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# Time-Parametrization Control of Quadrotors with a Robust Quaternion-based Sliding Mode Controller for Aggressive Maneuvering

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*Abstract*— **In this paper, we introduce the concept of time parametrization for Quadrotor flight maneuvers establishing a novel, robust and model free feedback controller based on a quaternion representation without singularities. Three prominent features are remarked: firstly, the control algorithm assures exponential stability of the full position/attitude dynamics of the system with smooth control efforts. Secondly, the closed-loop system is robust in presence of external forces and induced moments generated during the flight maneuvers. Finally, the controlled Quadrotor offers capabilities for aggressive maneuvers. Additionally, based on time-based generators, the sliding mode gives rise to well-posed terminal stability, parametrized by a desired convergence time defined by the user, independently of initial conditions and feedback gains. This allows tasks, such as cooperative Quadrotor, or force and interaction task, to be solved on a nonlinear and simplified setting. Simulations are presented in several scenarios in order to carry out a wide idea of our approach.**

## I. INTRODUCTION

The Quadrotor is considered as a highly underactuated nonlinear system with fast orientation and slow coupled position dynamics. Having four thrusters for performing hovering and navigation flight, the Quadrotor is a challenge for the stabilization in regulation and trajectory tracking.

Quadrotors can perform high spatial maneuverability; however time response is an overlooked characteristic. This time response possesses a new problem in Quadrotors, that is, to control the time attributes which would make possible that several Quadrotor meet in a given coordinate at a given time. Notice that, for cooperative or interaction tasks, asymptotic methods based on Lyapunov stability theory can demonstrate limited usefulness for Quadrotors due to the different convergence time in which all the Quadrotors meet at the desired point.

Fast orientation dynamics and slow translational dynamics have been studied in Quadrotors. This characteristic has been widely observed in practice and exploited in [1], wherein it is proposed a two-time scale controller based on singular perturbation analysis. This allows a concise approach to address the control design. Attitude based on quaternion representation has been studied for a well-posed representation absent of singularities inherent of the Euler angle representation, [6]. Nevertheless, for the coupled translation-orientation dynamics of a Quadrotor is poorly understood in terms of its time control. Typically, linearized models are obtained to design PID-based controllers for local regulation, [3]. Due to the underactuated nature of these systems, even PID regulators require model-based terms to cancel gravitational forces. In contrast to robot manipulators where model-free PID state feedback regulators exist, the Quadrotor is subject to the nonconservative gravitational field, computed torque or regressor-based controllers allow tracking to be completed at the expense of precise structural knowledge of the plant, which is rarely available, [2]. Nonetheless, computed torque-based controllers are very useful to study the subtlety complexities of compensating complex dynamics, but it does not stand as a viable option for control of Quadrotors. Being the Quadrotor an underactuated system, formal methods have been proposed to deal with stabilization of underactuated dynamics, such as Backsteeping [2], which is highly dependent on the exact model of the system, difficult to implement in practice. Model-free controllers have been proposed such as iterative learning control, [5], which has produced impressive experiments at the expense of trial-and-error which may lead to catastrophic results in the first tests, and also it is difficult to tune due its overparametrized control.

The main contribution of this paper is a new sliding mode controller that enforces an integral sliding mode for any initial conditions for the full nonlinear model of a Quadrotor. Consequently, stemmed on the fact that sliding mode guarantees the order reduction and invariance to system model, and inspired by the work of chattering free and model-free second order sliding mode for robot manipulatiors, [7], time-base generators are introduced to induce terminal stability, [8]. This parametrizes a spatial convergence of tracking errors to zero in a given desired convergence time defined by the user, independently of initial condition and feedback gains. Remarkably, although the general modeling of Quadrotors has been mastered including the electrical dynamics of thrusters, mapping the thrust-torque to the input current, and the aerodynamic effects. Such models are hardly available in practice to be used for control purposes; our controller can be synthesized by purely state feedback kinematic calculations. Further, preliminary developments follow the direction of [4] for quaternion-based control and full modeling of [1]. An immediate implication of this feature

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is the ability to carry out loops based on a singularity-free quaternion representation as well as a cooperative stiffness control of Quadrotor by defining a desired meeting point in space and in a given time. Indeed, this allows the cooperative Quadrotor problem to be solved on a nonlinear and simplified setting.

## II. System Modeling

The dynamic model of an Quadrotor is basically obtained representing the aircraft as a rigid body evolving in 3D and subject to one force and three moments.

### A. Dynamics

Let us consider earth fixed frame $\mathcal{I} = \{e_x, e_y, e_z\}$ and body fixed frame $\mathcal{A} = \{e_x^b, e_y^b, e_z^b\}$, as seen in Figure 1. The center of mass and the body fixed frame origin are assumed to coincide. The orientation of the rigid body is given by a rotation $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{I}$, where $\mathcal{R} \in SO(3)$ is an orthogonal rotation matrix, parametrized by the Euler angles $\psi, \theta, \phi$ (yaw, pitch, roll). Newtons-Euler equations of motion state the dynamics of the quad-rotor as follows:

$$\begin{aligned} m\ddot{\xi} &= -\mathcal{T}\mathcal{R}e_z + F(t) & (1)\\ \dot{\mathcal{R}} &= \mathcal{R}\omega^\times & (2)\\ \mathbf{J}\dot{\omega} &= -\omega \times \mathbf{J}\omega + \tau + d(t) & (3)\end{aligned}$$

where $\xi = (x, y, z)^T$ denotes the position of the center of mass of the airframe in the frame $\mathcal{I}$ relative to a fixed origin, $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathcal{A}$ denotes the angular velocity of the airframe expressed in the body fixed frame. $m$ denotes the mass of the rigid object and $\mathbf{J} \in \mathbb{R}^{3\times3}$ denotes the constant inertia matrix around the center of mass (expressed in the body fixed frame $\mathcal{A}$). $\omega^\times$ denotes the skew-symmetric matrix of the vector $\omega$, which is given by

$$\omega^\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$\mathcal{T} \in \mathbb{R}_+$ represents the magnitude of the principal non-conservative forces applied to the object. $F(t)$ represents the external forces applied to the aerial vehicle, such that in the absence of forces exerted by the ambient (aerodynamic reaction forces, etc.) $F(t) = mge_z$. $\tau \in \mathcal{A}$ is the control torque. $d(t) \in \mathbb{R}^3$ represents the external torque disturbances induced by $F(t)$.

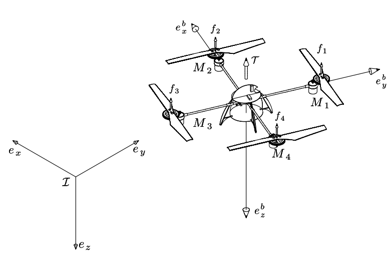


Fig. 1. The Quadrotor system. $f_i$ represents the thrust of motor $M_i$ and $\mathcal{T}$ is the main thrust.

### B. The open loop extended error

Let us define a parametrization $Y_r$ in terms of a nominal references $\omega_r$, to be defined, and its derivative $\dot{\omega}_r$, as follows

$$Y_r = \mathbf{J}\dot{\omega}_r + \omega^\times \mathbf{J}\omega_r + \omega_r^\times \mathbf{J}\omega - \omega_r^\times \mathbf{J}\omega_r \quad (4)$$

Introducing (4) into (3) yields

$$\mathbf{J}\dot{S}_r + S_r^\times \mathbf{J} S_r = \tau + d(t) - Y_r \quad (5)$$

where the error coordinates $S_r$ are defined by

$$S_r = \omega - \omega_r \quad (6)$$

At this point, the control objective is to design a $\tau$ such that $S_r$ will be stable despite the presence of bounded disturbances.

### C. The attitude error

We employ the unit quaternion as the attitude representation. Using this representation the attitude control design does not suffer from singularities. The unit quaternion satisfies the following constraint

$$\mathbf{q}^T\mathbf{q} = \begin{pmatrix} q_0 & q^T \end{pmatrix} \begin{pmatrix} q_0 \\ q \end{pmatrix} = q_0^2 + q^T q = 1 \quad (7)$$

and it is related to the angular velocity $\omega$ by the following differential equations

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_0 \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} q^T \omega \\ \frac{1}{2}\left(q_0 I + q^\times\right)\omega \end{pmatrix} \quad (8)$$

Let us define the angular velocity error $\omega_e$ as follows

$$\omega_e = \omega - \omega_d \quad (9)$$

where $\omega_d$ is the desired angular velocity.

The relative attitude error is defined as

$$\mathbf{q}_e = (q_{0e}\ q_e^T)^T = \mathbf{q} \otimes \mathbf{q}_d^* \quad (10)$$

where $\otimes$ denotes the operator for quaternion multiplication, $\mathbf{q}_d = (q_{0d}\ q_d^T)^T$ is the desired attitude, such that $q_{0d}(t), q_d(t)$ are once differentiable functions and $\mathbf{q}_d^*$ is the inverse of $\mathbf{q}_d$. The vectorial part $q_e$ given by

$$q_e = -q_0 q_d + q_{0d} q - q^\times q_d \quad (11)$$

### D. The sliding surface

In order to induce an attractive point on the extended error $S_r$, let us consider the following nominal reference

$$\begin{aligned} \omega_r &= \omega_d - \alpha q_e + S_d - \gamma\sigma & (12)\\ \dot{\sigma} &= \text{sgn}(S_q) & (13)\end{aligned}$$

where feedback gains, $\alpha > 0$ and $\gamma$ is diagonal positive definite matrix; the function $\text{sgn}(X) = (\text{sgn}(x_1), \text{sgn}(x_2), \text{sgn}(x_3))^T$ stands for the input wise discontinuous function of $X$.

The sliding surface $S_q$ defined as

$$\begin{aligned} S_q &= S - S_d & (14)\\ S &= \omega_e + \alpha q_e & (15)\\ S_d &= S(t_0)e^{-k(t-t_0)} & (16)\end{aligned}$$

where $S$ is the error manifold, $S_d$ is a vanishing term to induce a sliding mode for any initial condition, so as to eliminates the reaching phase with $k > 0$ and $S(t_0)$ stands for $S(t)$ at $t = t_0$.

From (6), (9), (12) and (15) the dynamic extended error $S_r$ is given by

$$S_r = S_q + \gamma\sigma \tag{17}$$

### E. Structural properties for the stability analysis

Let us now introduce some structural properties of (3) and (12), which will be used in the stability analysis.

There exist positive scalars $\beta_i$ for $i = 0, \ldots, 4$, such that

$$\begin{aligned} 0 < \beta_0 < \lambda_{\min}(\mathbf{J}) \leq \quad & \|\mathbf{J}\| \leq \lambda_{\max}(\mathbf{J}) < \beta_1 < \infty \\ & \|q_e\| < 1 \\ & \|\omega_r\| \leq \beta_2 + \|\gamma\|\|\sigma\| \\ & \|\dot{\omega}_r\| \leq \beta_3 + \beta_4\|\omega_e\| \end{aligned} \tag{18}$$

where $\lambda_{\min}(\mathbf{J}), \lambda_{\max}(\mathbf{J})$ stand for the minimum and maximum eigenvalues of matrix $\mathbf{J} \in \mathbb{R}^3$, $\|\mathbf{J}\| = \sqrt{\lambda_{\max}(\mathbf{J}^T\mathbf{J})}$ and $\|\cdot\|$ stands for the vector Euclidean norm.

From (4), (3) and using (18), $d(t) - Y_r$ can be bounded as

$$\begin{aligned} \|d(t) - Y_r\| \leq & \|d(t)\| + \|\mathbf{J}\|\|\dot{\omega}_r\| + 2\|\omega\|\|\mathbf{J}\|\|\omega_r\| \\ & + \|\mathbf{J}\|\|\omega_r\|^2 \\ \leq & \|d(t)\| + \beta_1\beta_3\|\omega_e\| + 2\beta_1\beta_2\|\sigma\|\|\omega\| \\ & + 2\beta_1\beta_2\|\gamma\|\|\sigma\| + \beta_1\|\gamma\|^2\|\sigma\|^2 + \beta_5 \\ \leq & \eta(t) \end{aligned} \tag{19}$$

where $\beta_5 = \beta_1\beta_4 + \beta_1\beta_2^2$, and $\eta(t)$ is a state-dependent function. Notice that, $\eta(t)$ considers all the external torques including state-dependence of $d(t)$.

### F. Practical zero

We define the practical zero as the minimal value measured of a variable from a digital sensor

$$\mathbf{0_p} = \frac{\epsilon_{ref}}{2^b} \tag{20}$$

where $\epsilon_{ref} > 0$ is, in general, a reference voltage and $b$ is the number of bits of resolution of the sensor.

For instance, consider the vanishing term $S_d$, from (16). There exists a finite time $t_1 > 0$ and a $k \gg 1$ such that

$$\|S_d(t)\| \leq \mathbf{0_p} \tag{21}$$

for all $t > t_1$. This condition represents the case when a sensor cannot reconstruct any element of $S_d(t)$ and therefore we will consider

$$S_d(t) = 0 \tag{22}$$

for all $t > t_1$. Note that, this consideration does not represent an assumption since it occurs in the practice.

## III. Control Design

In this section we present the non trivial extension of [7] for the non inertial underactuated Quadrotor dynamical system.

### A. Attitude Control

Consider the following control law

$$\tau = -K_d S_r \tag{23}$$

where $K_d$ is a diagonal positive definite matrix. We now have the following result.

*Theorem 1:* Consider the attitude dynamics (3) in closed loop with the controller (23). Then, semiglobal exponential tracking is assured, provided that $\gamma$ in (17) and $K_d$ are large enough, for small initial errors.

*Proof:* [Proof of Theorem 1] Substituting (23) into (5) yields

$$\mathbf{J}\dot{S}_r = -\left(K_d S_r + S_r^\times \mathbf{J} S_r\right) + d(t) - Y_r \tag{24}$$

Let us consider the following Lyapunov function

$$V = \frac{1}{2} S_r^T \mathbf{J} S_r \tag{25}$$

The total derivative of (25) along its solution (24) gives rise to

$$\dot{V} = -S_r^T K_d S_r + S_r^T\left(d(t) - Y_r\right) \tag{26}$$

Since $K_d$ is a diagonal positive definite matrix, then $S_r^T K_d S_r$ satisfies

$$\lambda_{\min}(K_d)\|S_r\|^2 \leq S_r^T K_d S_r \leq \lambda_{\max}(K_d)\|S_r\|^2 \tag{27}$$

Using (19) and (27), (26) becomes

$$\dot{V} \leq -\|S_r\|\left(\lambda_{\min}(K_d)\|S_r\| - \eta(t)\right) \tag{28}$$

Let $c = \sup_{t \geq 0} \eta(t)$. Note that if $\|S_r\| > (c/\lambda_{\min}(K_d))$ then $\dot{V} < 0$. This implies that exists a time $t_1$ such that

$$\|S_r\| \leq \frac{c}{\lambda_{\min}(K_d)} \quad \forall t > t_1 \tag{29}$$

In this way $S_r$, is upper bounded by $c/\lambda_{\min}(K_d)$. Boundedness of $S_r$ implies the boundedness of the state which includes $\sigma$. Therefore, it is concluded the boundedness of $\dot{S}_r$ as follows:

From (24), we have

$$\begin{aligned} \|\dot{S}_r\| \leq & \lambda_{\max}(\mathbf{J}^{-1})\left[\left(\lambda_{\max}(K_d) + \tfrac{c\lambda_{\max}(\mathbf{J})}{\lambda_{\min}(K_d)}\right)\tfrac{c}{\lambda_{\min}(K_d)} + c\right] \\ \leq & \bar{c} \end{aligned}$$

for some real $\bar{c} > 0$.

Now, for a given $\gamma$, sliding mode is induced on $S_q = 0$. Consider the following dynamic system defined by (17)

$$\dot{S}_q = -\gamma \operatorname{sgn}(S_q) + \dot{S}_r \tag{30}$$

with the following positive definite function

$$V_q = \frac{1}{2} S_q^T S_q \tag{31}$$

The total derivative of (31) , along its solution (30) gives rise to

$$\begin{aligned} \dot{V}_q &= -S_q^T \gamma \operatorname{sgn}(S_q) + S_q^T \dot{S}_r \\ &\leq -\lambda_{\min}(\gamma)|S_q| + |S_q||\dot{S}_r| \\ &\leq -\left(\lambda_{\min}(\gamma) - \sqrt{3}\bar{c}\right)|S_q| \end{aligned}$$

Thus, in order to prove that $S_q \to 0$ in a finite time, $\lambda_{\min}(\gamma) > \sqrt{3}\bar{c}$ is chosen for guaranteing the existence of a sliding mode condition

$$S_q^T \dot{S}_q \leq -\nu |S_q| \tag{32}$$

where $\nu = \left(\lambda_{\max}(\gamma) - \sqrt{3}\bar{c}\right) > 0$. This implies that a sliding mode is established at time $t_s \leq (|S_q(t_0)|/\nu)$, and since $S_q(t_0) = 0$ for any initial condition, then the sliding mode in $S_q(t) = 0$ is enforced for all time.

Considering that $S_d(t) = 0$, in the sense of subsection II-F, and provided that $S_q = 0$, then from (14) we have

$$\omega_e = -\alpha q_e \tag{33}$$

for all $t > t_1$.

According to (8), the time derivative of $\mathbf{q}_e$ is given by

$$\dot{\mathbf{q}}_e = \begin{pmatrix} -\frac{1}{2} q_e^T \omega_e \\ \frac{1}{2}\left(q_{0e} I + q_e^{\times}\right) \omega_e \end{pmatrix} \tag{34}$$

Then, from (33) it gets

$$\dot{q}_{0e} = \frac{\alpha}{2} f_1(t) \tag{35}$$

$$\dot{q}_e = -\frac{\alpha}{2} f_2(t) q_e \tag{36}$$

where $f_1(t) = q_e^T q_e$ and $f_2(t) = q_{0e}$. Note that, $f_1(t)$ is a positive definite function of $q_e$. Then the solution of (35) is given by

$$q_{0e}(t) = q_{0e}(t_1) + \frac{\alpha}{2}\int_{t_1}^{t} f_1(\zeta) d\zeta \tag{37}$$

Assuming, without loss of generality, that $q_{0e}(t_1) > 0$, a simple and yet practical constraint easy to find for small initial errors. Then we can conclude that

$$q_{0e}(t) > 0 \tag{38}$$

for all $t > t_1$.

The solution of (36) is given by

$$q_e(t) = e^{-\frac{\alpha}{2} f_2 (t - t_1)} q_e(t_1) \tag{39}$$

for $t > t_1$. From (38), we conclude that

$$q_e(t) \to 0 \tag{40}$$

exponentially. From the constraint of a unit quaternion (7), it follows that $q_{0e} \to 1$ exponentially. This completes the proof of Theorem 1. ■

### B. Position Control

Consider the translational dynamics (1). Now, let us define the following virtual control

$$u = \mathcal{T}\mathcal{R} e_z \tag{41}$$

The main idea is to design a controller by using the virtual control $u$ as a full vectorial term. Then, system (1) can be rewritten as follows

$$m\ddot{\xi} = -u + F(t) \tag{42}$$

Following the similar procedure described in section III-A, the control design becomes straightforward. The parametrization can be written as follows

$$m\ddot{\xi}_r = \bar{Y}_r \tag{43}$$

Introducing (43) into (42) yields

$$m\dot{\bar{S}}_r = -u + F(t) - \bar{Y}_r \tag{44}$$

where $\bar{S}_r = \dot{\xi} - \dot{\xi}_r$.

Consider the following nominal reference $\dot{\xi}_r$

$$\dot{\xi}_r = \dot{\xi}_d - \bar{\alpha}\xi_e + \bar{S}_d - \bar{\gamma}\bar{\sigma} \tag{45}$$

$$\dot{\bar{\sigma}} = \operatorname{sgn}\left(\bar{S}_q\right) \tag{46}$$

where $\xi_e = \xi - \xi_d$ is the tracking error, $\xi_d(t) \in C^2$ is the reference trajectory, feedback gains $\bar{\alpha}$ and $\bar{\gamma}$ are diagonal positive definite matrices, and

$$\bar{S}_q = \bar{S} - \bar{S}_d \tag{47}$$

$$\bar{S} = \dot{\xi}_e + \bar{\alpha}\xi_e \tag{48}$$

$$\bar{S}_d = \bar{S}(t_0) e^{-\bar{k}(t - t_2)} \tag{49}$$

for $\bar{k} > 0$, $t_2 > 0$ and $\bar{S}_r$ rewritten as

$$\bar{S}_r = \bar{S}_q + \bar{\gamma}\bar{\sigma} \tag{50}$$

As before, $F(t) - \bar{Y}_r$ can be bounded as

$$\begin{aligned} \|F(t) - \bar{Y}_r\| &\leq \|F(t)\| + m\|\ddot{\xi}_r\| \\ &\leq \|F(t)\| + m(\bar{\beta} + \bar{\alpha}\|\dot{\xi}_e\|) \\ &\leq \bar{\eta}(t) \end{aligned}$$

where $\bar{\beta}$ is a positive constant, and $\bar{\eta}(t)$ is a state-dependent function. Notice that, $\bar{\eta}$ not only includes all the external forces affecting the aircraft (buoyancy forces, aerodynamic forces, gravity, etc.) but also a general state-dependence of $F(t)$.

Now, consider the following position control law

$$u = \bar{K}_d \bar{S}_r \tag{51}$$

where $\bar{K}_d$ is a diagonal positive definite matrix. We have the following result.

*Theorem 2:* Consider the translational dynamics (1) in closed-loop with the controller (51). Then, semiglobal exponential tracking is assured, provided that $\bar{\gamma}$ and $\bar{K}_d$ are large enough, for small initial errors.

*Proof:* [Proof of Theorem 2] In a similar way to the proof of Theorem 1, it follows that $\bar{S}_r$ and $\dot{\bar{S}}_r$ are upper bounded, and the sliding mode in $\bar{S}_q(t) = 0$ is enforced for all time. In this way, tracking errors are constrained to evolve on a manifold that has exponential solution toward the desired trajectory $\xi_d(t)$ for designer parameters $\bar{k}$ and $\bar{\alpha}$ in the following way

$$\xi(t) = \xi_d(t) + \zeta(t) \tag{52}$$

where $\zeta(t) = (\bar{\alpha} - \bar{k}I)^{-1}\left[e^{-\bar{k}I(t-t_0)} - e^{-\bar{\alpha}(t-t_0)}\right]\bar{S}(t_0) + e^{-\bar{\alpha}(t-t_0)}\xi_e(t_0) \to 0$. Equation (52), in turn, establishes the exponential convergence of tracking errors

$$\xi(t) \to \xi_d(t), \quad \dot{\xi}(t) \to \dot{\xi}_d(t) \tag{53}$$

regardless of uncertainty of system parameters. This completes the proof of Theorem 2. ■

### C. Desired attitude trajectories

Let us now present the deduction of the desired attitude trajectories to satisfy (41).

In order to compute the desired Euler angles and angular velocity, let $\mathcal{T}_d$ be defined as the magnitude of $u$, and $\mathcal{R}_d e_z$ as a unit vector, representing the direction, as follows

$$\mathcal{T}_d = \|u(t)\|, \; \mathcal{R}_d e_z = u(t)/\mathcal{T}_d \tag{54}$$

Then, by solving equation (54), for yaw $\psi_d = 0$, the desired Euler angles are obtained as follows

$$\theta_d = \arctan(u_1/u_3), \; \phi_d = -\arcsin(u_2/\mathcal{T}_d) \tag{55}$$

where $u_1$, $u_2$ and $u_3$ are the components of the control input $u$. The desired angular velocity $\omega_d = (\omega_{1_d} \; \omega_{2_d} \; \omega_{3_d})^T$ is deduced from the relationship between the Euler angles (and its derivatives) and the angular velocity $\omega$ of equation (2), as follows

$$\omega_{1_d} = -(\dot{u}_2 \mathcal{T}_d - \dot{\mathcal{T}}_d u_2) / \left( \mathcal{T}_d \sqrt{u_1^2 + u_3^2} \right) \tag{56}$$

$$\omega_{2_d} = \sqrt{1 - (u_2^2/\mathcal{T}_d^2)}\,(u_3 \dot{u}_1 - \dot{u}_3 u_1) \,/\, \left(u_1^2 + u_3^2\right) \tag{57}$$

$$\omega_{3_d} = \frac{u_2(u_3 \dot{u}_1 - \dot{u}_3 u_1)}{\mathcal{T}_d(u_1^2 + u_3^2)} \tag{58}$$

where $\dot{\mathcal{T}}_d = \frac{u_1\dot{u}_1 + u_2\dot{u}_2 + u_3\dot{u}_3}{\mathcal{T}_d}$ and $\dot{u}_1$, $\dot{u}_2$ and $\dot{u}_3$ are the elements of the time-derivative of $u$. $\mathbf{q}_d$ is obtained, from the desired Euler angles, by using the conversion between quaternion and Euler angles.

Notice that, equations (55)-(58) are well-posed, since the third component of the control $u_3$ is counteracting the gravity, and therefore $u_3 > 0$ for all time.

*1) Time-derivative of the control input $u$:* From (51) it follows that

$$\dot{u} = \bar{K}_d \dot{\bar{S}}_r \tag{59}$$

Then, from (50), equation (59) can be rewritten as follows

$$\dot{u} = \bar{K}_d \dot{\bar{S}}_q + \bar{K}_d \bar{\gamma} \operatorname{sgn}\left(\bar{S}_q\right) \tag{60}$$

but, since the sliding mode in $\bar{S}_q(t) = 0$ is enforced for all time, implies $\dot{\bar{S}}_q(t) = 0$, and therefore we conclude that

$$\dot{u} = \bar{K}_d \bar{\gamma} \operatorname{sgn}(\bar{S}_q) \tag{61}$$

## IV. Time Base Generator for finite time tracking

In order to implement the control law (51), the vectorial term $\mathcal{T}\mathcal{R}e_z$ must converge to $u$ in finite-time. This is done by assuring a finite-time convergence of attitude tracking errors through a desired rotation matrix $\mathcal{R}_d$ and thrust $\mathcal{T}_d$, computed from the control $u$. However, the result presented in Theorem 1 only provides an exponential convergence of the attitude tracking errors, and does not guarantee a finite-time convergence.

Now, a new sliding surface is proposed and parametrized by a Time Base Generator (TBG) which moves and rotates continuously the nominal sliding surface through a known, state-independent, vanishing vector to achieve finite time convergence of the tracking errors, with an arbitrary convergence time. This methodology is based on [8] and can also be applied to the position control as it will be shown in simulation results.

Consider the following first order differential equation

$$\dot{z} = -\rho(t) z \tag{62}$$

where

$$\rho(t) = \rho_0 \frac{\dot{\chi}}{(1-\chi) + \delta} \tag{63}$$

with $\rho_0 = 1 + \varepsilon$, $0 < \varepsilon \ll 1$, and $0 < \delta \ll 1$. The time base generator $\chi(t) \in C^2$ must be provided by the user so as to $\chi$ goes smoothly from 0 to 1 in finite time $t = t_b > 0$, and $\dot{\chi}(t)$ is a bell-shaped derivative of $\chi$ such that $\dot{\chi}(t_0) = \dot{\chi}(t_b) \equiv 0$. Under these conditions, a solution of (62) is given by

$$z(t) = z(t_0)\left[(1-\chi) + \delta\right]^{1+\varepsilon} \tag{64}$$

with $\rho(t_b) > 0$. Note that, $t_b$ is independent of any initial conditions and hence

$$\chi(t_b) = 1 \Rightarrow z(t_b) = z(t_0)\delta^{1+\varepsilon} > 0$$

can be made arbitrarily small in arbitrarily finite time $t_b$.

Thus, the key idea is to bring the solution of the attitude tracking errors to an equation similar to (64), defining $\rho_0 = \frac{1+\varepsilon}{q_{0e}}$. Notice that, from Theorem 1, $q_{0e}(t)$ is positive and converges to 1, so $\rho_0$ is well possed.

Consider that the sliding mode is induced on $S_q(t) = 0$, for all time, and $S_d(t) = 0$ (with a gain $k$ tuned large enough and in the sense of subsection II-F) as has been proved in Theorem 1. Then, from (36), and replacing the gain $\alpha$ by $2\rho(t)$, it results

$$\frac{d}{d\chi} q_e = -\rho_0 q_{0e} \frac{q_e}{(1-\chi) + \delta}$$

which attains the following solution $q_e(t) = q_e(t_0)\left[(1-\chi) + \delta\right]^{1+\varepsilon} = q_e(t_0)\delta^{1+\varepsilon}$ at time $t = t_b$, where by assumption $\chi(t_b) = 1$. Considering that $\delta$ and $\varepsilon$ are very small, then at $t = t_b$, tracking errors belong to a very small vicinity $\varepsilon$ of the origin, which in practice may stand for the required precision or practical zero error. Note that at $t > t_b$, the time varying feedback gain $\rho(t)$ must be reset to the desired constant $\alpha > 0$. Thus, the convergence of the attitude tracking errors is guaranteed in finite-time.

## V. Simulation Study

Two scenarios are considered to illustrate the closed-loop performance, those are tracking and cooperative cases.

It is considered a Quadrotor with $m = 500gr$, and a non diagonal inertial tensor matrix.

### A. Cooperative Quadrotors with terminal stability

The proposed controller based on TBG is implemented for both regulation and tracking tasks in cooperative Quadrotors. Desired task is set up to catch cooperatively and gently a floating ballon, located at a given cartesian position

$(1,1,1)m$ at $t=t_b$. Then, TBG is tuned to induce a convergence along a trajectory $x_d = 0.5\cos(t-5+\pi)+1, y_d = 0.5\sin(t-5+\pi)+1, z_d = 1$ in meters, whose derivative stand for the desired velocities, avoiding some collisions. Then, two Quadrotors are programmed with different initial conditions, and disturbances are introduced at $t=5s$. Noise of sensors is introduced as white noise with mean zero, see Fig. 2.

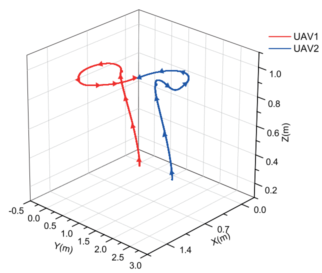


Fig. 2. Cooperative flights with terminal stability. Both Quadrotors reaches the defined spatial coordinate precisely at a given time to catch the object.

### B. Aggressive maneuvering

Looping and spiral flights are tested under ideal conditions, that is no sensor noise and no wind disturbances. Fig. 3 shows how a spiral is successfully performed departing from ground.

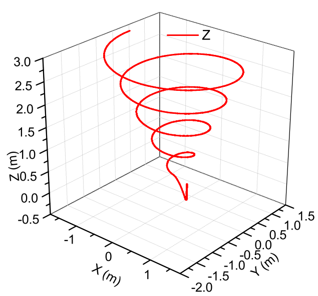


Fig. 3. Ascending helix at a frequency of $\frac{1}{2\pi}$ Hz, and increasing amplitude.

Fig. 4 indeed shows a very aggressive maneuver, a loop where the Quadrotor frame is flipping several times, due to the well-posed quaternion representation and the fast and robust controller.

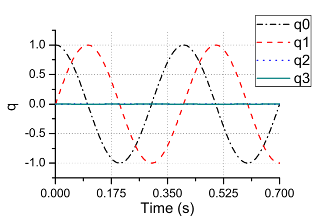


Fig. 4. Quadrotor loops. This extreme flight takes the control proposal to the limit.

### C. Discussions

Numerical simulations, based on Matlab with ODE8, fixed step at $1ms$ sampling rate, show the Quadrotor behaviors. The set of feedback gains are tuned case by case.

Aggressive maneuvers such as ascending in spiral with increasing amplitude, and loops several times, are obtained. It is clear that the tests are achieved in the ideal environment of the numerical world of Matlab, however these were obtained when the Quadrotors are subject to disturbances, sensor noise and even at different latencies.

For cooperative flight, a simple path parametrization in space-time allows two Quadrotors to catch and handle an object using the time parametrization of the proposed controller.

Finally, preliminary experiments for one Quadrotor has been obtained, however those are not shown in this paper because of space limitations to describe clearly the full experimental setting. Those are based on fast sampling rate ensured with a Vicon inertial measurement system at $75Hz$, implemented in Linux Debian platform with RTAI (video available online at `http://anand.sanchez.voila.net/`).

## VI. Conclusions

A novel quaternion-based sliding mode surface is proposed for a model-free sliding mode control of the full dynamic model of a Quadrotor. Exponential stabilization and terminal stability are achieved for a desired cooperative time, where Quadrotor catches and handles cooperatively a massless object. Stiffness control is reached at the contact points. Stability analysis establishes a robust and fast convergence error, with well-posed transformations even in aggressive maneuverings. Simulations show the capabilities of the closed-loop performance under a variety of conditions. Preliminary experiments with one Quadrotor has been obtained, but not included due to space limitations.

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