A NEW WAY FOR ROBUSTNESS ANALYSIS OF NONLINEAR CONTROL SYSTEMS: APPLICATION TO A MAGNETIC SUSPENSION DEVICE

A. Henni and H. Siguerdidjane

Supelec, Plateau de Moulon, 91192 Gif-sur-Yvette, France
fax: +33 1 69 85 13 89
e-mail: assia.henni@supelec.fr, houria.siguerdidjane@supelec.fr

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Abstract

The stability robustness analysis for a class of nonlinear systems with bounded structured uncertainties is characterized by Nonlinear Matrix Inequalities (NLMI). By introducing scaling to reduce conservatism arising from the uncertainty structure, the problem turns out to be still convex. However, it is shown that it is as hard as Lyapunov stability analysis. In this paper, it is proposed a new way for the stability robustness analysis. It consists of splitting the problem into two steps and treating it on the basis of Lyapunov designs.

As application, we have applied the nonlinear backstepping based approach to deal with the problem of stabilizing a magnetic suspension system with bounded structured uncertainties. Numerical simulations as well as experimental results have been performed, they are shown to be satisfactory and are in concordance.

1 Introduction

In this paper, robustness analysis for a class of uncertain nonlinear systems with bounded structured uncertainty is considered. On the basis of $\mathcal{L}_2$-gain analysis and small-gain theorem for nonlinear systems [5,6,7], the problem can be characterized by nonlinear matrix inequalities which are convex. The NLMI conditions involve neither a finite number of unknowns nor finite number of constraints. In [3,6], NLMI computations have been proposed, by using finite differences and finite elements approximations, in order to solve robustness analysis problem. Theses schemes are anyhow useful tools only in the case of low dimensional problems. It is proved that the NLMI characterization for robustness analysis is as hard as Lyapunov stability analysis.

Therefore, we herein propose an easier way to overcome this difficulty. It consists of a reformulation of the stability robustness analysis problem by splitting it into two steps, one for checking a Lyapunov function of the nominal system by employing techniques proposed in [4], and another step, for analyzing the stability robustness of the uncertain nonlinear system by solving an LMI.

In order to illustrate this approach, an application is described. It deals with the stabilization of a magnetic suspension system with bounded structured uncertainties.

This paper is organized as follows: In *Section 2*, some results on $\mathcal{L}_2$-gain analysis for nonlinear system are reviewed. In *Section 3*, NLMI characterization of stability robustness analysis for nonlinear uncertain systems is emphasized and a new approach is proposed to deal with stability robustness analysis in an easier way. In *Section 4*, the model of a magnetic suspension is presented with nonlinear backstepping design controller and the simulation results are provided out for the nominal system and for the one with uncertainties as well. *Section 5* is devoted to the experimental results. In *Section 6*, the proposed approach for stability robustness analysis are applied to the magnetic suspension system with structured uncertainty. Finally, some concluding remarks are given in *Section 7*.

2 The $\mathcal{L}_2$-gain of nonlinear systems and small-gain theorem

Let us consider the following control affine system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathbf{R^n}$ is the state vector, $u \in \mathbf{R^m}$ is the input vector and $y \in \mathbf{R^p}$ is the output vector. It is assumed that : $f, g, h \in C^0$ are vectors or matrices valued functions and $f(0) = 0$, $h(0) = 0$. The system evolves on a convex open subset $\mathbf{X} \subset \mathbf{R^n}$ containing the origin. Thus, $0 \in \mathbf{R^n}$ is the equilibrium of the system with $u = 0$.

Definition 2.1
System (1) with the initial condition $x(0) = 0$ is said to have $\mathcal{L}_2$-gain less than or equal to 1 if

$$\int_0^T \|y(t)\|^2 \, dt \leq \int_0^T \|u(t)\|^2 \, dt \tag{2}$$

for all $T \geq 0$ and $u(t) \in \mathcal{L}_2^e(\mathbf{R}^+)$. Where $\mathcal{L}_2^e(\mathbf{R}^+)$ is the extended space of $\mathcal{L}_2(\mathbf{R}^+)$ which is defined as the set of all vector-valued functions $u(t)$ on $\mathbf{R}^+$ such that $\|u(t)\|_2 := (\int_0^\infty \|u(t)\|^2\, dt)^{1/2} < \infty$.

The following theorem characterizes $\mathcal{L}_2$-gains for a class of nonlinear systems which are asymptotically stable in terms of NLMIs [5,10].

Theorem 2.1
Consider system G given by (1), it is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1 if there exists a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \to \mathbf{R}^+$ such that

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) \\ \frac{1}{2}g^T(x)\frac{\partial^T V}{\partial x}(x) & -I \end{bmatrix} < 0 \quad (3)$$

for all $x \in \mathbf{X}\backslash\{0\}$. The upperscript T denotes the transposition sign.

Note that the condition in theorem (2.1) is affine in $V(x)$ and all solutions form convex sets. This inequality is the so-called **Nonlinear Matrix Inequalities** (NLMIs) [5,6].

3 Robustness analysis for nonlinear uncertain systems

Consider the uncertain nonlinear system described in Fig.1 as a feedback system, where G is the nominal system and has realization similar to system (1). Δ is the uncertainty matrix. It belongs to a bounded-norm structured set:

$$\mathbf{B\Delta} := \{\Delta = diag\{\Delta_1, \Delta_2, ..., \Delta_N\} : \ \Delta \text{ is causal stable and has } \mathcal{L}_2\text{-gain} \leq 1\ \}$$

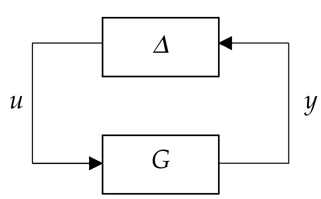

Figure 1: The uncertain system

Definition 3.1

The uncertain system is robustly stable if for each $\Delta \in \mathbf{B\Delta}$, the feedback system is well-posed and is asymptotically stable around the origin.

So, it is assumed that the nominal system G is well-posed and a stronger assumption is made for each uncertainty Δ :

Assumption 3.1
For each $\Delta = diag\{\Delta_1, \Delta_2, ..., \Delta_N\} \in \mathbf{B\Delta}$, Δ_i, $i = 1, 2, ..., N$ has the following realization:

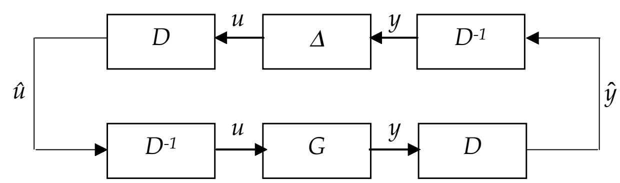

Figure 2: The uncertain system with scaling

$$\begin{cases} \dot{\xi}_i = f_{\Delta_i}(\xi_i) + g_{\Delta_i}(\xi_i).y_i \\ u_i = h_{\Delta_i}(\xi_i) \end{cases} \quad (4)$$

which evolves on $\mathbf{X_i}$ with $f_{\Delta_i}(0) = 0$ and $h_{\Delta_i}(0) = 0$. In addition, there is a $\mathbf{C}^1$ storage function U_i such that

$$\dot{U}_i(\xi_i) \leq \|y_i\|^2 - \|u_i\|^2.$$

In order to reduce possible conservatism arising from the uncertainty structure, scaling manipulation can be used as in figure 2, where D is some real invertible matrix such that $D \in \mathcal{D}$ with the bloc diagonal matrix set

$$\mathcal{D} := \{diag\{d_1 I, d_2 I, ..., d_N I\} : d_i \in \mathbf{R}, d_i > 0\}$$

where each of identity matrix is compatible with the corresponding uncertainty Δ_i. It is noted that for each $D \in \mathcal{D}$, $\Delta \in \mathbf{B\Delta}$ if and only if $D\Delta D^{-1} \in \mathbf{B\Delta}$, since the $\mathcal{L}_2$-gains of Δ and $D\Delta D^{-1}$ are the same.

Therefore, $D\Delta D^{-1}$ is a legal uncertainty structure and satisfies assumption (3.1).

Thus, we consider the scaled system DGD^{-1} instead of G. The obtained uncertain system is the same as the original one represented in figure 1.

We have the following theorem on robustness analysis for structured nonlinear uncertain systems [5,6,7,8].

Theorem 3.1
Consider the uncertain nonlinear system represented in figure 2 where G is the nominal system as in (1) and Δ is the structured uncertainty in the admissible uncertainty set $\mathbf{B\Delta}$ under assumption (3.1).

*The uncertain nonlinear system is robustly stable if there exists a positive definite function $V : \mathbf{X} \to \mathbf{R}^+$ and a positive definite matrix $Q \in \mathcal{D}$ such that the following "***NLMI***" holds*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) \\ \frac{1}{2}g^T(x)\frac{\partial^T V}{\partial x}(x) & -Q \end{bmatrix} < 0 \quad (5)$$

for all $x \in \mathbf{X}\backslash\{0\}$.

For more details, one may find the proof of the theorem in [5].

By Schur complements argument, inequality (5) can be rewritten as follows:

$$\begin{cases} \mathcal{H}(V,Q,x) = \dfrac{\partial V}{\partial x}(x) f(x) + h^T(x) Q h(x) \\ \qquad\qquad + \dfrac{1}{4}\dfrac{\partial V}{\partial x}(x) g(x) Q^{-1} g^T(x) \dfrac{\partial^T V}{\partial x}(x) < 0 \\ Q > 0 \end{cases} \tag{6}$$

The inequality $\mathcal{H}(V,Q,x)$ is called Hamilton-Jacobi inequality. Its solutions also form a convex set but it is not easy to employ it in numerical computation. In [2,6], Huang, Lu and Doyle propose computational issues using finite differences and finite elements approximations. Theses schemes are useful tools in the case of low dimensional problems only. Moreover, the solutions depend on the initial conditions, which is nevertheless not obvious. With the NLMI characterization, it is shown that the NLMI computation for robustness analysis is as hard as Lyapunov stability analysis.

Therefore, we propose herein an easier way. Why not splitting the problem into two steps: in the first step, one may check a Lyapunov function $V(x)$ for the nominal system by using techniques proposed in [4] by Krstic, Kanellakopoulos and Kokotovic, and in the second step, we look for a positive definite matrix Q by solving the LMI $\mathcal{H}(Q,x)$.

Let us apply this suggestion to a magnetic suspension system. A Lyapunov function is determined for the nominal system by backstepping approach.

4 Application to a magnetic suspension device

4.1 System modelling

Consider the magnetic suspension shown in figure 3. It consists of an iron pendulum in a vertical magnetic field created by a single electromagnet. The closed loop block diagram is depicted in figure 4, where z is the measured position of the rotor about the sensor position center in an absolute reference frame and i is the output current signal of the actuator.

The dynamics between the input voltage u of the actuator and its output current i are a first order differential equation. By letting the state vector as $x = [i, z, m\dot{z}]^T$, the modelling of the system is

$$\begin{cases} \dot{x}_1 = (-x_1 + k_\nu u)/\tau \\ \dot{x}_2 = x_3/m \\ \dot{x}_3 = -mg + k\dfrac{x_1^2}{(c - x_2)^2} \end{cases} \tag{7}$$

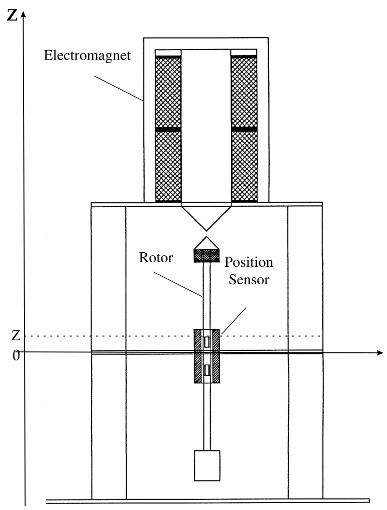

Figure 3: Magnetic suspension device

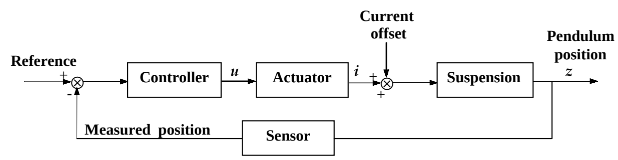

Figure 4: Block diagram of the magnetic suspension system

where m is the mass of the pendulum, τ, k_ν are the time constant and the amplification factor of the actuator respectively, k is a positive constant and c denotes the nominal air gap.

For a given desired constant position of the pendulum x_{2*}, the equilibrium we want to stabilize is

$$x_* = [\sqrt{mg/k}(c - x_{2*}), x_{2*}, 0]^T$$

Magnetic suspension system is an open loop unstable, so then in order to guarantee stable feedback, suitable control is needed. Let us first determine the controller by using a nonlinear backstepping control design that has been developed in [4].

4.2 Backstepping controller design

In nonlinear systems, the backstepping procedure is consisting of finding strictly assignable Control Lyapunov Function (CLF), positive definite and radially unbounded guarantees the global asymptotic stability of the system (see [4,9]).

<u>*Step 1*</u>: Consider the subsystem:

$$\dot{x}_2 = v_1(x_2)/m \tag{8}$$

In order to find the virtual control law $v_1(x_2)$, we introduce the CLF

$$V_1(x_2) = \frac{1}{2}\alpha_1 e^2 \tag{9}$$

where e being the error signal $e = x_2 - x_{2*}$ and α_1 is a positive constant. So, we have

$$\dot{V}_1(x_2) = \alpha_1 e.v_1/m \quad (10)$$

By taking $v_1 = -e$, we strictly assign the CLF $V_1(x_2)$.

Step 2: Now we consider the subsystem:

$$\begin{cases} \dot{x}_2 = x_3/m \\ \dot{x}_3 = -mg + k\dfrac{v_2(x_2,x_3)}{(c-x_2)^2} \end{cases} \quad (11)$$

with $v_2(x_2, x_3)$ as the second virtual control law.
Taking the CLF

$$V_2(x_2,x_3) = V_1(x_2) + \frac{1}{2}\alpha_2(x_3 - v_1(x_2))^2,\ \alpha_2 > 0 \quad (12)$$

As in *step 1*, we strictly assign CLF $V_2(x_2, x_3)$ by setting

$$v_2(x_2,x_3) = \frac{-1}{k}(c-x_2)^2(-mg + \frac{(1+\frac{\alpha_1}{\alpha_2})e + 2x_3}{m}) \quad (13)$$

Step 3: Finally, we consider the whole magnetic suspension system (1).
Introducing the CLF

$$V(x_1,x_2,x_3) = V_2(x_2,x_3) + \frac{1}{2}\alpha_3(x_1^2 - v_2(x_2,x_3))^2,\ \alpha_3 > 0 \quad (14)$$

and similarly as made in the preceding steps, we can get the control input u that renders CLF $V(x_1, x_2, x_3)$ strictly assignable, thus achieving the global asymptotic stability of the magnetic suspension system. The control law is found to be having the following expression :

$$\begin{aligned} u = {} & h_0 x_3 + (c-x_2)^2 h_1(-2mg + h_2 x_1) \\ & + h_3 \frac{x_1 x_3}{(c-x_2)} \\ & + h_4 (c-x_2)^2 (h_5 - h_6 x_3 - h_7(x_2 - x_{ref})) \end{aligned} \quad (15)$$

where the h_i's are constant.

Two parametric uncertainties are considered, one on the pendulum mass m and the other one on the actuator constant k. Thus, structured matrix uncertainty is obtained as shown below:

$$\Delta = \begin{bmatrix} \delta_m & 0 \\ 0 & \delta_k \end{bmatrix} \quad (16)$$

4.3 Simulation results

The required performance are commonly expressed in terms of the overshoot $(< 20\%)$, the settling time $(< 1s)$ and the tracking error accuracy. First, numerical simulations have been performed for both nominal system and under uncertainties.

Figure 5 exhibits the set point position z (a), the input voltage u (b). It may be observed that concerning the position the overshoot , the settling time and the accuracy are within the required performance. The control input lies within the experimental device constraints, namely $(\pm 10V)$.

The control law for the nominal system does not guarantee the global asymptotic stability in the presence of parametric uncertainties, $i.e.$ there is a steady state error compared to the equilibrium we want to stabilize. In order to “robustify ” the backstepping control design, we propose to redesign the controller by adding a PI regulator in parallel with the nonlinear controller.

The parameters of the PI controller have been determined based on the local linearization of magnetic suspension system (7) around the operating point $z = 0$, such that the integral action operates far from the desired band-width of the system in order to do not jeopardize the performance of stabilization brought by the nonlinear backstepping controller.

In doing so, the simulation results are shown to be satisfying the required performance in the case of parametric uncertainties $\delta_m = 10\%$ and $\delta_k = 10\%$ in $(Fig.6)$. The control input still remains within the required voltage bounds.

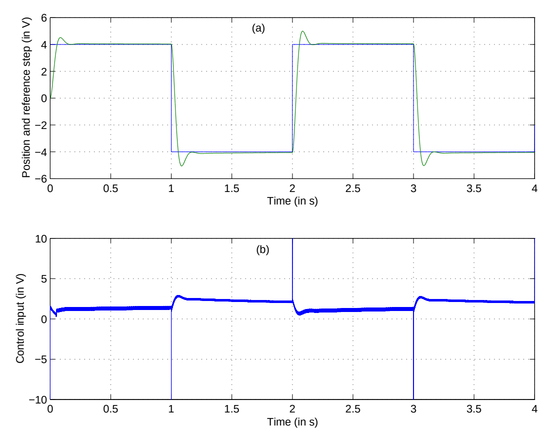

Figure 5: Simulation results with nominal parameters.

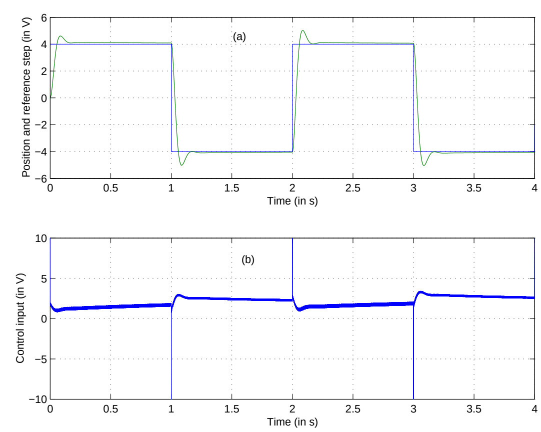

Figure 6: Simulation results with parametric uncertainties $\delta_m = 10\%$ and $\delta_k = 10\%$

5 Experimental results

5.1 Platform description

The experimental bench available in our laboratory $(fig.7)$ is composed of a magnetic suspension unit and a personal computer with an interface.

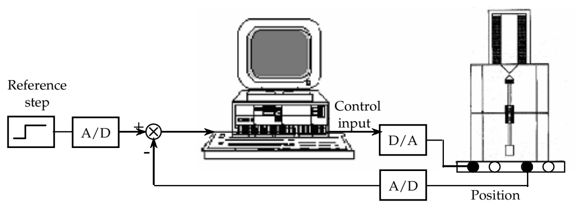

Figure 7: Experimental bench.

The magnetic suspension device is made of an actuator, a rotor and a sensor position $(Fig.3)$. The actuator consists of a coil and a corresponding voltage controlled current source. Thus, the electromagnet and the controller current are considered as a unit. The later is assumed to be a first order lag where the parameters τ and k_ν of system (1) result from an experimental identification. The rotor axis is in line with the acceleration due to the gravity, it is subject to the force exerted by the actuator, thus causing a displacement of the pendulum. The sensor position is based on the differential transformer principle. Sensors of this type offer a good linearity as well as an infinite resolution. Moreover, they are robust with respect to electrical disturbances. The main characteristics of the actuator and the rotor are:

$$\begin{array}{ccc} m = 0.0844kg & k = 0.005 & g = 9.81m/s^2 \\ \tau = 1ms & k_\nu = .1 & c = 0.011m \end{array}$$

Concerning the interface, it contains three A/D converters inputs $(12\ bits)$ and two D/A converters outputs $(12\ bits)$.

5.2 Experimental results

In order to implement the nonlinear backstepping controller we have estimated the velocity and the current of the actuator by using a linear observer. The dynamics of the observer are chosen faster than the magnetic suspension dynamics. The initial state condition is $x(0) = [x_{1*}, 0, 0]^T$.

The experimental results are shown to be satisfactory with respect to the nominal mass and to the considered uncertainties. Hereafter, are some results corresponding to system performance with nominal parameters $(Fig.8)$ and with parametric uncertainties $\Delta m = 10\%$ and $\Delta k = 10\%$ $(Fig.9)$. One may observe that the desired performances are met. For more results, the reader may see [11].

Bear in mind that our aim is to solve the robustness analysis problem (6). This problem is more significant for industrial applications when tests are particularly quite expensive.

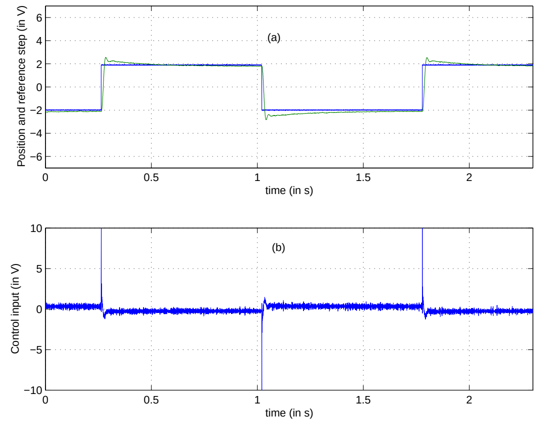

Figure 8: Experimental results with nominal parameters.

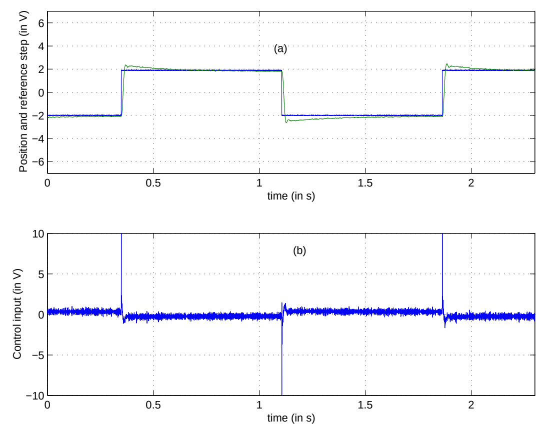

Figure 9: Experimental results with parametric uncertainties $\delta_m = 10\%$ and $\delta_k = 10\%$.

6 Robustness analysis for the uncertain system

The parametric uncertainties form a structured matrix uncertainty as follows:

$$\Delta = \begin{bmatrix} \delta_m & 0 \\ 0 & \delta_k \end{bmatrix} \tag{17}$$

By standard manipulation, the structured uncertain system may be represented as shown in figure 10, where G is the nominal system interconnected with the controller K, y is the measured output, u the control input, z and v are the input and output of the system uncertainty respectively. Finally, d is a perturbation depending on the pendulum mass parametric uncertainty. The conservatism may be reduced by introducing scaling as is described in *Section 3*.

By applying the nonlinear backstepping design, the Lyapunov function of the nominal system is derived from the procedure of *Section 4.2*.

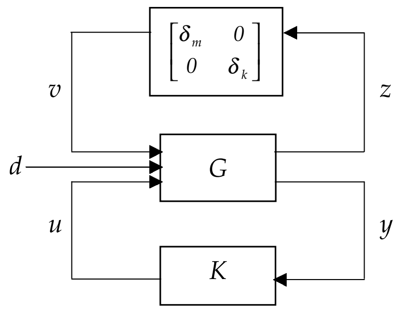

Figure 10: The structured uncertain system

Thus, the new problem on robustness analysis for the feedback system depicted in Figure 10 consists of solving LMI (6), where

$$Q = \begin{bmatrix} q_m & 0 \\ 0 & q_k \end{bmatrix} \tag{18}$$

By using the LMI control toolbox (see [1, 2]), the following optimal matrix Q is obtained:

$$Q = \begin{bmatrix} 1.4764 & 0 \\ 0 & 1.4572 \end{bmatrix} \tag{19}$$

Consequently, since Q is positive definite matrix we may conclude that the nonlinear uncertain system is robustly stable, this is validated by robustness tests carried out on the experimental bench.

7 Conclusion

Robustness analysis for nonlinear systems with bounded structured uncertainties consists of solving the Hamilton-Jacobi inequality which can be characterized by NLMI. With this characterization, it is shown that the NLMI computation for robustness analysis is as hard as Lyapunov stability analysis.

Therefore, in order to over pass this difficulty we have proposed an easier way that split the problem into two steps. The first one checks a Lyapunov function for the nominal system, and the other one checks the positive definite matrix of the theorem statement by solving an LMI.

As application, we have applied the nonlinear backstepping based approach to deal with the problem of stabilizing a magnetic suspension system and then to get a Lyapunov function for the nominal system. The obtained experimental results are satisfactory and are in concordance with the theoretical ones, the required performances are achieved.

References

[1] S.P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. "Linear Matrix Inequalities in System and Control Theory", *SIAM*, (1994).

[2] P. Gahinet, A. Arkadi, A.J. Laub and M. Chilali "LMI Control Toolbox", *The Mathworks Inc.*, The Mathworks Partner series, (1995).

[3] D.J. Huang and W.M. Lu. "Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation", *Proceedings of IEEE Conference on Decision and Control*, Kroben, Japan, (1996).

[4] M. Krstic, I. Kanellakopoulos and P. V. Kokotovic. "Nonlinear and Adaptive Control Design", *Wiley*, New York, (1995).

[5] W.M. Lu, and J.C. Doyle . "Robustness analysis and synthesis for nonlinear uncertain systems", *Proceedings of the 33rd Conference on Decision and Control*, Lake Buena vista, FL , pp. 787-792, (1994).

[6] W.M. Lu, and J.C. Doyle. "Robustness analysis and synthesis for nonlinear uncertain systems", *IEEE Transactions on Automatic Control*, **42(42)**, pp. 1654–1662, (1997).

[7] W.M. Lu, and J.C. Doyle. "$\mathcal{H}_\infty$ Control of nonlinear systems: A convex characterization", *Proceedings of the American Control Conference*, Baltimore, Maryland , pp. 2098–2102, (1994).

[8] W.M. Lu, and J.C. Doyle. "$\mathcal{H}_\infty$ Control of nonlinear systems via output feedback: A class of controllers", *Proceedings of the 32nd Conference on Decision and Control*, San Antonio, Texas , pp. 166–171, (1993).

[9] L. Praly. "Survey on Lyapunov designs of stabilizing state and output feedback", *Lecture notes*, (1996).

[10] A. J. van der Schaft. "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control", *IEEE Transactions on Automatic Control*, **37(6)**, June(1992).

[11] A. Henni and H. Siguerdidjane. "Robust Nonlinear Control of a Magnetic Suspension System", *Proceedings of the IEEE Conference on Control Applications*, Istanbul, Turkey, June(2003).