2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Adaptive Symbolic Feedback for One-Dimensional Discrete-Time Uncertain Systems

Tomohisa Hayakawa
Department of Mechanical and Environmental Informatics
Tokyo Institute of Technology, Tokyo 152-8552, JAPAN
hayakawa@mei.titech.ac.jp

Abstract

An adaptive control framework for stabilization of 1-dimensional linear uncertain discrete-time dynamical systems with symbolic feedback is developed. The plant output is assumed to be quantized and only its symbolic quantity is fed back to the controller. In the case of multi-dimensional systems, ideal system matrix is constructed so that the approach for the scalar systems can be applied in separate subspaces in the state space.

1. Introduction

With the increasing demand of implementing control systems over the communication network, it is important to develop a feedback protocol that requires low data rate to give the control information to the actuators. To this end, it is common to quantize the input variables and encode them so that they are sent through the communication channels and decoded at the actuator side.

In a recent series of papers [1–4], a direct adaptive control framework for adaptive stabilization of multivariable linear and nonlinear uncertain systems with input quantization requirement was developed. In this paper we develop analogous results for discrete-time linear uncertain systems with quantized *output* available from the plant. Specifically, a Lyapunov-based direct adaptive control framework is developed that guarantees asymptotic stability with respect to part of the closed-loop system states associated with the plant.

The idea is predicated on the framework of symbolic dynamics [5–7] where the behavior of the dynamics is described by the sequence of symbols. The symbolic dynamics are typically employed to characterize the distance between two different fluctuating chaotic trajectories with the shift operators and not frequently used in the context of stable system trajectories.

In the conventional adaptive control framework [8–11] the feedback gain update algorithm is *always* adjusting the gain matrix. Even though the use of symbolic feedback is inspired by the possible requirement when a controller is placed in the network and available information of the plant state is limited in communication between the plant and the controller, the resulting adaptive controller is simpler in that if the system trajectory is likely to be converging in a desired rate, the feedback gain update does not take place.

In this paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $(\,)^{\mathrm{T}}$ denote transpose, let $(\,)^{\dagger}$ denote the Moore-Penrose generalized inverse, and let $\mathcal{N}$ denote the set of nonnegative integers. Furthermore, we write $\mathrm{tr}(\cdot)$ for the trace operator, and $\ln(\cdot)$ for the natural log operator.

2. Symbolic Dynamics

In this section we consider the problem of characterizing symbolic adaptive feedback control laws for linear uncertain discrete-time systems. Specifically, consider the 1-dimensional controlled linear uncertain discrete-time system $\mathcal{G}_1$ given by

$$x(k+1) = ax(k) + bu(k), \quad x(0) = x_0, \quad k \in \mathbb{N}_0, \quad (1)$$

where $x(k) \in \mathbb{R}$, $k \in \mathbb{N}_0$, is the state vector, $u(k) \in \mathbb{R}$, $k \in \mathbb{N}_0$, is the control input, and $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are unknown constants. First assume that $u(k) \equiv 0$ and $a = a_{\mathrm{s}}$, where $a_{\mathrm{s}} \in \mathbb{R}$ is such that $0 < a_{\mathrm{s}} < 1$, and consider the state-space partition

$$\mathcal{D}_i \triangleq (a_{\mathrm{s}}^{i+1}, a_{\mathrm{s}}^{i}], \quad i \in \mathbb{Z}, \quad (2)$$

so that the boundary of the domains $\mathcal{D}_0$ and $\mathcal{D}_{-1}$ is 1. In this case, if $x_0 \in \mathcal{D}_i$, then the sequence of domain that the state $x(k)$ belongs to is given by

$$.\mathcal{D}_i\mathcal{D}_{i+1}\mathcal{D}_{i+2}\mathcal{D}_{i+3}\cdots. \quad (3)$$

On the other hand, if $a \neq a_{\mathrm{s}}$, then the sequence with $x_0 \in \mathcal{D}_i$ must not be given by (3); i.e., there must exist $k \in \mathbb{N}_0$ such that $x(k) \notin \mathcal{D}_{i+k}$. Specifically, consider the sequence

$$.\mathcal{D}_i\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\cdots. \quad (4)$$

This research was supported in part by the Aihara Innovative Mathematical Modelling Project, the Japan Society for the Promotion of Science (JSPS) through the Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program), initiated by the Council for Science and Technology Policy (CSTP).

978-3-952-41734-8/©2013 EUCA

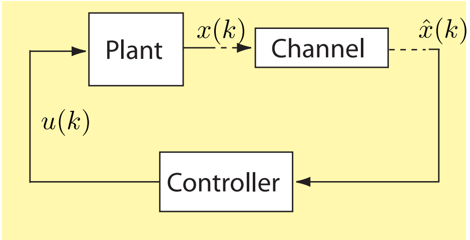

Figure 2.1: Block diagram of the closed-loop system. The output of the plant $x(t)$ is symbolized (quantized) to $\hat{x}(k)$

where $i_j \in \mathbb{Z}$. Note that if the second domain is such that $i+1 \neq i_1$, then the relationship between a and $a_{\rm s}$ is characterized by $a_{\rm s}^{i_1-i+2} \leq a < a_{\rm s}^{i_1-i}$. For example, for $x_0 \in \mathcal{D}_i$, if $i_1 = i+2$ instead of $i+1$, then $a_{\rm s}^3 \leq a < a_{\rm s}$. Alternatively, suppose $i_1 = i+1, \cdots, i_j = i+j$, but $i_{j+1} \neq i+j+1$. Note that in this case i_{j+1} is either $i+j$ or $i+j+2$. When $i_{j+1} = i+j+2$ (resp., $i_{j+1} = i+j$), the unknown parameter a must satisfy $a_{\rm s}^{\frac{j+3}{j+1}} \leq a < a_{\rm s}$ (resp., $a_{\rm s} < a \leq a_{\rm s}^{\frac{j}{j+1}}$).

For the control purpose, we assume that the label (subscript) of the observed state is available for feedback (Figure 2.1). This corresponds to the case where the observed state is quantized and sent through a communication channel. At the controller side, we reconstruct the state signal so that the quantized state $\hat{x}(\cdot)$ is such that $\hat{x}(\cdot) = a_{\rm s}^i$ if $x(\cdot) \in \mathcal{D}_i$. In this case, if the parameter a is known, then with any $0 < a_{\rm s} < 1$ we can employ the controller

$$u(k) = K_{\rm g}\hat{x}(k), \tag{5}$$

where

$$-\frac{a}{b} < K_{\rm g} < \frac{(1-a)a_{\rm s}}{b}, \tag{6}$$

in order to stabilize the system (1), which follows from the fact that $x \leq \hat{x} < \frac{1}{a_{\rm s}}x$ so that

$$\begin{aligned} 0 < ax(k) + bu(k) &= ax(k) + bK_{\rm g}\hat{x}(k) \\ &< ax(k) + (1-a)a_{\rm s}\frac{1}{a_{\rm s}}x(k) \\ &= x(k), \quad x(k) \neq 0. \end{aligned} \tag{7}$$

3. Adaptive Symbolic Feedback for 1-Dimensional Linear Systems

For the 1-dimensional system (1) where the parameter a is unknown, we cannot employ the controller (5) because the knowledge of a is required to determine the feedback gain $K_{\rm g}$. In the following theorem we provide an adaptive control algorithm for the feedback gain to be adjusted.

Theorem 3.1. Consider the 1-dimensional linear uncertain system $\mathcal{G}_1$ given by (1). Let $Q \in \mathbb{R}$ be $0 < bQ < \frac{1}{a_{\rm s}^{-1}+1}$. Then the adaptive feedback control law

$$u(k) = K(k)\hat{x}(k), \tag{8}$$

where $K(k) \in \mathbb{R}$, $k \in \mathbb{N}_0$, with update law

$$K(k+1) = \begin{cases} 0, & \text{if } \hat{x}(k+1) \leq a_{\rm s}\hat{x}(k), \\ K(k) - \frac{Q}{1+\hat{x}^2(k)}[\hat{x}(k+1) - a_{\rm s}\hat{x}(k)]\hat{x}(k), & \text{otherwise}, \end{cases} \tag{9}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\rm g})$ of the closed-loop system given by (1), (8), and (9) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. First, define $\tilde{K}(k) \triangleq K(k) - K_{\rm g}$. Note that with $u(k)$, $k \in \mathbb{N}_0$, given by (8) it follows from (1) that

$$\begin{aligned} x(k+1) &= ax(k) + bK(k)\hat{x}(k) \\ &= ax(k) + b\tilde{K}(k)\hat{x}(k) + bK_{\rm g}\hat{x}(k) \\ &= a(x(k) - \hat{x}(k)) + b\tilde{K}(k)\hat{x}(k) + a_{\rm s}\hat{x}(k), \\ &\qquad x(0) = x_0, \quad k \in \mathbb{N}_0, \end{aligned} \tag{10}$$

or, equivalently,

$$\begin{aligned} b\tilde{K}(k)\hat{x}(k) &= [\hat{x}(k+1) - a_{\rm s}\hat{x}(k)] + a(\hat{x}(k) - x(k)) \\ &\quad - (\hat{x}(k+1) - x(k+1)), \\ &\qquad x(0) = x_0, \quad k \in \mathbb{N}_0. \end{aligned} \tag{11}$$

To show Lyapunov stability of the closed-loop system (9) and (10), consider the Lyapunov function candidate

$$V(x, K) = \ln(1 + x^2) + \alpha Q^{-1}\tilde{K}^2, \tag{12}$$

where

$$\alpha > \frac{b(a_{\rm s}^{-2}+1)}{(a_{\rm s}^{-1}-1)^2(2-(a_{\rm s}^{-1}+1)bQ)}. \tag{13}$$

Note that $V(0, K_{\rm g}) = 0$ and $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\rm g})$. Furthermore, $V(x, K)$ is radially unbounded. Now, letting $x(k)$, $k \in \mathbb{N}_0$, denote the solution to (10), it follows that the Lyapunov difference along the closed-loop system trajectories is given by:
if $\hat{x}(k+1) \leq a_{\rm s}\hat{x}(k)$, which implies $x(k+1) < x(k)$ and $K(k+1) = K(k)$, then

$$\begin{aligned} &\Delta V(x(k), K(k)) \\ &\quad \triangleq V(x(k+1), K(k+1)) - V(x(k), K(k)) \\ &\quad = \ln\left(1 + x^2(k+1)\right) - \ln\left(1 + x^2(k)\right) \\ &\quad = \ln\frac{1 + x^2(k+1)}{1 + x^2(k)} \\ &\quad \leq \frac{x^2(k+1) - x^2(k)}{1 + x^2(k)} \\ &\quad \leq 0; \end{aligned} \tag{14}$$

Alternatively, if $\hat{x}(k+1) > a_s \hat{x}(k)$, then assume that $x(k) \in \mathcal{D}_i$ and $x(k+1) \in \mathcal{D}_{i-j+1}$ for some $j \in \mathbb{N}$ so that $\hat{x}(k) = a_s^i$ and $\hat{x}(k+1) = a_s^{i-j+1}$. In this case, it follows from (10) that

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad = \ln \frac{1+x^2(k+1)}{1+x^2(k)} + \alpha Q^{-1}(\tilde{K}^2(k+1) - \tilde{K}^2(k)) \\
&\quad \leq \frac{x^2(k+1) - x^2(k)}{1+x^2(k)} + aQ^{-1}\tilde{K}^2(k) \\
&\qquad - \tfrac{2\alpha}{1+\hat{x}^2(k)}[\hat{x}(k+1) - a_s\hat{x}(k)]\tilde{K}(k)\hat{x}(k) \\
&\qquad + \tfrac{\alpha Q}{(1+\hat{x}^2(k))^2}[\hat{x}(k+1) - a_s\hat{x}(k)]^2\hat{x}^2(k) \\
&\qquad - aQ^{-1}\tilde{K}^2(k) \\
&\quad \leq \frac{x^2(k+1) - x^2(k)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+\hat{x}^2(k)} \tfrac{[\hat{x}(k+1) - a_s\hat{x}(k)]}{b}(2[\hat{x}(k+1) - a_s\hat{x}(k)] \\
&\qquad + 2a(\hat{x}(k) - x(k)) - 2(\hat{x}(k+1) - x(k+1)) \\
&\qquad - bQ[\hat{x}(k+1) - a_s\hat{x}(k)]).
\end{aligned} \tag{15}$$

Furthermore,

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad \leq \frac{(a_s^{i-j+1})^2 - (a_s^{i-1})^2}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{a_s^{i+1}(a_s^{-j}-1)}{b}((2-bQ)a_s^{i+1}(a_s^{-j}-1) \\
&\qquad + 2a(\hat{x}(k) - x(k)) - 2a_s^{i-j+2}(a_s^{-1}-1)) \\
&\quad \leq \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)}{b}((2-bQ)(a_s^{-j}-1) \\
&\qquad - 2a_s^{-j+1}(a_s^{-1}-1)) \\
&\quad = \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)}{b}(2(a_s^{-j+1}-1) \\
&\qquad - bQ(a_s^{-j}-1)) \\
&\quad = \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)^2}{b}(\frac{2(a_s^{-j+1}-1)}{a_s^{-j}-1} - bQ) \\
&\quad = \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)^2}{b}(\frac{2(a_s^{-j+1}-1)}{a_s^{-j}-1} - bQ),
\end{aligned} \tag{16}$$

where in (16) we used $\hat{x}(k+1) - a_s\hat{x}(k) = (a_s^{i+1})(a_s^{-2j} - 1) > 0$ and $0 \leq \hat{x}(k+1) - x(k+1) \leq a_s^{i-j+2}(a_s^{-1} - 1)$ for $x(k+1) \in \mathcal{D}_{i-j+1}$. When $j \geq 2$, it follows that

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad \leq \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)^2}{b}(\frac{2(a_s^{-j+1}-1)}{a_s^{-j}-1} - bQ) \\
&\quad \leq \frac{(a_s^{i+1})^2(a_s^{-2j}-1)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+x^2(k)} \tfrac{(a_s^{i+1})^2(a_s^{-j}-1)^2}{b}(\frac{2(a_s^{-1}-1)}{a_s^{-2}-1} - bQ) \\
&\quad \leq -\frac{(a_s^{i+1})^2}{1+x^2(k)}[-(a_s^{-2j}-1) \\
&\qquad - \alpha\tfrac{(a_s^{-j}-1)^2}{b}(\frac{2}{a_s^{-1}+1} - bQ)] \\
&\quad \leq 0.
\end{aligned} \tag{17}$$

On the other hand, when $j = 1$ it follows from (15) that if $\hat{x}(k+1) - x(k+1) \leq \frac{1}{2}(\frac{1}{a_s} - 1)a_s^{i-j+1}$, then

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad \leq -\frac{(a_s^{i+1})^2}{1+x^2(k)}[-(a_s^{-2}-1) - \alpha\tfrac{a_s^{-1}+1}{b}(2-bQ)] \\
&\quad \leq 0,
\end{aligned} \tag{18}$$

and if $\hat{x}(k+1) - x(k+1) > \frac{1}{2}(\frac{1}{a_s} - 1)a_s^{i-j+1}$, then

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad \leq \frac{(\frac{1}{2})^2(a_s^i - a_s^{i+1})^2 - x^2(k)}{1+x^2(k)} \\
&\qquad - \tfrac{\alpha}{1+\hat{x}^2(k)} \tfrac{a_s^{i+1}(a_s^{-1}-1)}{b}[2(a_s^i - x(k)) \\
&\qquad - bQa_s^{i+1}(a_s^{-1}-1)] \\
&\quad \leq 0.
\end{aligned} \tag{19}$$

In any case, $\Delta V(x(k), K(k)) \leq 0$, which proves that the the solution $(x(k), K(k)) \equiv (0, \hat{K}_g)$ to (10) and (9) is Lyapunov stable. Furthermore, it follows from that $\Delta V(x(k), K(k))$ is negative when ever $x(k) \neq 0$ so that $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

4. Conclusion

A discrete-time symbolic adaptive linear control framework for adaptive stabilization of linear uncertain dynamical systems was developed. The plant output was assumed to be quantized and only its symbolic quantity is fed back to the controller. Future research will involve incorporation of input quantization requirement assuming the controller is communicated with the plant with band limited communication channels.

References

[1] T. Hayakawa, H. Ishii, and K. Tsumura, "Adaptive quantized control for linear uncertain discrete-time systems," *Automatica*, vol. 46, no. 3, pp. 692–700, 2009.

[2] T. Hayakawa, H. Ishii, and K. Tsumura, "Adaptive quantized control for nonlinear uncertain systems," *Sys. Contr. Lett.*, vol. 58, no. 9, pp. 625–632, 2009.

[3] H. Ishii, K. Tsumura, and T. Hayakawa, "Coarseness in quantization for stabilization of linear systems over networks," in *Proc. Math. Theory Netw. Syst.*, (Budapest, Hungary), pp. 1211–1218, July 2010.

[4] M. Siami, T. Hayakawa, H. Ishii, and K. Tsumura, "Adaptive quantized control for linear uncertain discrete-time systems over channels subject to packet loss," in *Proc. IEEE Conf. Dec. Contr.*, (Atlanta, GA), pp. 4655–4660, December 2010.

[5] M. Coornaert and A. Papadopoulos, *Symbolic Dynamics and Hyperbolic Groups.* Springer, 1996.

[6] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, 1996.

[7] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos.* CRC Press, 1998.

[8] R. Johansson, "Global Lyapunov stability and exponential convergence of direct adaptive control," *Int. J. Contr.*, vol. 50, no. 3, pp. 859–869, 1989.

[9] P.-C. Yeh and P. V. Kokotović, "Adaptive control of a class of nonlinear discrete-time systems," *Int. J. Contr.*, vol. 62, pp. 303–324, 1995.

[10] M. R. Rokui and K. Khorasani, "An indirect adaptive control for fully feedback linearizable discrete-time non-linear systems," *Int. J. Adapt. Control Signal Process.*, vol. 11, pp. 665–680, 1997.

[11] R. Venugopal, V. G. Rao, and D. S. Bernstein, "Lyapunov-based backward-horizon adaptive stabilization," *Int. J. Adapt. Control Signal Process.*, vol. 17, no. 1, pp. 67–84, 2003.