SIMPLE OUTPUT-FEEDBACK 2-SLIDING CONTROLLER FOR SYSTEMS OF RELATIVE DEGREE TWO

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Keywords: Variable Structure Control, Robust Control, Higher order sliding modes

Abstract

A 2-sliding algorithm to stabilise a class of nonlinear systems of relative degree two with respect to the input is presented. It does not require the output derivative to be measured or observed.

1 Introduction

Sliding Mode Control (SMC) is known to be a robust control method that is appropriate for controlling uncertain systems. High robustness is maintained against various kinds of uncertainties such as external disturbances and measurement error [4, 12]. It is also straightforward to implement the resulting algorithms.

In traditional sliding mode control, or first order sliding mode (1-sliding mode, 1-SM) controller design, the sliding variable is selected such that it has relative degree one with respect to the control. The control acts on the first derivative (with respect to time) of the sliding variable ($\dot{s}$) to keep the system trajectories in the sliding set $s = 0$. Essentially, the discontinuous control signal acts on the first derivative of s. This condition will of course limit the choice of sliding variable.

The notion of 1-SM control has recently been extended and the concept of higher order sliding modes (HOSM) [5, 11] as the generalisation of first order sliding modes (FOSM) has been developed. In HOSM control, the control acts on higher derivatives of the sliding variable. For example, the case of second order sliding modes (2-SM) corresponds to the control acting on the second derivative of the sliding variable, namely $\ddot{s}$, and the sliding set is defined as $s = \dot{s} = 0$. It is readily seen that such a HOSM control provides a natural means to avoid the chattering of the control signal in 1-SM control. Several such 2-sliding algorithms have been presented in the literature [1, 2, 7]. Levant [7] presented 2-sliding algorithms to stabilise second order uncertain nonlinear systems but these use knowledge of the output-derivative, $\dot{s}$, to implement so called twisting or drift algorithms [9, 10]. Bartolini et al. [3] presented an optimised version of the twisting algorithm. However, this requires at least knowledge of the sign of the output-derivative which is implemented by incorporating a memory element into the controller.

The super twisting algorithm [6], however, does not require this output derivative to be measured but it has been originally developed and analysed for systems with relative degree one with respect to the input. A robust exact finite-time-convergent differentiator is proposed in [9] which is based on this controller. An arbitrary-order sliding differentiator with similar features was developed in [8]. Using such a differentiator, an output feedback 2-sliding controller is shown to be effective for systems with relative degree 2. An alternative output feedback 2-sliding controller for relative degree 2 systems is proposed in this paper. This new controller does not include any explicit differentiator. This simplification yields higher robustness with respect to input noise.

Section 2 provides a statement of the original super twisting algorithm. The behaviour of this super twisting algorithm in the presence of unmodelled dynamics is addressed in Section 3. The new algorithm is proposed in Section 4 and comments on its stability are presented in Section 5.

2 Super Twisting Algorithm

The super twisting algorithm [6] has been developed and analysed for systems with relative degree one with respect to the input as in (1)

$$\dot{s} = \phi(s,t) + \gamma(s,t)u \tag{1}$$

where, $0 < |\phi(\cdot)| \leq \Phi$ and $0 < \Gamma_m \leq \gamma(\cdot) \leq \Gamma_M$. The super twisting algorithm defines the control law, $u(t)$, as the combination of two terms.

$$\left.\begin{array}{l} u(t) = u_1(t) + u_2(t) \\ \dot{u}_1 = \left\{ \begin{array}{ll} -u, & |u| > 1 \\ -W\,\text{sign}(s), & |u| \leq 1 \end{array} \right. \\ u_2 = \left\{ \begin{array}{ll} -\lambda|s_0|^\rho \text{sign}(s), & |s| > s_0 \\ -\lambda|s|^\rho \text{sign}(s), & |s| \leq s_0 \end{array} \right. \end{array}\right\} \tag{2}$$

where, $|s| < s_0$. The trajectories of the algorithm 'twist' around the origin in the phase portrait of the sliding variable. The super twisting algorithm converges in finite time and the corresponding sufficient conditions for finite time convergence are:

$$\left.\begin{array}{l} W > \dfrac{\Phi}{\Gamma_m} > 0; \quad \lambda^2 \geq \dfrac{4\Phi\Gamma_M(W+\Phi)}{\Gamma_m^3(W-\Phi)} \\ 0 < \rho \leq 0.5 \end{array}\right\} \tag{3}$$

For $\rho = 1$, this algorithm converges to the origin exponentially.

3 Unmodelled dynamics

Consider a first order system to be controlled

$$\dot{x} = \phi(x) + \Gamma(x)v + d(t)$$

with an actuator having first order dynamics

$$\dot{v} = \gamma(v) + \eta(v)u$$

where $d(t)$ represents a bounded external disturbance. The complete dynamics of the system with actuator can be written as

$$\ddot{x} = \left(\frac{\partial\phi}{\partial x} + \frac{\partial\Gamma}{\partial x}v\right)(\phi + \Gamma v + d(t)) + \dot{d}(t) + \Gamma\gamma + \eta\Gamma u$$

which may be represented by a second order SISO system of the following type

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, t) + g(x_1, x_2, t)u \\ y &= x_1 \end{aligned} \tag{4}$$

where $0 \leq |f(\cdot)| \leq F$ and $0 < G_{min} \leq g(\cdot) \leq G_{max}$ are uncertain, bounded functions. The time derivative of f and g, namely $\dot{f}$ and $\dot{g}$, are also bounded.

This type of system appears naturally due to the presence of actuator or sensor dynamics [13]. It is required to stabilize the output y of this system using a sliding mode control with the condition that neither measured nor observed x_2 is available to the controller. The system output y can be considered as a suitable sliding variable s.

It has been noticed that the algorithm (2) is not robust against such unmodelled dynamics. A simple example is simulated here to support this claim. Three cases for the term $f(x,t)$ in equation (4) are considered:

- $f(x,t) = 0$. The system model becomes similar to that of a double integrator.
- $f(x,t) = 2$. The system has constant matched uncertainty.
- $f(x,t) = 2\sin(t)$. The system has variable but bounded and matched uncertainty.

The controller (2) with parameters $\lambda = 7$, $W = 3$, $k = 1$ and $\rho = 0.5$ is used to simulate the system with an integration step size of 0.1 millisecond. The simulation results show the existence of a limit cycle. The limit cycle is due to the increased system order due to the unmodelled dynamics.

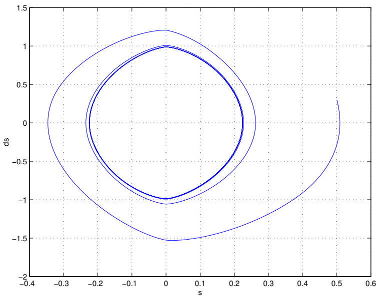

Figure 1: Limit cycle for $f(x,t) = 0$

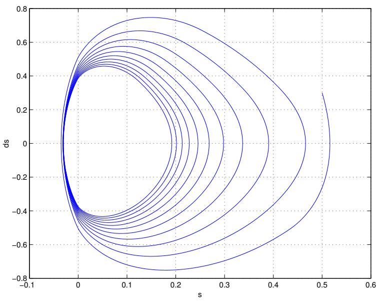

Figure 2: Limit cycle for $f(x,t) = 2$

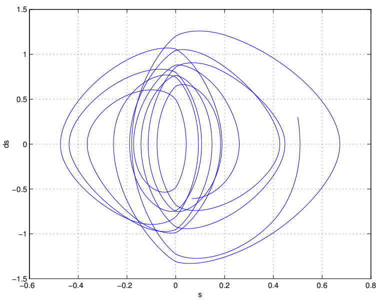

Figure 3: Limit cycle for $f(x,t) = 2\sin(t)$

The simulation results show limit cycle behaviour for all three cases in Figures 1, 2 and 3. It has been noticed that if ρ in (2) is reduced to 0 rather than the recommended value of 0.5, the modified algorithm may be successfully applied to systems of type (4). The results of this modification are presented in this paper.

4 The modified algorithm

The modified algorithm is defined by the following control law with respect to systems of type (4). The phase plot of the algorithm is shown in Figure 14.

$$\left.\begin{aligned} u(t) &= -\lambda\,\mathrm{sign}(y) + u_1(t) \\ \dot{u}_1 &= \begin{cases} -k\,u, & |u| > u_0 \\ -W\,\mathrm{sign}(y), & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{5}$$

where $\lambda, W, k, u_0 > 0$ are positive constants.

It has been observed that the following conditions assure local convergence at least:

$$\left.\begin{aligned} \lambda > 2u_0;\ u_0 > F/G_{min} \\ kG_{min}u_0 > \sup(|\dot{f}| + (2\lambda - u_0)|\dot{g}|) \\ W > 0 \end{aligned}\right\} \qquad (6)$$

Consider a system of type (4) having relative degree two with respect to the output. The 2-sliding algorithm (5) is suggested to steer y to zero and achieve the control task. For tracking problems, the controller can be defined in a similar manner, where $y(e)$ in the error space is used as the sliding variable. All cases proposed in Section 3 will be discussed in turn and simulation results presented.

Case 1: *f(x,t) = 0*

This case corresponds to the control of an ideal double integrator. The controller coefficients are selected as $\lambda = 6$, $W = 0.5$, $k = 4$ and $u_0 = 1$ with initial conditions [0.1, -0.01, 0]. The simulation is carried out at a fixed step size of 0.1 milliseconds. The steady state error in the output $|e|$, as shown in Figure 4, is of the order of 10^{-8} and that of $\dot{s}$ is of the order of 10^{-4}.

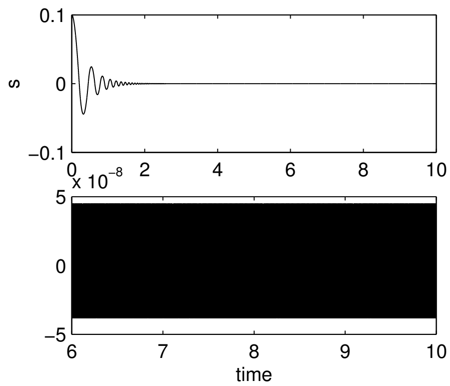

Figure 4: Sliding variable,s

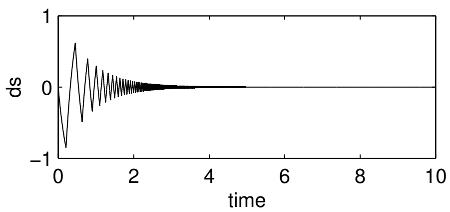

Figure 5: Sliding variable derivative $\dot{s}$

Case 2: *f(x,t) = a constant*

The value of $f(x,t)$ in the system dynamics (4) is considered as 2, which is independent of the system parameters and does not vanish as $y \to 0$ but has derivative equal to zero. The controller coefficients are selected as $\lambda = 4$, $W = 0.5$, $k = 3$

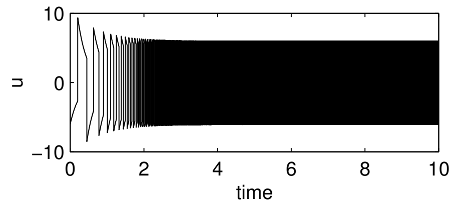

Figure 6: Control effort u

and $u_0 = 3$ with initial conditions [0.1, -0.01, 0]. The results are shown in Figures 7- 9. The simulation is carried out at the same fixed step size of 0.1 milliseconds. The steady state error in the output $|e|$, as shown in Figure 7, is of the order of 10^{-8} and that of $\dot{s}$ is of the order of 10^{-4}.

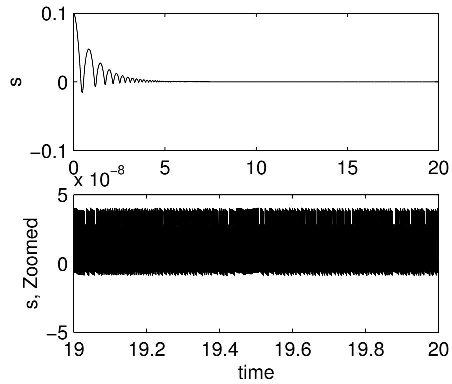

Figure 7: Sliding variable, s

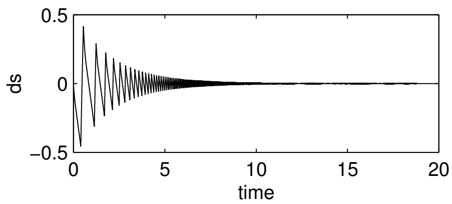

Figure 8: Sliding variable derivative $\dot{s}$

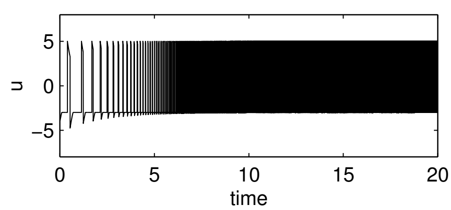

Figure 9: Control Effort, $u(t)$

Case 3: *f(x,t) = sin(t)*

The value of $f(x,t)$ considered in the system dynamics (4) is $\sin(t)$ which is independent of the system parameters, has bounded derivative and does not vanish as $y \to 0$. The controller coefficients are selected as $\lambda = 7$, $W = 0.5$, $k = 3$ and $u_0 = 1$ with initial conditions [0.1, -0.01, 0]. The results are shown in Figures 10− 12. The simulation is carried out at the same fixed step size of 0.1 milliseconds. The steady state error in the output $|e|$, as shown in Figure 7, is of the order of 10^{-6} and that of $\dot{s}$ is of the order of 10^{-3}.

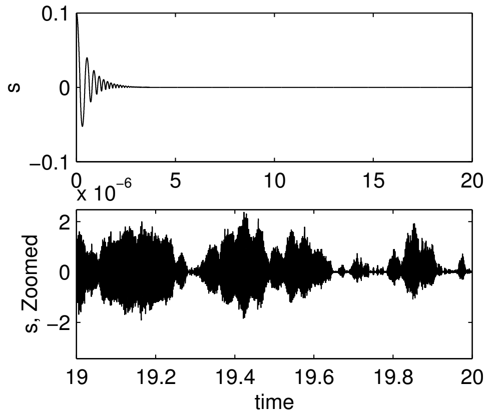

Figure 10: Sliding variable, s

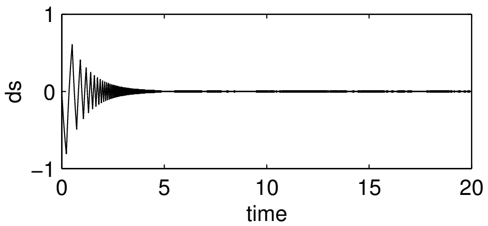

Figure 11: Sliding variable derivative $\dot{s}$

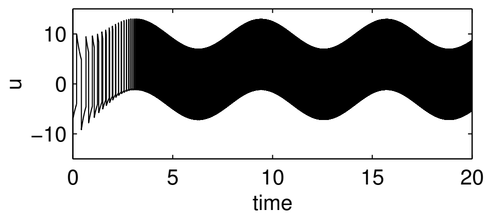

Figure 12: Control Effort

Even for very large values of initial conditions such as $[100, 50]$, the controller stabilizes the system output to zero. Controller coefficients chosen for this are $\lambda = 25$, $W = 0.5$, $k = 0.5$, $u_0 = 10$.

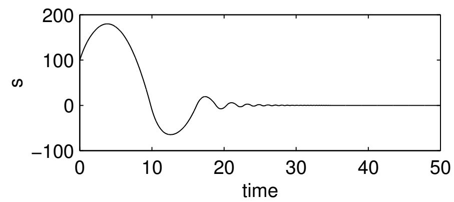

Figure 13: Large initial conditions

5 Comments on stability

The controller (5) is a 2-sliding controller as it ensures the convergence of s and $\dot{s}$ within a region of size τ^2 and τ respectively [6]. The ode45 integration algorithm from Matlab is used for the simulations presented in the paper, where τ is the sampling step. The accuracy, when the Euler method is used for integration, is of the order of the sampling step τ and $\tau^{\frac{1}{2}}$.

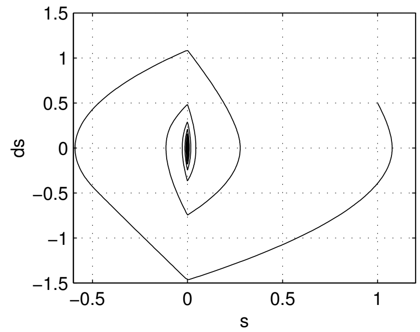

Figure 14: Phase plot of algorithm (5)

The control effort consists of two parts. The first part is a discontinuous switching signal while the second one (u_1) is smooth. In the $u-$scale, u_1 is bounded by $\pm(\lambda - u_0)$ and the control effort u remains bounded by $\pm(2\lambda - u_0)$ with $u_0 < |u| < (2\lambda - u_0)$. Hence, the optimal values of λ and u_0 are constrained by the permissible control. It is a 2-sliding mode with chattering because of the discontinuous control action.

In digital implementation, the sequence $\{\dot{s}_i\}$ of the intersection points with the axis $s = 0$, is a convergent series because it satisfies

$$\left| \frac{\dot{s}_{i+1}}{\dot{s}_i} \right| < 1 \quad (7)$$

before it settles into the real 2-sliding set. The equation (7) implies that $|\dot{s}| \to 0$ as $i \to \infty$. The settling time can be estimated as the sum of the encircling time sequence by a geometric series. The settling time is a function of $(\lambda - u_0)$. It seems that selection of λ and u_0 that give larger $(\lambda - u_0)$ and λ / u_0 will be more suitable. Increased values of $(\lambda - u_0)$ reduce the settling time but increase the amplitude of oscillations in x_2. Suitable choice of k helps in reducing oscillations in the output.

6 Conclusions

A new 2-sliding algorithm has been presented for systems with relative degree two with respect to the input. The only input to the controller is the measured output; the algorithm does not require measurement or estimation of the output derivative.

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