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Node Knock-out Based Structure Identification in Networks of Identical Multi-dimensional Subsystems*

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Abstract— **In this paper, based on the node knock-out procedure, for a networked system consisting of identical multi-dimensional subsystems in which the network structure is unknown and the number of input/output nodes is less than that of subsystems, we propose a novel method to identify the strength of the interaction between nodes even if information on the other nodes is not unknown.**

I. INTRODUCTION

Along with the increase of interest in analysis and control of large-scale network (NW) systems, developing methods to identify (reconstruct) the NW structures, which is completely/partially unknown, has been attracting much attention. In fact, interesting results are given in a lot of studies recently, e.g., [1]–[6]. However, many methods are on the basis of an assumption that one can manipulate or observe the state of every node [1], [2]. Since satisfying the above assumption is being difficult as the scale of the NW system is getting large, it is quite important to develop identification methods of the NW structures by relatively smaller actuators/sensors than the number of the nodes.

The identification of the NW structures belongs to a class of the identification of gray-box models, and it has some features different from that of black-box models. In particular, we encounter the problem in terms of the identifiability for gray-box models [7], [8]. For example, in [3], while a method to reconstruct NW topology, which can be applied even when some nodes cannot be controlled or observed directly, has been proposed, the class of NW topology has been restricted to the tree-type for the problem to be identifiable. In fact, Gonçalves and Warnick have shown that without prior information, one cannot uniquely reconstruct the NW structure from input/output data only for the NW system [4] (see also [5]). Therefore, we need to prepare some knowledge of the NW structure in advance, or to collect some other kinds of time-series data other than input/output data for the original NW system. In experimental biology, it is known as a technique following the later candidate that the gene-knock-out procedure works well for estimation of the connectivity between nodes. In the gene-knock-out procedure, some specified subsystems are killed (knocked out), and the NW structure is estimated by observing the difference between the original NW system and the knocked-out NW systems. Motivated by this operation, Nabi-Abdolyousefi and Mesbahi have proposed a method to identify the NW structure of a consensus-type NW system that consists of one-dimensional subsystems [6].

In this paper, we focus on a NW system that consists of identical multi-dimensional subsystems in which the dynamics of subsystems is known in advance but the NW structure is unknown. For this system, we propose a novel algorithm to identify the NW structure on the basis of the knocked-out procedure and the conventional input/output identification in the system engineering. First, we show what can be reconstructed from input/output data of the knocked-out NW systems and give some formulas. The key is by introducing the generalized frequency variable, we can treat the multi-dimensional subsystems as one-dimensional ones. Then, we demonstrate how the strength of a specified edge is estimated even if we do not know any information on the other nodes. Although we also use the same idea of knock-out technique as that of [6], the proposed method can be applied for a wider class of systems, and it provides a procedure to identify the connectivity between specified nodes.

This paper is organized as follows: In Section II, we formulate our system and identification problem. The proposed method is shown in Section III. Then, in Section IV, we confirm by a numerical simulation that the proposed method is effective. Some proofs for propositions are given in Appendix.

II. PROBLEM DESCRIPTION

Consider a NW system consisting of N identical SISO subsystems, which are denoted as Σ_i, $i \in \mathcal{I} := \{1, 2, \ldots, N\}$, and given as follows:

$$\Sigma_i : \begin{cases} \dot{x}_i = Ax_i + bv_i, \\ y_i = cx_i \end{cases}$$

where $x_i(t) \in \mathbb{R}^p$ is the state variable, $v_i(t) \in \mathbb{R}$ is the input from other nodes and the exterior of the NW system, and $y_i(t) \in \mathbb{R}$ is the output. Assume that these subsystems affect each other via a graph structure denoted by a graph Laplacian $\mathcal{L} \in \mathbb{R}^{N\times N}$. Furthermore, the input $u(t) \in \mathbb{R}^m$ to the NW system is supposed to be added to certain subsystems, which is specified by $\mathcal{B} \in \mathbb{R}^{N\times m}$, through the input ports of the subsystems. Similarly, the output $y(t) \in \mathbb{R}^r$ from the NW system is supposed to be the state of certain nodes specified

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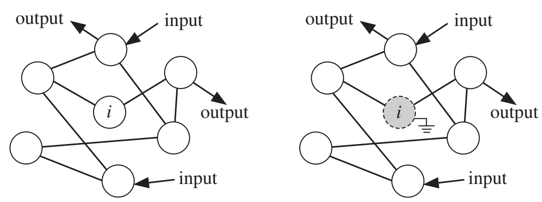

Fig. 1. A sketch of knocking out subsystem i: The left figure shows the original NW system and the right figure shows the knocked-out NW system.

$\mathcal{C} \in \mathbb{R}^{r\times N}$ through the output ports of the subsystems. Then, the whole NW system, denoted by Σ, can be represented by using the Kronecker product as follows:

$$\Sigma : \begin{cases} \dot{x} = (I_N \otimes A - \mathcal{L} \otimes bc)x + (\mathcal{B} \otimes b)u, \\ y = (\mathcal{C} \otimes c)x, \end{cases} \tag{1}$$

where $x(t) = [x_1^\top(t)\ x_2(t)^\top\ \cdots\ x_N(t)^\top]^\top \in \mathbb{R}^{Np}$.

Knocking out a node i is to keep the state variable x_i taking a stationary value (here, zero) by literally knocking out (or grounding) it (see Fig. 1). The case of knocking out multiple nodes is the same. Then, the interactions that relates to the knocked-out nodes are removed. Let $\Delta \in 2^{\mathcal{I}}$ be a set of indices of subsystems that are knocked out, and denote its cardinality by $|\Delta|$. Then, the knocked-out NW system evolves according to dynamics

$$\Sigma^\Delta : \begin{cases} \dot{x}^\Delta = (I_{N-|\Delta|} \otimes A - \mathcal{L}^\Delta \otimes bc)x^\Delta + (\mathcal{B}^\Delta \otimes b)u, \\ y = (\mathcal{C}^\Delta \otimes c)x^\Delta \end{cases}$$

where the state variable x^Δ is defined by removing the state variables of the knocked-out subsystems from the original one

$$\begin{aligned} x^\Delta(t) =& [x_1^\top(t)\ \cdots\ x_{i-1}^\top(t)\ x_{i+1}^\top(t)\ \cdots \\ & \cdots\ x_{j-1}^\top(t)\ x_{j+1}^\top(t)\ \cdots\ x_N(t)^\top]^\top,\ i, j, \ldots \in \Delta, \end{aligned}$$

and, $\mathcal{B}^\Delta$, $\mathcal{C}^\Delta$ and $\mathcal{L}^\Delta$ are matrices obtained from $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{L}$ after removing columns, rows and both columns and rows that correspond to Δ, respectively. Note that $\emptyset \in 2^{\mathcal{I}}$, and $\Sigma^\emptyset = \Sigma$. In this sense, we regard the original NW system Σ of (1) as one of knocked-out systems.

Now, we consider the following assumptions:

Assumption 1 *As a priori knowledge, we assume*

A1-1. the number of the subsystems[1] *N and the coefficients of the subsystems (A, b, c) are known,*

A1-2. $\mathcal{L}$ is symmetric, that is, the graph of the network is bidirectional, and

A1-3. the elements of $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{C}$ are unknown.

In this paper, we tackle the following problem.

Problem *Under Assumption 1, and given i, j ($\geq i$), identify the weight of edge $\mathcal{L}_{ij}$ between ith and jth subsystems from input-output relations for some of the knocked-out NW systems $\{\Sigma^\Delta\}_{\Delta\in 2^{\mathcal{I}}}$. More specifically, determine which knocked-out NW systems are needed, and give an algorithm to estimate $\mathcal{L}_{ij}$.*

[1]Here, we denote a linear system $\dot{x} = Ax + Bu$, $y = Cx$ by (A, B, C).

III. IDENTIFICATION OF THE NW STRUCTURE

Nabi-Abdolyousefi and Mesbahi proposed a method to identify the NW structure of a consensus-type NW system that consists of one-dimensional subsystems [6]. In [6], for NW system Σ with a setting $A = 0$, $b = c = 1$, $\mathcal{L} = \mathcal{L}^\top$ and $\mathcal{L} \cdot 1_{N\times 1} = 0$, and under Assumptions 1, they proposed a method to identify the NW structure $\mathcal{L}$ of the NW system $(-\mathcal{L}, \mathcal{B}, \mathcal{C})$ by deriving an estimate of a transfer function matrix $(sI_N + \mathcal{L})^{-1}$ from identification results of the knocked-out NW systems $\{\Sigma^\Delta\}$. More specifically, they utilized the following facts:

i) The ith diagonal element of $(sI_N + \mathcal{L})^{-1}$ is defined as the ratio between the determinant of $sI_N + \mathcal{L}$ and the minor determinant of $sI_N + \mathcal{L}$ from which ith column and ith row are removed. The former is the characteristic polynomial of the original NW system Σ, and the later is that of a knocked-out system $\Sigma^{\{i\}}$ in which ith subsystem is removed. If the systems Σ and $\Sigma^{\{i\}}$ are controllable and observable, their characteristic polynomials can be identified by existing methods.

ii) For off-diagonal elements, one need more information. The (i, j)-element of $(sI_N + \mathcal{L})^{-1}$ can be calculated from the characteristic polynomials of Σ, $\Sigma^{\{i\}}$, $\Sigma^{\{j\}}$ and $\Sigma^{\{i,j\}}$. This will be described in detail in the sequel.

The purpose of this paper is to extend the method in [6] so as to apply it for NW systems that consist of multi-dimensional subsystems, and to develop a procedure to estimate the connectivity between specified nodes without any information in terms of the other nodes. We do not restrict NW systems to consensus-type ones.

A. Proposed method using the generalized frequency variable

For the subsystem (A, b, c), we let $h(s)$, $d(s)$ and $n(s)$ be the transfer function, its denominator and numerator, that is,

$$\begin{aligned} h(s) &= c(sI_p - A)^{-1}b, \\ d(s) &= \det[sI_p - A], \\ n(s) &= c\,\mathrm{adj}[sI_p - A]b. \end{aligned}$$

The transfer function of th NW system Σ is equal to

$$G(s) := (\mathcal{C} \otimes c)(sI_{Np} - (I_N \otimes A - \mathcal{L} \otimes bc))^{-1}(\mathcal{B} \otimes b).$$

This can be represented in the following compact form by using the *generalized frequency variable* $\phi(s) := 1/h(s)$ [9], [10].

Theorem 1 *For NW system Σ of (1), the following holds.*

$$G(s) = \mathcal{C}(\phi(s)I_N + \mathcal{L})^{-1}\mathcal{B} \tag{2}$$

One can find the proof in [9]. Consider a system $(-\mathcal{L}, \mathcal{B}, \mathcal{C})$. Then this theorem implies that $G(s)$ is equal to a transfer function $\mathcal{C}(sI_N + \mathcal{L})^{-1}\mathcal{B} =: H(s)$ the variable s of which is replaced by the generalized frequency variable $\phi(s)$: i.e.,

$G(s) = H(\phi(s))$. Utilizing this similarity between $G(s)$ and $H(s)$ is a key point of our method.

To identify the NW structure of Σ, we give an estimate of $(\phi(s)I_N+\mathcal{L})^{-1}$. As well as the method in [6], we employ the calculation method using the adjugate of the matrix $\phi(s)I_N+\mathcal{L}$. The (i,i)-element of $(\phi(s)I_N+\mathcal{L})^{-1}$ is given by

$$[(\phi(s)I_N+\mathcal{L})^{-1}]_{ii} = \frac{\mathrm{adj}_{ii}[\phi(s)I_N+\mathcal{L}]}{\det[\phi(s)I_N+\mathcal{L}]}.$$

However, what is derived by identification from input/output data is the characteristic polynomial of the system, for example, in the case of the NW system Σ, it is $\det[sI_{Np} - (I_N \otimes A - \mathcal{L} \otimes bc)]$ rather than $\det[\phi(s)I_N+\mathcal{L}]$. Therefore, we have to clarify the relation between these determinants. In fact, the following holds.

Lemma 2 *Assume that $\mathcal{L}$ is symmetric (A1-2). Then*

$$\det[sI_{Np} - (I_N \otimes A - \mathcal{L} \otimes bc)] = n(s)^N \det[\phi(s)I_N+\mathcal{L}]. \tag{3}$$

The proof is given in Appendix I-A. From this lemma, we derive the following immediately.

Proposition 3 *Assume that $\mathcal{L}$ is symmetric (A1-2). Then*

$$\left[(\phi(s)I_N+\mathcal{L})^{-1}\right]_{ii} = \frac{n(s)\det\left[sI_{(N-1)p} - (I_{N-1}\otimes A - \mathcal{L}^{\{i\}}\otimes bc)\right]}{\det[sI_{Np} - (I_N \otimes A - \mathcal{L}\otimes bc)]}. \tag{4}$$

Remark 4 *A notable feature of (4) is that the polynomial $n(s)$ appears in the numerator, which can be regarded as a factor to fill a gap between the characteristic polynomial and the corresponding determinant in the form using the generalized frequency variable. In the case in which the subsystem is one-dimensional and $(A,b,c) = (0,1,1)$, we have $n(s) = 1$, and hence, this foctor does not have to taken into account.*

Next, we consider the off-diagonal elements of $(\phi(s)I_N+\mathcal{L})^{-1}$. The calculation is rather intricate. The following formula is proved by means of a formula in terms of minors of inverse matrices and Eq. (3), (4) (see Appendix I-B).

Proposition 5 *Assume that $\mathcal{L}$ is symmetric (A1-2). Then*

$$[(\phi(s)I_N+\mathcal{L})^{-1}]_{ij} = \frac{n(s)\hat{\Psi}_{ij}(s)}{\det[sI_{Np} - (I_N\otimes A - \mathcal{L}\otimes bc)]} \tag{5}$$

where

$$\begin{aligned}\hat{\Psi}_{ij}(s) = \big(&\det[sI_{(N-1)p} - (I_{N-1}\otimes A - \mathcal{L}^{\{i\}}\otimes bc)] \\ &\cdot\det[sI_{(N-1)p} - (I_{N-1}\otimes A - \mathcal{L}^{\{j\}}\otimes bc)] \\ &-\det[sI_{Np} - (I_N\otimes A - \mathcal{L}\otimes bc)] \\ &\cdot\det[sI_{(N-2)p} - (I_{N-2}\otimes A - \mathcal{L}^{\{i,j\}}\otimes bc)]\big)^{\frac{1}{2}}.\end{aligned} \tag{6}$$

Thus, to calculate (5), one needs the characteristic polynomials of four NW systems; the original NW system Σ, knocked-out NW systems $\Sigma^{\{i\}}$ and $\Sigma^{\{j\}}$ where ith and jth subsystems are removed, respectively, and a knocked-out system $\Sigma^{\{i,j\}}$ where both ith and jth subsystems are removed. As well as the case of the diagonal elements, there exists $n(s)$ in the numerator of (5).

B. Proposed method: Identification of a specified edge

Suppose that a given rational function matrix $F(s)$ satisfies

$$(\phi(s)I_N+\mathcal{L})^{-1} = F(s), \text{ for each } s. \tag{7}$$

Then, substituting some value into the variable s, $\mathcal{L}$ can be calculated: For example, letting $s = 0$, we have

$$\mathcal{L} = F(0)^{-1} - \phi(0)I_N.$$

Since this calculation takes the inverse of the matrix, all elements in $F(s)$ are used in general. This means that identification of all elements in $(\phi(s)I_N+\mathcal{L})^{-1}$ is needed. In fact, we often encounter the situations in which the identification of the whole NW structure is not necessary: For example, we just want to estimate the strength of a specified edge while we do not have any information of nodes around the corresponding node. In such cases, it is meaningful to identify specified objects by as few procedures as possible. From now, we show that the edge $\mathcal{L}_{ij}$ between subsystems i and j can be calculated only from the identified (i,j)-element of $(\phi(s)I_N+\mathcal{L})^{-1}$.

Assume that the transfer function $h(s)$ of the subsystems $\{\Sigma_i\}$ is given as follows:

$$h(s) = \frac{\sum_{k=0}^{q}\beta_k s^k}{\sum_{k=0}^{p}\alpha_k s^k},$$

where $\alpha_0 = 1$. Denote the relative degree of $h(s)$ by r, i.e., $r = p - q$. Then the following holds.

Proposition 6 *Assume that a rational function matrix $F(s)$ satisfies (7). Then we derive $\mathcal{L}$ as follows:*

$$\mathcal{L} = \frac{\xi_r I_N - \lim_{s\to\infty}[sH_r(s)]}{\beta_q^2} \tag{8}$$

where

$$H_n(s) = \begin{cases} s^{r-1}F(s), & n = 0, \\ sH_{n-1}(s) - \xi_{n-1}I_N, & n = 1,\dots,r, \end{cases} \tag{9}$$

$$\xi_n = \begin{cases} \beta_q, & n = 0, \\ \beta_{q-n} - \sum_{k=0}^{n-1}\alpha_{p-n+k}\xi_k, & n = 1,\dots,r \end{cases}$$

and β_l ($l < 0$) takes zero when $r > q$.

Proof: Applying (9) for $F(s) = (\phi(s)I_N+\mathcal{L})^{-1}$, we have

$$H_n(s) = (\kappa_n(s)I_N + \theta_n(s)\mathcal{L})(d(s)I_N + n(s)\mathcal{L})^{-1}, \quad n = 0,\dots,r$$

where

$$\begin{cases} \kappa_n(s) = \xi_n s^{p-1} + \sum_{k=0}^{p-2} \text{(some constant)}\cdot s^k, \\ \theta_n(s) = -\beta_q^2 s^{q+n-1} + \sum_{k=0}^{q+n-2} \text{(some constant)}\cdot s^k. \end{cases}$$

While $\kappa_n(s)$ is $(p-1)$th-order polynomial unless $\xi_n = 0$, the order of $\theta_n(s)$ is $q+n-1$ and, in particular, it is $p-1$ for $n = r$. By taking account in the fact the coefficient of the maximum order term of $d(s)$ is one (i.e., $\alpha_0 = 1$, it turns out

$$\begin{aligned}&\lim_{s\to\infty} sH_r(s)\\&= \lim_{s\to\infty} (\underbrace{s\kappa_r(s)}_{\mathcal{O}(p)} I_N + \underbrace{s\theta_r(s)}_{\mathcal{O}(p)} \mathcal{L})(\underbrace{d(s)}_{\mathcal{O}(p)} I_N + \underbrace{n(s)}_{\mathcal{O}(q)} \mathcal{L})^{-1}\\&= \xi_r I_N - \beta_q^2 \mathcal{L}.\end{aligned}$$

Eq. (8) is immediately derived. ■

The calculations of (8) and the recurrence equations (9) are done by element-wise operations. Therefore, in calculation of $\mathcal{L}_{ij}$, we use only $F_{ij}(s)$ in addition to the information of the subsystems, $\{\alpha_k\}$, $\{\beta_k\}$. In particular, for the off-diagonal elements, we can show the next proposition.

Proposition 7 *Assume that a rational function matrix $F(s)$ satisfies (7), and $i < j$. Then the relative degree of $F_{ij}(s)$ is equal to or greater than $2r$. Furthermore, the following holds:*

$$\mathcal{L}_{ij} = \begin{cases} -\mu_{ij}/\beta_q^2, & \text{when } \theta_{ij} = 2r, \\ 0, & \text{elsewhere} \end{cases}$$

where μ_{ij} is the coefficient of the highest order term in the numerator of $F_{ij}(s)$, and θ_{ij} is the relative degree of $F_{ij}(s)$.

Proof: First, we show that the relative degree of $F_{ij}(s)$ is equal to or greater than $2r$. From Eq. (13) in Appendix, we have

$$F_{ij}(s) = \frac{n(s)\hat{\Psi}_{ij}(s)}{\det[d(s)I_N + n(s)\mathcal{L}]}$$

where

$$\begin{aligned}&\hat{\Psi}_{ij}(s)\\&= \Big(\det\left[d(s)I_{N-1}+n(s)\mathcal{L}^{\{i\}}\right]\det\left[d(s)I_{N-1}+n(s)\mathcal{L}^{\{j\}}\right]\\&\quad -\det[d(s)I_N + n(s)\mathcal{L}]\det\left[d(s)I_{N-2} + n(s)\mathcal{L}^{\{i,j\}}\right]\Big)^{\frac{1}{2}}.\end{aligned}$$

Since we are treating a linear system, $F_{ij}(s)$ must be a rational function, so must $\hat{\Psi}_{ij}(s)$ (in fact, $\hat{\Psi}_{ij}(s)$ is a polynomial.). In addition, it turns out that the highest order of $\hat{\Psi}_{ij}(s)$ is equal to or less than $p(N-2)+q$. Then we find that the order of the numerator of $F_{ij}(s)$ is equal to or less than $p(N-2)+2q$ while that of its denominator is Np. Consequently, the first part of the proposition follows.

In calculation of (8) and (9), the auxiliary parameters $\{\xi_n\}$ relate only with the diagonal elements of $F(s)$, and hence, the calculation for the off-diagonal elements is quite simple as follows:

$$\mathcal{L}_{ij} = -\frac{\lim_{s\to\infty} s^{2r}F_{ij}(s)}{\beta_q^2}.$$

Since

$$F_{ij}(s) = \frac{\mu_{ij}s^{Np-\theta_{ij}} + \mathcal{O}(Np-\theta_{ij}-1)}{s^{Np} + \mathcal{O}(Np-1)}$$

and $\theta_{ij} \geq 2r$, the second part of the proposition follows. ■

From the argument in the proof, by letting ν_{ij} be the coefficient of the $(2p(N-2)+2q)$th term of $\hat{\Psi}_{ij}(s)^2$, $\mathcal{L}_{ij}$ can be represented as follows:

$$\mathcal{L}_{ij} = -\frac{\sqrt{\nu_{ij}}}{\beta_q}. \tag{10}$$

This can be used for a estimation of the edge in the case in which completing square in the square root cannot be done due to error coming of identification results.

We summarize the proposed algorithm as follows:

Algorithm *From the knowledge of the subsystems (A, b, c), calculate its transfer function $h(s) = n(s)/d(s)$.*
(Case $i = j$)

d1. For the NW system Σ and the knocked-out NW system $\Sigma^{\{i\}}$, identify the characteristic polynomials from the input/output relations, and denote them by $\tilde{p}(s)$ and $\tilde{p}_i(s)$, respectively.
d2. According to (4), calculate $n(s)\tilde{p}_i(s)/\tilde{p}(s)$.
d3. Then, regard the above calculation result as $F_{ij}(s)$, and apply (8).

(Case $i \neq j$)

o1. For the NW system Σ and the knocked-out systems $\Sigma_{\{i\}}$, $\Sigma_{\{j\}}$ and $\Sigma_{\{1,j\}}$, identify the characteristic polynomials from the input/output relations, and denote them by $\tilde{p}(s)$, $\tilde{p}_i(s)$, $\tilde{p}_j(s)$ and $\tilde{p}_{ij}(s)$, respectively.
o2. According to (6), calculate $\tilde{p}_i(s)\tilde{p}_j(s) - \tilde{p}(s)\tilde{p}_{ij}(s)$. Then, let the coefficient of the term $s^{2p(N-2)+2q}$ be $\tilde{\nu}_{ij}$.
o3. According to (10), calculate $-\sqrt{\tilde{\nu}_{ij}}/\beta_q$.

Thus we need two/four NW systems (experiments) to identify the weight of a specific edge.

Remark 8 *In applying input/output identification methods to estimate characteristic polynomials, it should be noted that the derived estimates may depend on the input and the identification method that one uses. If the NW system to be identified is controllable and observable, consistent estimates can be derived for appropriate inputs (e.g., M-sequence). However, it is sometimes difficult for, in particular, large-scale systems to satisfy the controllability/observability, and moreover, these properties cannot be checked in advance since $\mathcal{L}$, $\mathcal{B}$, $\mathcal{C}$ are unknown. It is one of the future works to study on what information can be reconstructed in the case in which controllable/observable are lost.*

IV. Numerical Example

In this section, we show a numerical example to outline to the proposed method. Consider a NW system that consists of five identical subsystems whose coefficients are given as

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

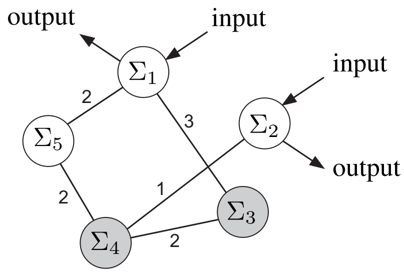

Fig. 2. A Network system consisting of five identical subsystems.

via the following graph structure and input/output matrices (see Fig. 2):

$$\mathcal{L} = \begin{bmatrix} 5 & 0 & -3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 \\ -3 & 0 & 5 & -2 & 0 \\ 0 & -1 & -2 & 5 & -2 \\ -2 & 0 & 0 & -2 & 4 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top, \ \mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We have $h(s) = (s+4)/(s^2+3s+2)$.

Suppose that we have the knowledge of the subsystems (A, b, c) but the NW structure $(\mathcal{L}, \mathcal{B}, \mathcal{C})$ is unknown. Although the controllability/observability for the NW system cannot be checked in advance, one can validate them from the identification results (In fact, the NW systems and the knocked-out NW systems used below were controllable and observable this time.). Now, we try to identify the edge between the subsystems 3 and 4 (the true value is $\mathcal{L}_{34} = -2$.). For this, we employ Algorithm-(Case $i \neq j$). First, identify the characteristic polynomials of the four NW systems Σ, $\Sigma^{\{3\}}$, $\Sigma^{\{4\}}$ and $\Sigma^{\{3,4\}}$. Here, we used the subspace identification method [11]. As a result, we derived

$$\begin{aligned} \tilde{p}(s) &= s^{10} + 35s^9 + 552s^8 + 5162s^7 + 31641s^6 \\ &\quad + 132480s^5 + 382470s^4 + 748710s^3 \\ &\quad + 946110s^2 + 689200s + 214820, \\ \tilde{p}_3(s) &= s^8 + 27s^7 + 327s^6 + 2303s^5 + 10300s^4 \\ &\quad + 29962s^3 + 55588s^2 + 60424s + 30064, \\ \tilde{p}_4(s) &= s^8 + 27s^7 + 323s^6 + 2235s^5 + 9812s^4 \\ &\quad + 28082s^3 + 51572s^2 + 56072s + 28272, \\ \tilde{p}_{34}(s) &= s^6 + 19s^5 + 158s^4 + 724s^3 + 1948s^2 \\ &\quad + 2924s + 1992. \end{aligned}$$

Then, calculating $\tilde{p}_3(s)\tilde{p}_4(s) - \tilde{p}(s)\tilde{p}_{34}(s)$, we have

$$(2.43 \times 10^{-9})s^{15} + (4.0 + 1.24 \times 10^{-7})s^{14} + \mathcal{O}(13).$$

Since $2p(N-2) + 2q = 14$, we let $\tilde{\nu}_{34} = 4.0 + 1.24 \times 10^{-7}$. The existence of 15th-order term is thought to be due to error coming of the input/output identifications, and so we ignore this term. Consequently, applying (10) with $\beta_q = 1$, we get an estimate $\mathcal{L}_{34} \approx -2.0 + 3.10 \times 10^{-8}$.

V. Conclusions

In this paper, for a NW system that consists of identical multi-dimensional systems, we proposed a method to identify the NW structure. The proposed method is based on the knocked-out scheme with input/output identification. We gave some formulas and showed polynomials that can be estimated from the input/output identification for the knocked-out NW systems. Moreover, we proposed a novel procedure to identify a specified unknown edge between two subsystems even if we do not have any information in terms of the other nodes. We also showed by a numerical example that the proposed method is effective.

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Appendix I
Proofs

A. *The proof of Lemma 2*

Proof: We use formulas about the Kronecker product

$$\begin{aligned} &(MN) \otimes Q = (M \otimes Q)(N \otimes I_l), \\ &Q \otimes (MN) = (I_q \otimes M)(Q \otimes N), \\ &X \otimes Y = (X \otimes I_n)(I_m \otimes Y) = (I_m \otimes X)(Y \otimes I_n) \end{aligned}$$

where $M \in \mathbb{R}^{m\times n}, N \in \mathbb{R}^{n\times k}, Q \in \mathbb{R}^{q\times l}, X \in \mathbb{R}^{m\times m}, Y \in \mathbb{R}^{n\times n}$, and a formula about determinant

$$\begin{aligned}&\det[sI_n - (A+BC)]\\&= \det[I_m - C(sI-A)^{-1}B]\det[sI_n - A]\end{aligned} \quad (11)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ [12], [13].

Let T be a diagonalization matrix for $\mathcal{L}$ (i.e., $\Lambda := T\mathcal{L}T^{-1}$). Then we can calculate the left hand side of (3) as follows:

$$\begin{aligned}\text{LHS} &= \det[sI_{Np} - \{(T^{-1}T)\otimes A - T^{-1}\Lambda T \otimes bc\}]\\&= \det[sI_{Np} - \{(T^{-1}\otimes I_p)(I_N \otimes A)(T\otimes I_p)\\&\qquad - (T^{-1}\otimes I_p)(\Lambda\otimes bc)(T\otimes I_p)\}]\\&= \det[sI_{Np}\\&\qquad - (T^{-1}\otimes I_p)\{I_N\otimes A - (\Lambda\otimes bc)\}(T\otimes I_p)]\\&= \det[sI_{Np} - (I_N\otimes A - \Lambda\otimes bc)]\\&\qquad \big(\text{because } (T^{-1}\otimes I_p)(T\otimes I_p) = I_{Np}\big)\\&= \det[\text{diag}\{sI_p - (A-\lambda_i bc)\}_{i=1}^N]\\&= \prod_{i=1}^N \det[sI_p - (A-\lambda_i bc)]\\&= \prod_{i=1}^N (1+\lambda_i c(sI_p - A)^{-1}b)\det[sI_p - A]\\&\qquad \big(\text{where we use (11)}\big)\\&= \prod_{i=1}^N (d(s)+\lambda_i n(s)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\text{RHS} &= \det[d(s)I_N + n(s)\mathcal{L}]\\&= \det[d(s)I_N + n(s)\Lambda]\\&= \det[\text{diag}\{d(s)+n(s)\lambda_i\}_{i=1}^N]\\&= \prod_{i=1}^N (d(s)+n(s)\lambda_i).\end{aligned}$$

Then, the lemma follows. ■

B. The proof of Proposition 5

Proof: For a $n\times n$ matrix M and integers $i,j \in \{1,\ldots,n\}$ $(i<j)$, We introduce the following notation:

$$M_{(i,j)} := \begin{bmatrix} M_{ii} & M_{ij}\\ M_{ji} & M_{jj}\end{bmatrix}.$$

Then, the following holds:

$$\det\left[(\phi(s)I_N + \mathcal{L})^{-1}\right]_{(i,j)} = \frac{\det\left[\phi(s)I_{N-2} + \mathcal{L}^{\{i,j\}}\right]}{\det[\phi(s)I_N + \mathcal{L}]}. \quad (12)$$

One can confirm this by taking account in

$$\text{LHS} = \frac{\begin{aligned}&\text{adj}_{ii}[\phi(s)I_N + \mathcal{L}]\,\text{adj}_{jj}[\phi(s)I_N + \mathcal{L}]\\&\quad - \text{adj}_{ij}[\phi(s)I_N + \mathcal{L}]\,\text{adj}_{ji}[\phi(s)I_N + \mathcal{L}]\end{aligned}}{(\det[\phi(s)I_N + \mathcal{L}])^2}$$

and repeating elementary operations on th matrix with respect to its column and row for the numerator.

Meanwhile, since

$$\begin{aligned}&\det[(\phi(s)I_N + \mathcal{L})^{-1}]_{(i,j)}\\&= [(\phi(s)I_N + \mathcal{L})^{-1}]_{ii}[(\phi(s)I_N + \mathcal{L})^{-1}]_{jj}\\&\quad - ([(\phi(s)I_N + \mathcal{L})^{-1}]_{ij})^2,\end{aligned}$$

using (4) and (12), we derive

$$\begin{aligned}&([(\phi(s)I_N + \mathcal{L})^{-1}]_{ij})^2\\&= [(\phi(s)I_N + \mathcal{L})^{-1}]_{ii}[(\phi(s)I_N + \mathcal{L})^{-1}]_{jj}\\&\quad - \frac{\det[\phi(s)I_{N-2} + \mathcal{L}^{\{i,j\}}]}{\det[\phi(s)I_N + \mathcal{L}]}\\&= \frac{\det[\phi(s)I_{N-1} + \mathcal{L}^{\{i\}}]\det[\phi(s)I_{N-1} + \mathcal{L}^{\{j\}}]}{(\det[\phi(s)I_N + \mathcal{L}])^2}\\&\quad - \frac{\det[\phi(s)I_{N-2} + \mathcal{L}^{\{i,j\}}]}{\det[\phi(s)I_N + \mathcal{L}]}.\end{aligned}$$

Therefore, the followings holds:

$$[(\phi(s)I_N + \mathcal{L})^{-1}]_{ij} = \frac{\Psi_{ij}(s)}{\det[\phi(s)I_N + \mathcal{L}]} \quad (13)$$

where

$$\begin{aligned}\Psi_{ij}(s) = \Big(&\det\left[\phi(s)I_{N-1} + \mathcal{L}^{\{i\}}\right]\det\left[\phi(s)I_{N-1} + \mathcal{L}^{\{j\}}\right]\\&- \det[\phi(s)I_N + \mathcal{L}]\det\left[\phi(s)I_{N-2} + \mathcal{L}^{\{i,j\}}\right]\Big)^{\frac{1}{2}}.\end{aligned}$$

Finally, applying (2), we get Proposition 5. ■