,\nu_1(\mathbf{x},\theta),\dots,\nu_{n-m}(\mathbf{x},\theta))^T$. For the sake of compactness when dealing with the partial derivatives of a function we will use the following notation: $L_{\mathbf{f}}\psi(\mathbf{x}) = \partial\psi/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x})$ to denote the Lie derivative of function $\psi(\mathbf{x})$ along the vector field $\mathbf{f}(\mathbf{x})$. It will be useful also to think of the state vector $\mathbf{x} \in \mathcal{L} \subseteq R^n$ as follows

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_1 \oplus \mathbf{x}_2, \\ \mathbf{x}_1 &= (x_1,\dots,x_m)^T,\ \mathbf{x}_2 = (x_{m+1},\dots,x_n)^T \qquad (5) \end{aligned}$$

where symbol $\oplus$ denotes concatenation of two vectors $\mathbf{x}_1 \in \mathcal{L}_1 \subseteq R^m$, $\mathbf{x}_2 \in \mathcal{L}_2 \subseteq R^{n-m}$, and $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ are linear spaces. Notice that time-derivative of $\mathbf{x}_1$ is independent on θ whereas time-derivative of vector $\mathbf{x}_2$ depends on unknown parameters θ explicitly. We refer to the spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ as *uncertainty-independent* and *uncertainty-dependent partitions* of system (3), respectively. To denote the right-hand sides of the partitioned system we use the following notations: $\mathbf{f}_1 = (f_1,\dots,f_m)^T$, $\mathbf{f}_2 = (f_{m+1},\dots,f_n)^T$, $\mathbf{g}_1 = (g_1,\dots,g_m)^T$, $\mathbf{g}_2 = (g_{m+1},\dots,g_n)^T$. In analogy with the definition of independence of a function on the components x_i of its argument $\mathbf{x}$ we would like to define a notion of independence of the function with respect to partition. Given that $\mathbf{x} \in \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ and function $\omega(\mathbf{x}) : R^n \to R^n$ be differentiable for any $\mathbf{x} \in R^n$. Function $\omega(\mathbf{x})$ is said to be *independent on partition* $\mathcal{L}_2$ iff $\partial\omega(\mathbf{x}_1 \oplus \mathbf{x}_2)/\partial\mathbf{x}_2 = 0$.

Similarly to [1, 2], we define the control goal as reaching a target manifold asymptotically. We assume that the target manifold can be given by the following equality $\psi(\mathbf{x},t) = 0$, where $\psi :\ R^n \times R \to R$, $\psi(\mathbf{x},t) \in C^1$. Additional restrictions on the function $\psi(\mathbf{x},t)$ are formulated in Assumptions 1, 2.

Assumption 1 *Function $\psi(\mathbf{x},t)$ is such that for any $\delta > 0$ there exists a function $\varepsilon :\ R_+ \to R_+$: $|\psi(\mathbf{x},t)| \le \delta \Rightarrow \|\mathbf{x}\| \le \varepsilon(\delta)$ along system (4) solutions.*

Assumption 2 *Functions $\psi(\mathbf{x},t)$ and $\mathbf{g}(x)$ satisfy the following inequality: $\forall\,\mathbf{x} \in R^n \Rightarrow |L_{\mathbf{g}}\psi(\mathbf{x},t))| > \delta_1 > 0,\ \delta_1 \in R_+$.*

Assumption 2 ensures existence of the feedback that transforms the original system into that of the error model with respect to the variable $\psi(\mathbf{x},t)$. We consider

$$\begin{aligned} \dot{\psi} \;=\;& L_{\mathbf{f}}\psi(\mathbf{x},t) + L_{\vartheta(\mathbf{x},\theta)}\psi(\mathbf{x},t) + (L_{\mathbf{g}}\psi(\mathbf{x},t))\,u + \\ & \partial\psi(\mathbf{x},t)/\partial t. \qquad (6) \end{aligned}$$

If Assumption 2 holds, there exists the control input

$$\begin{aligned} u(\mathbf{x},\hat{\theta},t) \;=\;& (L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t))^{-1}(-\varphi(\psi) - L_{\mathbf{f}}\psi(\mathbf{x},t) - \\ & L_{\vartheta(\mathbf{x},\hat{\theta})}\psi(\mathbf{x},t) - \partial\psi(\mathbf{x},t)/\partial t), \qquad (7) \end{aligned}$$

where $\hat{\theta} \in \Omega_{\hat{\theta}} \subset R^d$ is a vector of controller parameters, such that it transforms (6) into

$$\dot{\psi} = -\varphi(\psi) + z(\mathbf{x},\theta,t) - z(\mathbf{x},\hat{\theta},t) \qquad (8)$$

where $z(\mathbf{x},\theta,t) = L_{\vartheta(\mathbf{x},\theta)}\psi(\mathbf{x},t)$.

Let the closed-loop system satisfy the following additional requirements

[2] It has been reported, for instance, in [2, 1] that the derivative-dependent algorithms written in differential form (1) are able to deal with a class nonconvexly parameterized plant.

[3] To the best of our knowledge, such terminology in adaptive control literature has been proposed first by A. Fradkov in [8] in order to denote the speed-gradient based adaptive control schemes given by integral-algebraic, not differential equations.

Assumption 3 *For any $\theta \in \Omega_\theta$ there exists $\hat{\theta}^* \in \Omega_{\hat{\theta}} \subset R^d$, such that for all $\mathbf{x} \in R^n$, $t \in R_+$ the following holds*

$$\dot{\psi} + \varphi(\psi) + z(\mathbf{x}, \theta, t) - z(\mathbf{x}, \hat{\theta}^*, t) = \dot{\psi} + \varphi(\psi) = 0 \qquad (9)$$

Assumption 4 *Function $\varphi(\psi)$ in (9) satisfies*

$$\varphi(\psi) \in C^0,\ \varphi(\psi)\psi > 0\ \forall \psi \neq 0,\ \lim_{\psi \to \infty} \int_0^\psi \varphi(\xi) d\xi = \infty.$$

Assumption 5 *There exists function $\alpha(\mathbf{x}, t) : R^n \times R \to R^d$ such that $(z(\mathbf{x}, \hat{\theta}, t) - z(\mathbf{x}, \hat{\theta}^*, t))(\alpha(\mathbf{x}, t)^T(\hat{\theta} - \hat{\theta}^*)) > 0\ \forall\ z(\mathbf{x}, \hat{\theta}^*, t) \neq z(\mathbf{x}, \hat{\theta}, t)$; $\|z(\mathbf{x}, \hat{\theta}, t) - z(\mathbf{x}, \hat{\theta}^*, t)\| \leq D\|\alpha(\mathbf{x}, t)^T(\hat{\theta} - \hat{\theta}^*)\|,\ D \in R_+,\ D > 0$.*

Assumption 6 *There exists such positive constant $D_1 > 0$ that for any $\mathbf{x}$, $\hat{\theta}$, $\hat{\theta}^*$, $t > 0$ the following inequality holds*

$$\|z(\mathbf{x}, \hat{\theta}, t) - z(\mathbf{x}, \hat{\theta}^*, t)\| \geq D_1\|\alpha(\mathbf{x}, t)^T(\hat{\theta} - \hat{\theta}^*)\|.$$

Assumption 3 is a kind of certainty equivalence or matching condition. It simply states that for every unknown $\theta^* \in \Omega_\theta$ there exists such vector of the controller parameters $\hat{\theta}^*(\theta^*) \in \Omega_{\hat{\theta}}$ that the system dynamics with this control function satisfies the following equation $\dot{\psi} = -\varphi(\psi)$. Assumption 4 specifies the properties of function $\varphi(\psi)$, thus stipulating asymptotic stability of manifold $\psi(\mathbf{x}, t) = 0$ for $\hat{\theta} = \hat{\theta}^*$ and ensuring unbounded growth of integral $\int_0^\psi \varphi(\xi) d\xi$ as $\psi \to \infty$.

Assumption 5 is given to specify an admissible nonlinear parameterization of the controller. Notice that for linearly parameterized plants this assumption is automatically satisfied. Throughout the paper we will also assume that functions $\alpha(\mathbf{x}, t)$ and $u(\mathbf{x}, \hat{\theta}, t)$ both are bounded in t and function $\alpha(\mathbf{x}, t) \in C^1$.

As a candidate for the augmented error $\tilde{\psi}(\mathbf{x}, t)$ we select the following $\tilde{\psi}(\mathbf{x}, t) = \dot{\psi} + \varphi(\psi(\mathbf{x}, t))$. It has been proven in [2, 1] that the algorithm

$$\dot{\hat{\theta}} = \Gamma(\varphi(\psi) + \dot{\psi})\alpha(\mathbf{x}, t), \qquad (10)$$

with positive-definite matrix $\Gamma > 0$ guarantees $\psi(\mathbf{x}(t), t) \to 0$ as $t \to \infty$ for the closed loop system

$$\begin{aligned}
\dot{x}_i &= f_i(\mathbf{x}) + g_i(\mathbf{x})u, \quad i = 1, \ldots, m \\
\dot{x}_j &= f_j(\mathbf{x}) + \vartheta_{j-m}(\mathbf{x}, \theta) + g_j(\mathbf{x})u, \quad j = m+1, \ldots, n, \\
\dot{\psi} &= -\varphi(\psi) + z(\mathbf{x}, \theta, t) - z(\mathbf{x}, \hat{\theta}, t) \\
\dot{\hat{\theta}} &= \Gamma(\varphi(\psi) + \dot{\psi})\alpha(\mathbf{x}, t),\ \Gamma > 0 \qquad (11)
\end{aligned}$$

with control (7) under Assumptions 1, 3 – 5. In addition, however, it is possible to derive from [2, 1] that algorithms (10) guarantee $\dot{\psi} \in L_2 \cap L_\infty$ and $u(\mathbf{x}, \theta, t) - u(\mathbf{x}, \hat{\theta}, t) \in L_2$ [18].

Sometimes it is desirable to consider slightly different error model from that given by equations (8):

$$\dot{\psi} = -\varphi(\psi) + z(\mathbf{x}, \theta, t) - z(\mathbf{x}, \hat{\theta}, t) + \varepsilon(t), \qquad (12)$$

where function $\varepsilon : R_+ \to R$, $\varepsilon \in C^0$, $\varepsilon \in L_2$ models unknown disturbances due to the unmodeled dynamics or the measurement errors. The properties of algorithm (10) in this case are formulated in the next theorem:

Theorem 1 *Let the error model be given by equation (12) and Assumptions 1, 3-6 hold. Then $\psi(\mathbf{x}, t)$ is bounded and furthermore $\psi(x, t) \in L_2,\ \dot{\psi} \in L_2,\ z(\mathbf{x}, \theta, t) - z(\mathbf{x}, \hat{\theta}, t) \in L_2$. If function $\varepsilon(t)$ is bounded and function $z(\mathbf{x}, \hat{\theta}, t)$ be locally bounded with respect to $\mathbf{x}$, $\hat{\theta}$, uniformly bounded with respect to t then $\psi(\mathbf{x}, t) \to 0$ as $t \to \infty$.*

Its proof follows from analysis of time-derivative of the following function:

$$2(D - D_1) \int_0^\psi \varphi(\xi) d\xi + V_{\hat{\theta}} + 2\beta^2(D - D_1) \int_t^\infty \varepsilon^2(\tau) d\tau,$$

where $\beta \in R,\ \beta > 1$, and

$$V_{\hat{\theta}} = D_1/4 \int_t^\infty \varepsilon^2(\tau) d\tau + 0.5(\hat{\theta} - \hat{\theta}^*)^T \Gamma^{-1} (\hat{\theta} - \hat{\theta}^*)$$

Hence the question is how to realize this algorithm in a form that depends on neither time-derivative $\dot{\psi}$ nor its filtered estimate explicitly, nor on anything implying knowledge of unknown parameters θ. As mentioned in Section 1, we propose to use finite form (2) of the adaptive algorithms instead of the differential form (10). In the next section we study under what conditions one can represent algorithms (10) in finite forms.

3 Adaptive Algorithms in Finite Forms

First, we consider rather general case and formulate the conditions ensuring realization of algorithm (10) in the finite form explicitly, i.e., without any additional filters and further transformations of the closed loop system. Being nontrivial to solve for any admissible model of the plant given by equations (3), these conditions are satisfied for some special combinations of the plant models and goal functions $\psi(\mathbf{x}, t)$.

Second, we suggest to embed the plant dynamics into these extended systems for which the conditions sufficient for the finite-form realizations are always met. By doing so we no longer need to find a solution of the partial differential equations to realize the adaptation algorithms.

3.1 Explicit realization

Let us assume that in addition to the Assumptions 1–5, the following hold

Assumption 7 *For the given functions $\alpha(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ there exists function $\Psi(\mathbf{x})$ such that the following hold:*

$$\Psi(\mathbf{x}) : \partial\Psi(\mathbf{x}, t)/\partial\mathbf{x}_2 = \psi(\mathbf{x}, t)\partial\alpha(\mathbf{x}, t)/\partial\mathbf{x}_2 \qquad (13)$$

Then realizations of the adaptive scheme described by equations (10) follow from the next theorem:

Theorem 2 *Let Assumption 7 hold. Then there is a finite-form realization of the algorithms (10):*

$$\begin{aligned}
&\hat{\theta}(\mathbf{x},t)=\Gamma(\hat{\theta}_P(\mathbf{x},t)+\hat{\theta}_I(t));\\
&\hat{\theta}_P(\mathbf{x},t)=\psi(\mathbf{x},t)\alpha(\mathbf{x},t)-\Psi(\mathbf{x},t)\\
&\dot{\hat{\theta}}_I=\varphi(\psi(\mathbf{x},t))\alpha(\mathbf{x},t)+\partial\Psi(\mathbf{x},t)/\partial t-\\
&\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial t-(\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial \mathbf{x}_1-\\
&\partial\Psi(\mathbf{x},t)/\partial\mathbf{x}_1)(\mathbf{f}_1(\mathbf{x})+\mathbf{g}_1(\mathbf{x})u(\mathbf{x},\hat{\theta},t))
\end{aligned}\qquad(14)$$

Proof of Theorem 2. The theorem proof is quite straightforward and follows from explicit differentiation of function $\hat{\theta}(\mathbf{x},t)$ with respect to time: $\dot{\hat{\theta}}(\mathbf{x},t)=\Gamma(\dot{\hat{\theta}}_P+\dot{\hat{\theta}}_I)=\Gamma(\dot{\psi}\alpha(\mathbf{x},t)+\psi\dot{\alpha}(\mathbf{x},t)-\dot{\Psi}(\mathbf{x},t)+\dot{\hat{\theta}}_I)$. Notice that

$$\begin{aligned}
&\psi\dot{\alpha}(\mathbf{x},t)-\dot{\Psi}(\mathbf{x},t)+\dot{\hat{\theta}}_I=\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial\mathbf{x}_1\dot{\mathbf{x}}_1+\\
&\psi(\mathbf{x},t)\partial\alpha(\mathbf{x})/\partial\mathbf{x}_2\dot{\mathbf{x}}_2+\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial t-\qquad(15)\\
&\partial\Psi(\mathbf{x},t)/\partial\mathbf{x}_1\dot{\mathbf{x}}_1-\partial\Psi(\mathbf{x},t)/\partial\mathbf{x}_2\dot{\mathbf{x}}_2-\partial\Psi(\mathbf{x},t)/\partial t+\dot{\hat{\theta}}_I
\end{aligned}$$

According to Assumption 7 $\partial\Psi(\mathbf{x},t)/\partial\mathbf{x}_2=\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial\mathbf{x}_2$. Then taking into account (15) we can derive that

$$\begin{aligned}
&\psi\dot{\alpha}(\mathbf{x},t)-\dot{\Psi}(\mathbf{x},t)+\dot{\hat{\theta}}_I=(\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial\mathbf{x}_1-\\
&\partial\Psi/\partial\mathbf{x}_1)\dot{\mathbf{x}}_1+\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial t-\\
&\Psi(\mathbf{x},t)/\partial t+\dot{\hat{\theta}}_I
\end{aligned}\qquad(16)$$

Hance it follows from (14) and (16) that $\psi\dot{\alpha}(\mathbf{x},t)-\dot{\Psi}(\mathbf{x},t)+\dot{\hat{\theta}}_I=\varphi(\psi)\alpha(\mathbf{x},t)$. Therefore $\dot{\hat{\theta}}(\mathbf{x},t)=\Gamma(\dot{\psi}+\varphi(\psi))\alpha(\mathbf{x},t)$. *The theorem is proven.*

Theorem 2 provides us with an answer to the question of existence of the algorithms that being physically realizable satisfy differential equations (10). What is important is that the number of integrators for both algorithms (10) and (14) is the same. The disadvantage, however, is that the functions $\Psi(\mathbf{x},t)$ in Assumption 7 are not easy to find, if they exist at all. Nevertheless, despite the obvious difficulties in finding those functions $\Psi(\mathbf{x},t)$ that satisfy Assumption 7 there are several classes of the dynamical systems with certain structural properties that automatically reduce Assumption 7 to more easily verifiable requirements.

Corollary 1 *Let* $\dim(\mathbf{x}_2)=1$ *and function* $\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial x_n$ *be Riemann-integrable with respect to* x_n*, i.e. the following integral exist*

$$\Psi(\mathbf{x},t)=\int\psi(\mathbf{x},t)\frac{\partial\alpha(\mathbf{x},t)}{\partial x_n}dx_n\qquad(17)$$

Then there is a finite-form realization of algorithms (10).

Remark 1 Corollary 1 allows us to turn the problem of searching for a function $\Psi(\mathbf{x},t)$ satisfying equation (13) to that of existence of the indefinite integral of a function with respect to a single scalar argument. One example that fits the required assumptions is a special case of (3) for $m=n-1$, with function $\vartheta(\mathbf{x},\theta)$ satisfying Assumption 5. This assumption is automatically satisfied if $\vartheta(\mathbf{x},\theta)$ linearly parameterized or $\vartheta(\mathbf{x},\theta)=\vartheta(\mathbf{x}^T\theta)$ and $\vartheta(\cdot)$ is monotonic and belongs to a sector. Notice that indefinite integral in (17) can be replaced by $\Psi(\mathbf{x},t)=\int_{x_n(0)}^{x_n(t)}\psi(\mathbf{x},t)\partial\alpha(\mathbf{x},t)/\partial x_n dx_n$.

Another class of dynamical systems that automatically satisfy Assumption 7 is given by the following corollary.

Corollary 2 *Let function* $\alpha(\mathbf{x},t)$ *be independent on* $\mathcal{L}_2$*, i. e., for any* $\mathbf{x}_2\in\mathcal{L}_2$

$$\partial\alpha(\mathbf{x},t)/\partial\mathbf{x}_2=\partial\alpha(\mathbf{x}_1\oplus\mathbf{x}_2,t)/\partial\mathbf{x}_2=0\qquad(18)$$

then there is a finite-form realization of algorithms (10).

Notice that equality (18) is equivalent to the fact that the plant dynamics can be described by the following equations

$$\begin{aligned}
\dot{\mathbf{x}}_1&=\mathbf{f}_1(\mathbf{x}_1\oplus\mathbf{x}_2)+\mathbf{g}_1(\mathbf{x}_1\oplus\mathbf{x}_2)u\\
\dot{\mathbf{x}}_2&=\mathbf{f}_2(\mathbf{x}_1\oplus\mathbf{x}_2)+\vartheta(\mathbf{x}_1,\theta)+\mathbf{g}_2(\mathbf{x}_1\oplus\mathbf{x}_2)u
\end{aligned}\qquad(19)$$

and $\partial\psi(\mathbf{x}_1\oplus\mathbf{x}_2,t)/\partial\mathbf{x}_2=\lambda(t)$, where $\lambda:R\rightarrow R^{n-m}$ is a known function of time. On the other hand, one can conclude from Corollary 2 that every error model of the type

$$\dot{\psi}=-\varphi(\psi)+z(\omega(t),\theta)-z(\omega(t),\hat{\theta}),\qquad(20)$$

where $\omega(t):R\rightarrow R^n$, $\omega\in C^1$ is a function with known time-derivatives $\dot{\omega}(t)$ admits the sufficient conditions for the algorithms (10) to be realized in the finite form. Indeed, it follows directly from Assumption 5 as functions $\alpha(\mathbf{x},t)$ in this case are independent on $\mathbf{x}$. Therefore if the derivatives $\dot{\alpha(t)}$ are known, then the finite forms follow immediately from

$$\begin{aligned}
\hat{\theta}(\mathbf{x},t)&=\Gamma(\hat{\theta}_P(\mathbf{x},t)+\hat{\theta}_I);\hat{\theta}_P(\mathbf{x},t)=\psi(\mathbf{x},t)\alpha(t)\\
\dot{\hat{\theta}}_I&=\varphi(\psi(\mathbf{x},t)\alpha(\mathbf{x},t)-\psi(\mathbf{x},t)\dot{\alpha}(t)
\end{aligned}$$

This property along with decomposition (19) will be used later for the approximate realizations of algorithms (10).

So far the simplified conditions for existence of the adaptive algorithms in the finite form were derived from Theorem 2 for those classes of nonlinear systems that admit certain structural properties like single dimension uncertainty-dependent partition (Corollary 1 and equation (3) with $m=n-1$), independence of $z(\mathbf{x},\theta,t)$ on uncertainty-dependent partition $\mathbf{x}_2$ (Corollary 2, equation (19) and error model (20)). These structural properties allowed us to reduce Assumption 7 to at most integrability of a function with respect to the single scalar argument. This rather simple test, however, is only sufficient (but not necessary) to establish existence of the adaptive control algorithms with improved transient behavior and abilities to deal with nonconvex parameterization. On the other hand, it is natural to expect that there are classes of systems that can be reduced to the considered cases for which the algorithms in the finite forms are proven to exist. In the next section we present a technique that allows us to transform a nonlinear dynamical system into a form that obeys these sufficient conditions.

3.2 Asymptotic Design via Embedding

Let us introduce the following assumption

Assumption 8 *There exist*

1) partition of the state vector $\mathbf{x}$*:* $\mathbf{x}=\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{x}_2''$, $\dim\mathbf{x}_2'=m_1,\ \dim\mathbf{x}_2''=n-m-m_1,\ 0\leq m_1\leq n-m$

2) system of differential equations

$$\dot{\xi} \;=\; \mathbf{f}_\xi(\mathbf{x},\xi,t);\ \mathbf{y}_\xi=\mathbf{h}_\xi(\xi), \tag{21}$$

$\xi\in R^r,\ \mathbf{f}_\xi: R^n\times R^r\times R_+\rightarrow R^r,\ \mathbf{f}_\xi\in C^1; \mathbf{h}_\xi: R^r\rightarrow R^{n-m-m_1},\ \mathbf{h}_\xi\in C^1$*;*

3) function $\Psi(\tilde{\mathbf{x}},t)\in C^1$ *and partition* $\tilde{\mathbf{x}}=\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{h}_\xi$*, such that:*

$$z(\mathbf{x},\theta,t)-z(\tilde{\mathbf{x}},\theta,t)\in L_2\cap L_\infty; \tag{22}$$

$$\partial\Psi(\tilde{\mathbf{x}},t)/\partial\mathbf{x}_2'=\psi(\tilde{\mathbf{x}},t)\partial\alpha(\tilde{\mathbf{x}},t)/\partial\mathbf{x}_2' \tag{23}$$

for any $\theta\in\Omega_\theta$ *and* $t\in R_+$ *along the solutions of the original system (3).*

The sufficient conditions for the desired embedding follow from the next theorem.

Theorem 3 *Let function* $\psi(\mathbf{x},t)$ *be given and Assumptions 1–6, 8 hold for system (3). Then there exist control function* $u(\mathbf{x},\mathbf{h}_\xi,\hat{\theta},t)$

$$u(\mathbf{x},\mathbf{h}_\xi,\theta,t)=(L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x},t))^{-1}(-\varphi(\psi)-L_{\mathbf{f}}\psi(\mathbf{x},t)- L_{\vartheta(\tilde{\mathbf{x}},\hat{\theta})}\psi(\tilde{\mathbf{x}},t)-\partial\psi(\mathbf{x},t)/\partial t) \tag{24}$$

and adaptation algorithm

$$\begin{aligned}&\hat{\theta}(\tilde{\mathbf{x}},t)=\Gamma(\hat{\theta}_P(\tilde{\mathbf{x}},t)+\hat{\theta}_I(t)),\ \Gamma>0\\&\hat{\theta}_P(\tilde{\mathbf{x}},t)=\psi(\mathbf{x},t)\alpha(\tilde{\mathbf{x}},t)-\Psi(\tilde{\mathbf{x}},t)\\&\dot{\hat{\theta}}_I=\varphi(\psi(\mathbf{x},t))\alpha(\tilde{\mathbf{x}},t)+\partial\Psi(\tilde{\mathbf{x}},t)/\partial t-\partial\alpha(\tilde{\mathbf{x}},t)/\partial t-\\&(\psi(\mathbf{x},t)\partial\alpha(\tilde{\mathbf{x}},t)/\partial\mathbf{x}_1-\partial\Psi(\tilde{\mathbf{x}},t)/\partial\mathbf{x}_1)(\mathbf{f}_1(\mathbf{x})+\\&\mathbf{g}_1(\mathbf{x})u(\mathbf{x},\mathbf{h}_\xi,\hat{\theta},t))-(\psi(\mathbf{x},t)\partial\alpha(\tilde{\mathbf{x}},t)/\partial\mathbf{h}_\xi-\\&\partial\Psi(\tilde{\mathbf{x}},t)/\partial\mathbf{h}_\xi)\partial\mathbf{h}_\xi/\partial\xi\mathbf{f}_\xi(\mathbf{x},\xi,t)\end{aligned} \tag{25}$$

such that for the extended system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x})+\vartheta(\mathbf{x},\theta)+\mathbf{g}(\mathbf{x})u;\ \dot{\xi}=\mathbf{f}_\xi(\mathbf{x},\xi,t)\\ \mathbf{y}_\xi &= \mathbf{h}_\xi(\xi),\end{aligned} \tag{26}$$

the following statements hold:

1) $\psi(\mathbf{x},t),\dot{\psi}\in L_2\cap L_\infty$, $z(\tilde{\mathbf{x}},\theta,t)-z(\tilde{\mathbf{x}},\hat{\theta},t)\in L_2\cap L_\infty$

2) $\psi(\mathbf{x},t)\rightarrow 0$ *as* $t\rightarrow\infty$*.*

Proof of Theorem 3. To prove the theorem it is sufficient to notice that control function (24) provides the following error model dynamics $\dot{\psi}=-\varphi(\psi)+z(\mathbf{x},\theta,t)-z(\tilde{\mathbf{x}},\theta,t)+z(\tilde{\mathbf{x}},\theta,t)-z(\tilde{\mathbf{x}},\hat{\theta},t)$. Further, $z(\mathbf{x},\theta,t)-z(\tilde{\mathbf{x}},\theta,t)\in L_2\cap L_\infty$ by Assumption 8. One also can verify that algorithms (25) satisfy the following differential equations: $\dot{\hat{\theta}}=\Gamma(\dot{\psi}+\varphi(\psi))\alpha(\tilde{\mathbf{x}},t)$. Therefore, according to Theorem 1 we can conclude that $\psi(\mathbf{x},t)\in L_2\cap L_\infty$ and $\dot{\psi}\in L_2$. Due to Assumption 1 we can deduce that $\mathbf{x}$, $\hat{\theta}$ are bounded and furthermore $z(\mathbf{x},\theta,t)-z(\tilde{\mathbf{x}},\theta,t)\in L_2$. Then $z(\tilde{\mathbf{x}},\theta,t)$, $z(\tilde{\mathbf{x}},\hat{\theta},t)$ are bounded. Therefore $\dot{\psi}$ is bounded and $\psi\rightarrow 0$ as $t\rightarrow\infty$. *The theorem is proven.*

Theorem 3 provides us with a way to reduce the complexity of searching for the function $\Psi(\mathbf{x},t)$ defined in Assumption 7. It is suggested to replace the problem of searching for the suitable functions $\Psi(\mathbf{x},t)$ satisfying partial differential equation (13) by that of searching for the embedding (26) which ensures properties (22) and (23). The complexity of finding a solution to equation (13) is reduced as $\dim\mathbf{x}_2'<\dim\mathbf{x}_2$ if the embedding into higher-order dynamics is used. Indeed, according to Assumption 8 and notations introduced above one can describe dynamics of the extended system as follows

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x})+\mathbf{g}_1(\mathbf{x})u;\ \dot{\mathbf{h}}_\xi=(\partial\mathbf{h}_\xi\ /\ \partial\xi)\mathbf{f}_\xi(\mathbf{x},\xi,t)\\ \dot{\mathbf{x}}_{2'} &= \mathbf{f}_{2'}(\mathbf{x})+\vartheta'(\mathbf{x},\theta)+\mathbf{g}_{2'}(\mathbf{x})u\\ \dot{\mathbf{x}}_{2''} &= \mathbf{f}_{2''}(\mathbf{x})+\vartheta''(\mathbf{x},\theta)+\mathbf{g}_{2''}(\mathbf{x})u,\end{aligned} \tag{27}$$

where vector $\mathbf{x}_1\oplus\mathbf{h}_\xi$ stands for uncertainty-independent partition in the extended state space, and vector $\mathbf{x}_2'$ is chosen to satisfy equation (23). Observe, that function $z(\tilde{\mathbf{x}},\theta,t)$ is independent on $\mathbf{x}_2''$ and $\dim\mathbf{h}_\xi=\dim\mathbf{x}_2''$. Then for any $\mathbf{h}_\xi$: $\dim\mathbf{h}_\xi>0$ we can derive that $\dim\mathbf{x}_2'<\dim\mathbf{x}_2=\dim\mathbf{x}_2'\oplus\mathbf{x}_2''$.

Notice also that by the appropriate choice of the dimensions of vectors ξ and $\mathbf{h}_\xi$ ($\dim\mathbf{h}_\xi=\dim\mathbf{x}_2''$) in (21) one can reduce dimension of vector $\mathbf{x}_2'$ to the unity or try to annihilate the partial derivative $\frac{\partial\alpha(\tilde{\mathbf{x}},t)}{\partial\mathbf{x}_2'}$ in (23). Hence, eventually either Corollary 1 or Corollary 2 conditions will be satisfied for the extended system (27). The last in turn implies that we can replace assumption (23) by weaker requirement like integrability of a function with respect to a single scalar argument.

The remaining problem is that, having computable function $\Psi(\tilde{\mathbf{x}},t)$, one should still find an extension (21) that guarantees property (22) for the given partition $\tilde{\mathbf{x}}=\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{h}_\xi$. If such an extension exists, then Assumption 8 is automatically satisfied, and adaptive control algorithms follow immediately from Theorem 3. Nevertheless, finding extension (21) that ensures boundedness and squared integrability of the difference $z(\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{x}_2'',\theta,t)-z(\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{h}_\xi(\xi),\theta,t)$ is not an easy problem (considering that partition $\mathbf{x}_2''$ is uncertainty-dependent). It is possible to solve it by using specially designed *adaptive* or *high-gain* auxiliary subsystems that track the reference signals $\mathbf{x}_2''$ with the desired performance: $z(\mathbf{x},\theta,t)-z(\tilde{\mathbf{x}},\theta,t)\in L_2\cap L_\infty$. For example, if partition $\mathbf{x}_2''$ is linearly parameterized (i.e. $\vartheta''(\mathbf{x},\theta)=\eta''(\mathbf{x})\theta$) and function $z(\mathbf{x},\theta,t)$ is locally bounded in $\mathbf{x}_2''$: $|z(\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{x}_2'',\theta,t)-z(\mathbf{x}_1\oplus\mathbf{x}_2'\oplus\mathbf{h}_\xi(\xi),\theta,t)|\leq\lambda(\mathbf{x},\xi,\theta,t)\|\mathbf{x}_2''-\mathbf{h}_\xi(\xi)\|$, then the suitable extension is defined by the following system

$$\dot{\xi}_1 \;=\; \mathbf{f}_2''(\mathbf{x})+\eta''(\mathbf{x})\xi_2+\bar{\lambda}(\mathbf{x},\xi,t)^2(\mathbf{x}_2''-\xi_1)+\mathbf{g}_2''(\mathbf{x})u$$

$$\dot{\xi}_2 = \Gamma_1(\mathbf{x}_2'' - \xi_1)^T \eta''(\mathbf{x}), \ \Gamma_1 > 0, \ \mathbf{h}_\xi(\xi) = \xi_1,$$

where $\xi = \xi_1 \oplus \xi_2$ and $\bar{\lambda}(\mathbf{x}, \xi, t) = \sup_{\theta \in \Omega_\theta} \lambda(\mathbf{x}, \xi, \theta, t)$. To show this it is sufficient to consider the following Lyapunov's candidate $V(\mathbf{x}, \xi) = 0.5\|(\mathbf{x}_2'' - \xi_1)\|^2 + 0.5\|\theta - \xi_2\|^2_{\Gamma^{-1}}$ and observe that $\dot{V} \leq -\bar{\lambda}^2(\mathbf{x}, \xi, t)^2 \|\mathbf{x}_2'' - \xi_1\|^2 \leq -(z(\mathbf{x}, \theta, t) - z(\tilde{\mathbf{x}}, \theta, t))^2 \leq 0$.

4 Conclusion

In the paper we proposed new method to design adaptive control algorithms with the improved performance for nonlinear systems with linear and nonlinear parameterization of the admissible type. The admissible nonlinearities are those that have linear growth property and are "monotonic" with respect to their parameters. In contrast to the existing adaptive control schemes that start from designing the adaptive control algorithms in the differential form and prohibit any use of the derivatives in the parameter tuning procedures, we first search for the desired augmentation of the error which, if used in the adjustment algorithms in the differential form, may result in non-realizable schemes. Having obtained the desired augmentations which satisfy the given performance measure, we then search for the realizations of these algorithms in the integral-algebraic or finite form. The conditions obtained for explicit realization of the algorithms require to solve partial differential equation (13) for the functions $\psi(\mathbf{x}, t)$ and $\alpha(\mathbf{x}, t)$. To make the method applicable for more broad class of nonlinear systems, we propose to embed the system dynamics into that of the higher order. It is possible to show that this allows us to decrease the dimensionality of (13) in Assumption 7, thereby significantly reducing complexity of the problem. Furthermore, if it is possible to design the extension satisfying (22) for any partition of vector $\mathbf{x}$, then one can sequentially transform original equation (13) for the extended system into (23) which will eventually satisfy assumptions of Corollaries 1 and 2. These equations have been shown to admit the sufficient conditions for the finite-form realization of the adaptive algorithms. Further results will be reported in [19].

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