A DESCRIPTOR APPROACH TO SLIDING MODE CONTROL OF SYSTEMS WITH TIME-VARYING DELAYS

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Abstract

In this note we combine a descriptor approach to stability and control of linear systems with time-varying delays, which is based on the Lyapunov - Krasovskii techniques, with a recent result on sliding mode control of such systems. The systems under consideration have norm-bounded uncertainties and uncertain bounded delays. The solution is given in terms of linear matrix inequalities (LMIs) and improves the previous results, based on other Lyapunov techniques. A numerical example illustrates the advantages of the new method.

1 Introduction

The problem of reducing the conservatism entailed in applying finite dimensional techniques to assess the stability of linear systems with time delay has attracted much attention in the past few years [1]-[15]. All these techniques, delay-independent and less conservative delay-dependent, provide sufficient conditions only for the asymptotic stability of these systems and they entail a considerable conservatism. Delay-dependent stability conditions in terms of LMIs have been obtained for retarded and neutral type systems. These conditions are based on three main model transformations of the original system (see [10]).

Recently a new *descriptor* model transformation was introduced for delay-dependent stability of neutral systems [2] and of a more general class of differential and algebraic (descriptor) system with delay ([3]). Unlike previous transformations, the descriptor model leads to a system which is equivalent to the original one (from the point of view of stability) and requires bounding of fewer cross-terms.

Two main approaches for dealing with time-varying delays have been suggested in the past. The first is based on Lyapunov-Krasovskii functionals and the second is based on Razumikhin theory. Two main cases of time-varying delays have been considered:
1) differentiable uniformly bounded delays with derivatives upper bounded by one, and
2) continuous uniformly bounded delays.

To the best of our knowledge, the Razumikhin's approach was the only one that was able to cope with the second case, which allows fast time-varying delays. This method was applied in [8] for sliding mode design. The method introduced in [5] based on Lyapunov-Krasovskii functional via the descriptor model transformation seems to be the first of this type for the second case.

Sliding mode control is an efficient solution to many practical issues, such as robotics or control of induction motors, because it is insensitive to a wide class of disturbances and uncertainties. Design of a sliding mode controller for systems with delay was introduced in [8], where the case of time-varying delay was treated by Razumikhin approach, while Lyapunov-Krasovskii method via neutral model transformation was applied to the case of constant delay. It is the purpose of the present note to combine the sliding mode control of [8] with the descriptor model transformation and Lyapunov-Krasovskii technique of [2], [5] in order to design a more efficient sliding mode controller for uncertain systems with time-varying delays and norm-bounded uncertainties.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. I_n represents the $n \times n$ identity matrix.

2 Stabilization of linear systems with norm-bounded uncertainties by delayed feedback

In this section we consider the following uncertain linear system with a time-varying delay:

$$\begin{aligned}\dot{x}(t) \;\; = \;\; & (A_0 + H\Delta(t)E_0)x(t) \\ & +(A_1 + H\Delta(t)E_1)x(t-\tau(t)) \\ & +(B_0 + H\Delta(t)E_2)u(t) + B_1 u(t-\tau(t)), \quad (1)\end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $u(t) \in \mathcal{R}^m$ is the control input, h is an upper-bound on the time-delay function ($0 \le \tau(t) \le h, \forall t \ge 0$). The matrix $\Delta(t) \in \mathcal{R}^{p\times q}$ is a matrix of time-varying, uncertain parameters satisfying

$$\Delta^T(t)\Delta(t) \le I_q \ \ \forall\, t. \qquad (2)$$

For simplicity, we took only one delay, but the results may be easily generalized to the case of multiple delays.

We seek a control law

$$u(t) = Kx(t) \qquad (3)$$

that will asymptotically stabilize the system.

2.1 The stability issue

In this subsection, we consider the following equation:

$$\begin{aligned} \dot{x}(t) &= (\bar{A}_0 + H\Delta(t)\bar{E}_0)x(t) \\ &\quad +(\bar{A}_1 + H\Delta(t)\bar{E}_1)x(t-\tau(t)). \end{aligned} \qquad (4)$$

Representing (4) in an equivalent (from the point of view of stability) descriptor form [2]:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ 0 &= -y(t) + (\bar{A}_T + H\Delta\bar{E}_T)x(t) \\ &\quad -(\bar{A}_1 + H\Delta\bar{E}_1)\int_{t-\tau(t)}^{t} y(s)ds, \end{aligned}$$

where

$$\bar{A}_T = \bar{A}_0 + \bar{A}_1, \quad \bar{E}_T = \bar{E}_0 + \bar{E}_1,$$

or

$$\begin{aligned} E\dot{\bar{x}}(t) &= \begin{bmatrix} 0 & I_n \\ \bar{A}_T + H\Delta\bar{E}_T & -I_n \end{bmatrix}\bar{x}(t) \\ &\quad -\begin{bmatrix} 0 \\ \bar{A}_1 + H\Delta\bar{E}_1 \end{bmatrix}\int_{t-\tau(t)}^{t} y(s)ds, \end{aligned} \qquad (5)$$

with $\bar{x}(t) = col\{x(t),\ y(t)\},\ E = diag\{I_n, 0\}$, the following Lyapunov-Krasovskii functional is applied:

$$V(t) = \bar{x}^T(t)EP\bar{x}(t) + V_2(t), \qquad (6)$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 > 0, \quad EP = P^TE \ge 0, \qquad \text{(7a-d)}$$

$$V_2(t) = \int_{-h}^{0}\int_{t+\theta}^{t} y^T(s)[R + \delta_2\bar{E}_1^T\bar{E}_1]y(s)dsd\theta.$$

The following result is obtained:

Lemma 1 *The system (4) is asymptotically stable if there exist $n\times n$ matrices $0 < P_1,\ P_2,\ P_3,\ R > 0$ and positive numbers δ_1, δ_2 that satisfy the following LMI:*

$$\Gamma = \begin{bmatrix} \Psi & hP^T\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} & P^T\begin{bmatrix} 0 \\ H \end{bmatrix} & hP^T\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hR & 0 & 0 \\ * & * & -\delta_1 I_p & 0 \\ * & * & * & -\delta_2 hI_p \end{bmatrix} < 0 \qquad (8)$$

where

$$\Psi = \Psi_0 + \begin{bmatrix} \delta_1\bar{E}_T^T\bar{E}_T & 0 \\ 0 & h(R + \delta_2\bar{E}_1^T\bar{E}_1) \end{bmatrix},$$

$$\Psi_0 = P^T\begin{bmatrix} 0 & I_n \\ \bar{A}_T & -I_n \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ \bar{A}_T & -I_n \end{bmatrix}^T P,$$

and $*$ denotes symmetrical entries.

Proof. Note that

$$\bar{x}^T(t)EP\bar{x}(t) = x^T(t)P_1x(t)$$

and, hence, differentiating the first term of (6) with respect to t gives:

$$\frac{d}{dt}\{\bar{x}^T(t)EP\bar{x}(t)\} = 2x^T(t)P_1\dot{x}(t) = 2\bar{x}^T(t)P^T\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}.$$

Replacing $\left[\dot{x}^T(t)\ 0\right]^T$ by the right side of (5) we obtain:

$$\begin{aligned} \frac{dV(t)}{dt} &= \bar{x}^T(t)\Psi_0\bar{x}(t) + \eta_0 + \eta_1 + \eta_2 \\ &\quad +hy^T(t)[R + \delta_2\bar{E}_1^T\bar{E}_1]y(t) \\ &\quad -\int_{t-h}^{t} y^T(s)[R + \delta_2\bar{E}_1^T\bar{E}_1]y(s)ds, \end{aligned} \qquad (9)$$

where

$$\begin{aligned} \eta_0(t) &\triangleq -2\int_{t-\tau(t)}^{t}\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix}y(s)ds, \\ \eta_1(t) &\triangleq 2\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ H \end{bmatrix}\Delta(\bar{E}_0 + \bar{E}_1)x(t), \\ \eta_2(t) &\triangleq -2\int_{t-\tau(t)}^{t}\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ H \end{bmatrix}\Delta\bar{E}_1y(s)ds. \end{aligned}$$

Applying the standard bounding

$$a^Tb \le a^TRa + b^TR^{-1}b, \quad \forall a, b \in \mathcal{R}^n, \forall R \in \mathcal{R}^{n\times n} : R > 0,$$

and using the fact that $\tau(t) \le h$, we have

$$\begin{aligned} \eta_0(t) &\le h\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix}R^{-1}[0\ \bar{A}_1^T]P\bar{x}(t) \\ &\quad +\int_{t-h}^{t} y^T(s)Ry(s)ds. \end{aligned}$$

Similarly

$$\begin{aligned} \eta_1 &\le \delta_1^{-1}\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ H \end{bmatrix}[0\ H^T]P\bar{x}(t) \\ &\quad +\delta_1 x^T(t)\bar{E}_T^T\bar{E}_Tx(t), \end{aligned}$$

$$\begin{aligned} \eta_2 &\le h\delta_2^{-1}\bar{x}^T(t)P^T\begin{bmatrix} 0 \\ H \end{bmatrix}[0\ H^T]P\bar{x}(t) \\ &\quad +\delta_2\int_{t-h}^{t} y^T(s)\bar{E}_1^T\bar{E}_1y(s)ds. \end{aligned}$$

Substituting the right sides of the latter inequalities into (9), we obtain

$$\frac{dV(t)}{dt} \le \bar{x}^T(t)\bar{\Gamma}\bar{x}(t) \qquad (10)$$

where

$$\begin{aligned}\bar{\Gamma} &= \Psi + hP^T \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} R^{-1}[0\ \bar{A}_1^T]P \\ &\quad +(\delta_1^{-1} + h\delta_2^{-1})P^T \begin{bmatrix} 0 \\ H \end{bmatrix} [0\ H^T]P.\end{aligned}$$

Therefore, LMI (8) yields by Schur complements that $\bar{\Gamma} < 0$ and hence $\dot{V} < 0$, while $V \ge 0$, and thus (4) is asymptotically stable [12], [3]. ♣

2.2 State-feedback stabilization

The results of Lemma 1 can also be used to verify the stability of the closed-loop system (3)-(4) if we set in (8)

$$\bar{A}_i = A_i + B_i K,\ i = 0,1, \quad \bar{E}_0 = E_0 + E_2 K \qquad (11)$$

and verify that the resulting LMI is feasible. The problem with (8) is that it is linear in its variables only when the state-feedback gain K is given. In order to find K we apply again Schur formula to $\bar{\Gamma}$, the Ψ term being expanded. We thus obtain the following matrix inequality:

$$\begin{bmatrix} \Psi_0 & hP^T\begin{bmatrix} 0 \\ \bar{A}_1 R^{-1}\end{bmatrix} & \begin{bmatrix} 0 \\ hI_n \end{bmatrix} & \begin{bmatrix} \bar{E}_T^T \\ 0 \end{bmatrix} & h\begin{bmatrix} 0 \\ \bar{E}_1^T \end{bmatrix} & \delta_1^{-1}P^T\begin{bmatrix} 0 \\ H \end{bmatrix} & \delta_2^{-1}hP^T\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hR^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & -hR^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & -\delta_1^{-1}I_q & 0 & 0 & 0 \\ * & * & * & * & -\delta_2^{-1}hI_q & 0 & 0 \\ * & * & * & * & * & -\delta_1^{-1}I_p & 0 \\ * & * & * & * & * & * & -\delta_2^{-1}hI \end{bmatrix} < 0 \qquad (12)$$

Consider the inverse of P. It is obvious, from the requirement $P_1 > 0$ and the fact that in (8) $-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad M = diag\{Q,\ I_{2(n+p+q)}\} \qquad \text{(13a-b)}$$

we multiply (12) by M^T and M, on the left and on the right, respectively. Choosing

$$R^{-1} = Q_1\varepsilon,$$

where ε is a positive number, and introducing $\bar{\delta}_1 = \delta_1^{-1}$ and $\bar{\delta}_2 = \delta_2^{-1}$, we obtain the LMI

$$\begin{bmatrix} \Phi & h\begin{bmatrix} 0 \\ \bar{A}_1 Q_1\varepsilon \end{bmatrix} & Q^T\begin{bmatrix} 0 \\ hI_n \end{bmatrix} & Q^T\begin{bmatrix} \bar{E}_T^T \\ 0 \end{bmatrix} & hQ^T\begin{bmatrix} 0 \\ \bar{E}_1^T \end{bmatrix} & \bar{\delta}_1\begin{bmatrix} 0 \\ H \end{bmatrix} & h\bar{\delta}_2\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hQ_1\varepsilon & 0 & 0 & 0 & 0 & 0 \\ * & * & -hQ_1\varepsilon & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\delta}_1 I_q & 0 & 0 & 0 \\ * & * & * & * & -h\bar{\delta}_2 I_q & 0 & 0 \\ * & * & * & * & * & -\bar{\delta}_1 I_p & 0 \\ * & * & * & * & * & * & -\bar{\delta}_2 hI_p \end{bmatrix} < 0 \qquad (14)$$

where

$$\Phi = \begin{bmatrix} 0 & I_n \\ \bar{A}_T & -I_n \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & I_n \\ \bar{A}_T & -I_n \end{bmatrix}^T.$$

Substituting (11) into (14) and denoting $Y = KQ_1$, we obtain

Theorem 1 *The control law of (3) asymptotically stabilizes (1) if, for some positive number ε, there exist positive numbers $\bar{\delta}_1$,$\bar{\delta}_2$ and matrices $0 < Q_1,\ Q_2,\ Q_3 \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{m\times n}$ that satisfy the following LMI:*

$$\begin{bmatrix} Q_2+Q_2^T & Z_{12} & 0 & hQ_2^T & Z_{51} & hQ_2^T E_1^T & 0 & 0 \\ * & -Q_3-Q_3^T & Z_{31} & hQ_3^T & 0 & hQ_3^T E_1^T & \bar{\delta}_1 H & h\bar{\delta}_2 H \\ * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\delta}_1 I_q & 0 & 0 & 0 \\ * & * & * & * & * & -h\bar{\delta}_2 I_q & 0 & 0 \\ * & * & * & * & * & * & -\bar{\delta}_1 I_p & 0 \\ * & * & * & * & * & * & * & -\bar{\delta}_2 hI_p \end{bmatrix} < 0, \qquad (15)$$

where

$$\begin{aligned} B_T &= B_0 + B_1, \\ Z_{12} &= Q_1 A_T^T + Y^T B_T^T - Q_2^T + Q_3, \\ Z_{31} &= h\varepsilon(A_1 Q_1 + B_1 Y), \\ Z_{51} &= Q_1 E_T^T + Y^T E_2^T. \end{aligned}$$

The state-feedback gain is then given by

$$K = YQ_1^{-1}. \qquad (16)$$

3 Sliding mode controller

In this section, we focus on time-delay systems that can be represented, possibly, after a change of state coordinates and input, in the following regular form ([8],[16]):

$$
\begin{aligned}
\frac{dz_1(t)}{dt} &= (A_{11} + H\Delta(t)E_0)z_1(t) \\
&\quad + (A_{d11} + H\Delta(t)E_1)z_1(t-\tau(t)) \\
&\quad + (A_{12} + H\Delta(t)E_2)z_2(t) + A_{d12}z_2(t-\tau(t)) \\
\frac{dz_2(t)}{dt} &= \sum_{i=1}^{2}(A_{2i}z_i(t) + A_{d2i}z_i(t-\tau)) \\
&\quad + Du(t) + f(t, z_t),
\end{aligned} \tag{17}
$$

where $z(t) = (z_1, z_2)^T$, $z_1 \in \mathcal{R}^{n-m}$, $z_2 \in \mathcal{R}^m$, $A_{ij}, A_{dij}, i = 1,2,\ j = 1,2,\ E_k,\ k = 0,1,2, H$ are constant matrices of appropriate dimensions, D is a regular $m \times m$ matrix, $\Delta(t) \in \mathcal{R}^{p\times q}$ is a time-varying matrix of uncertain parameters, $u \in \mathcal{R}^m$ is the input vector, τ is a time-varying delay satisfying $0 \le \tau(t) \le h, \forall t \ge 0$, $z_t(\theta)$ is the function associated with z and defined on $[-h, 0]$ by $z_t(\theta) = z(t+\theta)$.

We will assume that:
A1) $(A_{11} + A_{d11}, A_{12} + A_{d12})$ is controllable.
A2) f is Lipschitz continuous and satisfies the inequality

$$\|f(t, z_t)\| < \psi(t, z_t), \quad \forall t \ge 0,$$

where $\psi(t, z_t)$ is a continuous functional assumed to be known *a priori*,
A3) $\Delta(t)$ is a time-varying matrix of uncertain parameters satisfying $\Delta^T(t)\Delta(t) \le I_q\ \ \forall\, t$.

Consider the following switching function:

$$s(z) = z_2 - Kz_1 \tag{18}$$

with $K \in \mathcal{R}^{m\times(n-m)}$. Let Ω and Θ be the linear functions defined by

$$
\begin{aligned}
\Omega(z(t)) &= \sum_{i=1}^{2}(A_{2i} - KA_{1i})z_i(t), \\
\Theta(z(t)) &= E_0 z_1(t) + E_2 z_2(t)
\end{aligned} \tag{19}
$$

and let D_M be the following functional:

$$
\begin{aligned}
D_M(z_t) &= (\|A_{d21} - KA_{d11}\| + \|KH\|\,\|E_1\|)\bar{z}_1(t) \\
&\quad + \|A_{d22} - KA_{d12}\|\,\bar{z}_2(t) \\
&\quad + (\psi(t, z_t) + \|KH\|\,\|\Theta(z(t))\| + M,
\end{aligned} \tag{20}
$$

where $M > 0$, $\bar{z}_i(t) = \sup\limits_{-h\le\theta\le 0} \|z_i(t+\theta)\|$.

Following [8] and using the results of previous section, we are able to design a sliding mode controller that will stabilize system (17) under less conservative assumptions on the delay law.

Theorem 2 *Assume A1-A3. If, for some positive number ε, there exist positive numbers $\bar{\delta}_1, \bar{\delta}_2$ and matrices $0 < Q_1,\ Q_2,$ $Q_3 \in \mathcal{R}^{(n-m)\times(n-m)},\ Y \in \mathcal{R}^{m\times(n-m)}$ that satisfy the following LMI:*

$$
\left[\begin{array}{cccccccc}
Q_2{+}Q_2^T & X_{12} & 0 & hQ_2^T & X_{51} & hQ_2^T E_1^T & 0 & 0 \\
* & -Q_3{-}Q_3^T & X_{23} & hQ_3^T & 0 & hQ_3^T E_1^T & \bar{\delta}_1 H & h\bar{\delta}_2 H \\
* & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\bar{\delta}_1 I_q & 0 & 0 & 0 \\
* & * & * & * & * & -h\bar{\delta}_2 I_q & 0 & 0 \\
* & * & * & * & * & * & -\bar{\delta}_1 I_p & 0 \\
* & * & * & * & * & * & * & -\bar{\delta}_2 h I_p
\end{array}\right] < 0, \tag{21}
$$

where

$$X_{12} = Q_1(A_{11}^T + A_{d11}^T) + Y^T(A_{12}^T + A_{d12}^T) - Q_2^T + Q_3,$$
$$X_{23} = h\varepsilon(A_{d11}Q_1 {+} A_{d12}Y)$$
$$X_{51} = Q_1 E_T^T {+} Y^T E_2^T$$

then the sliding mode control law

$$u(t) = -D^{-1}\left[\Omega(z(t)) + D_M(z_t)\frac{s(z(t))}{\|s(z(t))\|}\right], \tag{22}$$

where $K = YQ_1^{-1}$, and s, Ω, D_M are defined in (18)-(20), asymptotically stabilizes zero solution of system (17) for any delay function $\tau(t) \le h$.

Proof : The proof is divided into two parts. The first one is dedicated to the proof of the existence of an ideal sliding motion on the surface $s(z) = 0$, the second part to the proof of the stability of the reduced system.

Attractivity of the manifold:

Consider the Lyapunov-Krasovskii functional

$$V(t) = s^T(z(t))s(z(t)) = \|s(z(t))\|^2. \tag{23}$$

Differentiating (23) on the trajectories of the closed-loop system gives

$$
\begin{aligned}
\dot{V}(t) &= 2s^T(t)(\Omega(z(t)) + \sum_{i=1}^{2}[A_{d2i} - KA_{d1i}]\, z_i(t-\tau) \\
&\quad + Du(t) + f(t, z_t) - KH\Delta(t)[\Theta(z(t)) \\
&\quad + E_1 z_1(t-\tau(t))]),
\end{aligned}
$$

Using the expression of the control law (22), we get

$$
\begin{aligned}
\dot{V}(t) &= 2s^T(t)(\sum_{i=1}^{2}(A_{d2i} - KA_{d1i})z_i(t-\tau) \\
&\quad + f(t, z_t) - KH\Delta(t)[\Theta(z(t)) + E_1 z_1(t-\tau(t))] \\
&\quad - D_M(z_t)\frac{s}{\|s\|})
\end{aligned}
$$

then we derive that:

$$\dot{V} \leq -2M\,\|s(z(t)\| = -2MV(t)^{\frac{1}{2}}.$$

This last inequality is known to prove the finite-time convergence of the system (17) into the surface $s = 0$ ([16]).

Stability of the reduced system:

On the sliding manifold $s(z) = 0$, the system is driven by the following reduced system:

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} + A_{12}K + H\Delta(t)(E_0 + E_2K))z_1(t)\\ &\quad +(A_{d11} + A_{d12}K + H\Delta(t)E_1)z_1(t-\tau(t))\end{aligned}$$

According to Theorem 1, this system is asymptotically stable for any delay law $\tau(t) \leq h$ if, for some positive number ε, there exist positive numbers $\bar{\delta}_1, \bar{\delta}_2$ and matrices $0 < Q_1,\ Q_2,\ Q_3,\ Y \in \mathcal{R}^{m\times(n-m)}$ that satisfy the LMI (21). ♣

Remark 1 *Note that the explicit knowledge of the time-dependance of the delay is not required in the expression of the control law $u(t)$, all is needed is the knowledge of an upper bound h.*

4 Example

We demonstrate the applicability of the above theory by solving the example from [8] for a system without uncertainty. Consider system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B[u(t) + f(x,t)], \qquad (24)$$

with a time-varying delay, where

$$A = \begin{bmatrix} 2 & 0 \\ 1.75 & 0.25 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.25 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (25)$$

By an appropriate change of variables, this system is equivalent to:

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{A}_d z(t-\tau) + \tilde{B}[u(t) + f(x,t)],$$

where

$$\tilde{A} = \begin{bmatrix} 0.25 & 0 \\ 1.75 & 2 \end{bmatrix}, \tilde{A}_d = \begin{bmatrix} -0.9 & -0.65 \\ -0.1 & -0.35 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is not controllable, the system cannot be stabilized independently of the delay.

For this system, previous published works give the following results:

— In the case of a constant delay and $f = 0$, the system is proved to be stabilized using a linear memoryless controller $u(t) = Kx(t)$ for the following maximum values of h: $h = 0.51$ by [13], $h = 0.984$ by [7] and $h = 1.46$ by [9]. By sliding mode control for the case of constant delay and $f \neq 0$ by [8] the maximum value of $h = 1.65$.

— Applying Theorem 2 in the case of a time-varying delay and $f \neq 0$, the corresponding value of $h = 3.999$ is achieved.

This is summarized in Table 1.

	delay upper bound	type of delay
Theorem 2	3.999	time-varying
Gouaisbaut et al [8]	1.65	constant
Ivanescu et al [9]	1.46	constant
Fu et al [7]	0.984	constant
Li and De Souza[13]	0.51	constant

Table 1: Results for example (23)-(24)

5 Conclusions

The problem of finding a sliding mode controller that asymptotically stabilizes a system with time-varying delay and norm-bounded uncertainty has been solved. A delay-dependent solution has been derived using a special Lyapunov-Krasovskii functional. The result is based on a sufficient condition and it thus entails an overdesign. This overdesign is considerably reduced due to the fact that the method is based on the descriptor representation. As a byproduct for the first time on the basis of the descriptor model transformation the solution to the stabilization problem by the feedback, which depends on both, non-delayed and delayed state is solved. Finally, some numerical examples show the effectiveness of the combined method: sliding mode and descriptor representation.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequality in Systems and Control Theory,* SIAM Frontier Series, 1994

[2] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems", *Systems & Control Letters*, vol. **43**, pp. 309–319, 2001.

[3] E. Fridman, "Stability of Linear Descriptor Systems with Delay: A Lyapunov-Based Approach", *J. Math. Anal. Appl.*, **273**, No 1, pp. 24-44, 2002.

[4] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems", *IEEE Trans. on Automat. Contr.*, **47**, No 2, pp. 253-270, 2002.

[5] E. Fridman and U. Shaked, "Delay dependent stability and H_∞ control l: Constant and time-varying delays", *Int. J. of Control*, 76, No. 1, 48-60, 2003.

[6] E. Fridman and U. Shaked, "An improved stabilization method for linear time-delay systems", *IEEE Trans. on Automat. Contr.*, vol. **47**, No 11, 1931-1937, 2002.

[7] M. Fu, H. Li and S.I. Niculescu, "Robust Stability and Stabilisation of Time-Delay System Via Integral Quadratic Constraint Approach", in *LNCIS Springer-Verlag*, Vol **228**, pp 101–116, 1997.

[8] F. Gouaisbaut, M. Dambrine and J.P. Richard, "Robust control of delay systems: a sliding mode control design

via LMI", Systems & Control Letters, Vol. **46(4)**, pp. 219–230, 2002.

[9] D. Ivanescu, "Sur la stabilisation des systèmes à retard : théorie et applications", *PhD Thesis in Institut National Polytechnique de Grenoble*, 2000.

[10] V. Kolmanovskii and J-P. Richard, "Stability of some linear systems with delays", *IEEE Trans. on Automat. Control*, **AC-44**, pp. 984-989, 1999.

[11] V. Kolmanovskii, S.I. Niculescu and J. P. Richard, "On the Liapunov-Krasovskii functionals for stability analysis of linear delay systems", *Int. J. Control* , vol. **72**, pp. 374–384, 1999.

[12] V. Kolmanovskii and A. Myshkis, *Applied Theory of functional differential equations,* Kluwer, 1999.

[13] X. Li, C. de Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delay", Automatica, **33**, pp. 1657–1662, 1997.

[14] M. Mahmoud, *Robust Control and Filtering for Time-Delay Systems*, Marcel Dekker, New-York, 2000.

[15] Y. S. Moon, P. Park, W. H. Kwon and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems" Int. J. Control, vol. **74** , pp. 1447–1455, 2001.

[16] W. Perruquetti and J-P. Barbot (Eds.), *Sliding mode control in engineering*, Marcel Dekker, New-York, 2002.