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A discrete time-varying internal model based approach for high precision tracking

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Abstract— **In this paper, we consider the discrete time case of a time-varying internal model-based control design for high precision tracking of frequency-varying reference trajectories. Thanks to a recently proposed parallel time-varying internal model structure, the asymptotic tracking conditions for the design of internal model units are developed, and a low order robust stabilizer is synthesized. In a discrete time setting, the high precision tracking control architecture is deployed on a Voice Coil Motor (VCM) actuated servo gantry system, where numerical simulation and real time experimental results are given to validate the proposed method.**

I. INTRODUCTION

One of the central topics in high precision mechatronics is trajectory tracking, which has important applications to a large class of high precision manipulations [2], such as assembly, IC chip inspection, track seeking of HDDs [11]. Lots of efforts have been devoted to various aspects of tracking control theory in the past several decades (see, for example [5], [17], [20], [7], [21], [14]). One of the most investigated approaches is internal model-based control design, which has emerged as a fundamental technique for tracking and/or rejecting periodic signals generated by automatous systems.

The internal model-based control theory for LTI systems has been well established with a complete solution, see [5]. However the available results in the literature cannot be applied to the case where the frequencies of the periodic reference are time varying, due to the fundamental challenges of the construction of a time-varying internal model which renders the error-zeroing subspace invariant, and a robust time-varying stabilizer which keeps the augmented time-varying system asymptomatically stable. We refer to [20], [7] for some recent advances of internal-model based design for LTV systems.

It is worth mentioning that a systematic design method for the construction of time-varying internal model has been proposed in [16], [21] in both input/output and state-space representations. However the implementations of the above algorithms are still complicated, because the robust and low-order stabilizer design has not been developed. Notice that some attempts have been made via LPV (Linear Parameter Varying) design approach, for example [22], [14], [23] in continuous time settings. In order to make the control architecture more implementable for the tracking of sophisticated periodic signals in reality, it is desirable to cast the internal-model based control framework in a discrete time setting, which would greatly reduce the computational burdens and avoid numerical issues. Very recently a discrete tracking controller design has been proposed in [15], which is an extension of the results in continuous time settings [14].

Based on the recently developed parallel structure for time varying internal model-based design in [23], we investigate in this paper the time-varying internal model-based design in a discrete time fashion, which can be considered as the counter part of the continuous time domain results in [23]. The tracking algorithm is considered for a high precision X-Y servo gantry with linear VCM motors, which represents many important industrial applications such as laser beam steer, PCB laser marking, and advanced imaging systems [12].

The rest of the paper is organized as follows: in Section II, we give the problem formulation of tracking. In Section III, we briefly recall the results of time-varying internal model design. In Section IV, we investigate the the discrete robust stabilizer design based on the parallel connection with the internal model unit. The simulation and experimental results for the servo platform are given in Section V to demonstrate the proposed control algorithm in discrete time followed by conclusions given in VI.

II. PROBLEM FORMULATION

In this paper we consider the SISO discrete LTI plant models driven by a discrete LTV exosystem in the following form

$$\begin{aligned} w(k+1) &= S(k)w(k) \\ r(k) &= Q(k)w(k) \end{aligned} \tag{1}$$

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k) \\ e(k) &= y(k)+r(k) \end{aligned} \tag{2}$$

with exogenous state $w \in \mathbb{R}^{\rho}$, plant state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and regulated error $e \in \mathbb{R}$.

The interconnection of system (1)–(2) represents a typical tracking problem of letting actuator systems track or reject angle depended or linear motion dependent signals. Specifically, the signal $r(k)$ is periodic with respect to angular position or linear position, but not periodic in temporal variable as (angular) velocity varies in real time. Detailed explanation of generating angle-dependent signals can be referred to [22].

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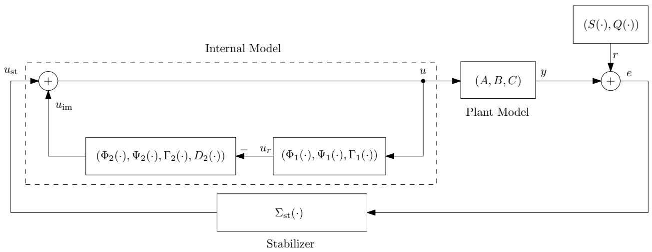

Fig. 1. Block diagram of a parallel connected error-feedback compensator.

To begin with, we make the standing assumptions below:

Assumption 2.1: The trajectory $w(k)$ in forward and backward directions of time are stable in the sense of Lyapunov, and the pair $(Q(\cdot), S(\cdot))$ is uniformly observable.

Assumption 2.2: The triplet (A, B, C) is controllable and observable.

Problem 2.1: The tracking problem under consideration is to find an error feedback controller of the form

$$\begin{aligned} \xi(k) &= G(k)\xi(k) + F(k)e(k) \\ u(k) &= H(k)\xi(k) + K(k)e(k) \end{aligned} \tag{3}$$

with state $\xi \in \mathbb{R}^{n_\xi}$ such that the following conditions hold:

1) The origin of closed-loop unforced system ($w(k) = 0$) is a uniformly asymptotically stable equilibrium.
2) The trajectories of closed-loop system (1)–(2)–(3) are all bounded and satisfies that

$$\lim_{k\to\infty} |e(k)| = 0 \,.$$

III. PRELIMINARIES OF TIME-VARYING INTERNAL MODEL

It is shown in [16], [21], the construction of the time-varying internal model unit can be designed by two steps: the first one is to immerse exogenous signal r in the place of u_r (see Figure 1), the second one is to make the I/O mappings between system $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ and the plant model the same.

In this work, we consider the counter part of [23], where in the compensator the internal model unit and the stabilizer are connected in parallel (see Figure 1). It is readily seen an internal model unit admits the following form

$$\begin{aligned} \textit{Subsystem 1:}\ \xi_1(k+1) &= \Phi_1(k)\xi_1(k) + \Psi_1(k)u(k) \\ u_r(k) &= \Gamma_1(k)\xi_1(k) \end{aligned} \tag{4}$$

and *Subsystem 2:*

$$\begin{aligned} \xi_2(k+1) &= \Phi_2(k)\xi_2(k) + \Psi_2(k)(-u_r(k)) \\ u_{\mathrm{im}}(k) &= \Gamma_2(k)\xi_2(k) + D_2(k)(-u_r(k)) \end{aligned} \tag{5}$$

with the state $(\xi_1,\ \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $u_r \in \mathbb{R}$, and $u_{\mathrm{im}} \in \mathbb{R}$.

Remark 3.1: The advantage of the parallel connection between the internal model unit and the stabilizer is to simplify the stabilization design, which is explained in [23].

We recall the results for the design of the time-varying internal model unit. The detailed design of system (4)– (5) can be referred to [21], where 1) by solving an algebraic Silvester equation, the signal r is embedded in the place of u_r; 2) by choosing the nominal values of the plant model as the subsystem $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$, the exosystem with the required error zeroing input u_{ff} , i.e.,

$$\begin{aligned} w(k+1) &= S(k)w(k) \\ u_{\mathrm{ff}}(k) &= R(k)w(k) \end{aligned} \tag{6}$$

is immersed [9], [16], [21], into

$$\begin{aligned} \begin{pmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{pmatrix} &= \Phi(k) \begin{pmatrix} \xi_1(k) \\ \xi_2(k) \end{pmatrix} \\ u_{\mathrm{im}}(k) &= \Gamma(k) \begin{pmatrix} \xi_1(k) \\ \xi_2(k) \end{pmatrix} , \end{aligned} \tag{7}$$

where

$$\Phi(k) = \begin{pmatrix} \Phi_1(k) - \Psi_1(k)D_2(k)\Gamma_1(k) & \Psi_1(k)\Gamma_2(k) \\ -\Psi_2(k)\Gamma_1(k) & \Phi_2(k) \end{pmatrix} ,$$

$$\Gamma(k) = \begin{pmatrix} -D_2(k)\Gamma_1(k) & \Gamma_2(k) \end{pmatrix} ,$$

and u_{im} is the desired input to keep error $e(k) = 0$ (see [21] in detail). A simplest choice of such $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ is to set

$$(\Phi_1(\cdot)\,,\ \Psi_1(\cdot)\,,\ \Gamma_1(\cdot)) = (A_{\mathrm{o}}\,,\ B_{\mathrm{o}}\,,\ C_{\mathrm{o}})\,,$$

where the triplet $(A_{\mathrm{o}}\,,\ B_{\mathrm{o}}\,,\ C_{\mathrm{o}})$ is the observer canonical realization of $(A\,,\ B\,,\ C)$.

IV. DISCRETE TIME ROBUST STABILIZER

With the internal model unit available, the task left is to design a time-varying stabilizer. In what follows, we show how to design a low order time-varying robust stabilizer such that the augmented plant and internal model is asymptotically stable. By design of the internal model 1 ($\xi_1(k+1) = A_{\mathrm{o}}\xi_1(k) + B_{\mathrm{o}}u(k)$), it is sufficient to consider the stabilization of the augmented internal model 2 and plant model (2), by assuming that

Assumption 4.1: Suppose that system $x(k+1) = A_{\mathrm{o}}x(k)$ is asymptotically stable.

If not, one can stabilize it first. The augmented system to be stabilized reads as

$$\begin{pmatrix} \xi_2(k+1) \\ x(k+1) \end{pmatrix} = \begin{pmatrix} \Phi_2(k) & -\Psi_2(k)C_{\mathrm{o}} \\ B_{\mathrm{o}}\Gamma_2(k) & A_{\mathrm{o}} - B_{\mathrm{o}}D_2(k)C_{\mathrm{o}} \end{pmatrix} \begin{pmatrix} \xi_2(k) \\ x(k) \end{pmatrix} + \begin{pmatrix} 0 \\ B_{\mathrm{o}} \end{pmatrix} u_{\mathrm{st}} \tag{8}$$

where quadruplet $(\Phi_2(\cdot), \Psi_2(\cdot), \Gamma_2(\cdot))$ is in controller canonical form by design.

Notice that the pair

$$\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} ,\ \begin{pmatrix} \Phi_2(\cdot) & -\Psi_2(\cdot)C_{\mathrm{o}} \\ B_{\mathrm{o}}\Gamma_2(\cdot) & A_{\mathrm{o}} - B_{\mathrm{o}}D_2(\cdot)C_{\mathrm{o}} \end{pmatrix} \right) \tag{9}$$

is uniformly observable due to the lower triangular form by design of $(\Phi_2(\cdot), \Psi_2(\cdot), \Gamma_2(\cdot), D_2(\cdot))$.

System (8) with output in (9) can be transformed to the observer canonical form as

$$\begin{aligned} x_{\rm a}(k+1) &= A_{\rm a}(k)x_{\rm a}(k)+B_{\rm a}(k)u_{\rm st}(k) \\ v(k) &= \begin{pmatrix}1 & 0\cdots & 0\end{pmatrix} x_{\rm a}(k)\,, \end{aligned} \tag{10}$$

where v is the first state of ξ_2

$$A_{\rm a}(k)=\begin{pmatrix} -\mathcal{O}_S(k)\begin{pmatrix}1\\ q(k)\end{pmatrix} & \begin{matrix} I\\ 0\end{matrix}\end{pmatrix}\,,\; B_{\rm a}(k)=\mathcal{C}_B\begin{pmatrix}1\\ q(k)\end{pmatrix}$$

and the definitions of $\mathcal{O}_S(t)$ and $\mathcal{C}_B$ are referred to [21] and omitted due to the space limit, while an illustrative example is given in Section V-A for the reader's convenience.

Also note that

$$\begin{aligned} \xi_2(k+1) &= \Phi_2(k)\xi_2(k)+\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} y(k) \\ v(k) &= \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}\xi_2(k)\,. \end{aligned} \tag{11}$$

With equation (11), system (10) can be reduced to

$$\begin{aligned} x_{\rm o}(k+1) &= A(k)x_{\rm o}+B(k)u_{\rm st}(k) \\ y(k) &= \begin{pmatrix}1 & 0\cdots & 0\end{pmatrix} x_{\rm o}(k)\,, \end{aligned} \tag{12}$$

where

$$A(k)=\begin{pmatrix} -\alpha(k) & \begin{matrix} I\\ 0\end{matrix}\end{pmatrix}\,, \quad B(k)=B\,,$$

with $\alpha(t)$ collecting the coefficients of the first column of exosystem $S(k)$ in observer canonical form. The stabilization of system (8) can be converted to the stabilization of system (12), and system (12) can be split as

$$\begin{aligned} x_1(k+1) &= A_{11}(k)x_1(k)+A_{12}x_b(k)+B_1u_{\rm st}(k) \\ x_b(k+1) &= A_{21}(k)x_1(k)+A_{22}x_b(k)+B_2u_{\rm st}(k) \\ y(k) &= x_1(k)\,, \end{aligned} \tag{13}$$

where x_1 is the first state of $x_{\rm o}$, and x_b collects the rest of $x_{\rm o}$, and

$$A_{11}(k)=-\alpha_{\rho-1}(k)\,,\quad A_{12}=\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}\,,$$

$$A_{21}(k)=\begin{pmatrix}-\alpha_{\rho-2}(t)\\ \vdots\\ -\alpha_0(t)\end{pmatrix}\,,\quad A_{22}=\begin{pmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{pmatrix}\,,$$

$$B_1=b_{\rho-1}\,,\quad B_2=\begin{pmatrix}b_{\rho-2} & \cdots & b_0\end{pmatrix}'\,.$$

with $\alpha_i(k)$ the coefficients of exosystem $S_{\rm o}(k)$ in observer canonical form.

Now we have converted the augmented system to a relative low order system (13) which also admits a special structure. Note that y is the only information available for feedback, we introduce a reduced observer of the state x_b below

$$\begin{aligned} \hat{z}(k+1) &= (A_{22}-HA_{12})\hat{z}(k)+(B_2-HB_1)u_{\rm st}(k) \\ &\quad +((A_{22}-HA_{12})H+A_{21}-HA_{11}(k))y(k) \\ \hat{x}_b(k+1) &= \hat{z}(k)+Hy(k)\,, \end{aligned} \tag{14}$$

with $z=x_b-Hx_1$ and $\hat{z}=\hat{x}_b-Hx_1$.

It is worth noting that the output injection gain can be chosen as time-invariant vector by leveraging on the observable pair $(A_{12}\,,\;A_{22})$. With the estimation of state x_b, we can further apply control

$$u_{\rm st}=K(k)\begin{pmatrix}x_1\\ \hat{x}_b\end{pmatrix}$$

to stabilize system (13). Then the left is to make the closed-loop system

$$x_{\rm o}(k+1)=(A(k)+BK(k))x_{\rm o}(k) \tag{15}$$

asymptotically stable.

We consider a Lyapunov function based method to design $K(k)$ such that system (15) is asymptotically stable and robust with respect to uncertainties in $A(k)$.

Note that so far the design is straightforward from the counter part of the continuous setting in [23]. In what follows, we discuss the main challenge for discrete time-varying stabilization design. The feedback gain $K(k)$ can be designed by finding a symmetric matrix $P(k)>0$ satisfying

$$A_{\rm c}'(k)P(k+1)A_{\rm c}(k)-P(k)<0\,, \tag{16}$$

with $A_{\rm c}(k)=A(k)+BK(k)$.

Notice that the inequality in the above form cannot be solved in practice, as the crossing term $B(k)K(k)P(k+1)$ involves $k+1$ index. Removing $(k+1)$-index in matrix P is main result in this work. To make the presentation streamlined, we dropped index k, but keep $k+1$ in those related matrices.

Apply Schur complement to inequality (16), which yields

$$\begin{pmatrix}P & A_{\rm c}'\\ A_{\rm c} & P^{-1}(k+1)\end{pmatrix}>0\,. \tag{17}$$

By setting $Q=P^{-1}$, matrix inequality (17) reads

$$\begin{pmatrix}Q & (A_{\rm c}Q)'\\ A_{\rm c}Q & Q(k+1)\end{pmatrix}>0\,. \tag{18}$$

Now the difficulty turns to remove $Q(k+1)$. In the literature, there are some ways to tackle the difficulty. For example, seek a matrix $\bar{Q}$ such that $\bar{Q}(k)=Q(k+1)$. In general, however, there is no systematic way to find such a matrix as a function of Q. Most of the existing results restrict the design to linear parameter-varying systems where the time-varying parameters belong to a polytope. See references [3], [10] and the references therein.

In specific, matrix $A(k)$ in (16) is assumed that:

Assumption 4.2: The time-varying terms of $A(k)$ in system (16) are parameter σ-dependent and σ belongs to a polytope, that is,

$$A(k) = A(\sigma(k)) = \sum_{i=1}^{N} \sigma_i(k) A_i \,,$$

where

$$\sigma_i(k) \geq 0 \,, \quad \sum_{i=1}^{N} \sigma_i(k) = 1 \,,$$

and A_i's are constant matrices.
We will show in Section V-A that the system under consideration satisfies the above property.

If this is the case, then

$$A_c(k) = \sum_{i=1}^{N} \sigma_i(k) A_i(k) + Bu(k) \,.$$

Note that B is constant matrix in our design.

Now we apply the following result to design the feedback gain $K(k)$.

Lemma 4.1: [3] If there exist symmetric matrices $Q_i > 0$, $Q_j > 0$, and matrices G_i, $\bar{K}_i$, and the following matrix inequalities

$$\begin{pmatrix} G_i + G_i' - Q_i & (A_i G_i + B\bar{K}_i)' \\ A_i G_i + B\bar{K}_i & Q_j \end{pmatrix} > 0 \qquad (19)$$

for all $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N$ are feasible, then $K = \bar{K}P$ with $\bar{K} = \sum_{j=1}^{N} \sigma_j(k) \bar{K}_j$ and $P = \sum_{j=1}^{N} \sigma_j(k) P_j$ is the feedback gain to stabilize system (15).

By solving inequalities (19), one removes the $(k+1)$-index in matrix Q.

Proposition 4.2: If all the above assumptions hold, then the augmented system (8) can be stabilized by the following stabilizer

$$\begin{aligned} \hat{z}(k+1) &= (A_{22} - HA_{12})\hat{z}(k) + (B_2(k) - HB_1(k))u_{\text{st}}(k) \\ &\quad + ((A_{22} - HA_{12})H + A_{21} - HA_{11}(t))\, e(k) \\ \hat{x}_b(k) &= \hat{z}(k) + He(k) \\ u_{\text{st}}(k) &= K_1(k)x_1(k) + K_2(k)\hat{x}_b(k) \,. \end{aligned} \qquad (20)$$

And the gain $K(k) = \begin{pmatrix} K_1(k) & K_2(k) \end{pmatrix}$ can be explicitly given if inequalities (19) are solvable.

Proof: The proof follows by the above design. ■

V. APPLICATION TO A SERVO GANTRY SETUP

A benchmark problem in high precision mechatronics is the analysis and control of servo gantry systems. We would like to refer to [13], [18] and references therein for advanced control algorithms and their industrial applications. As depicted in Figure 2, the servo stage is constructed by assembling two modular single-axis stages perpendicularly, with actuators of linear VCMs including cylindrical permanent magnet stators and electromagnetic rotors (designed by *H2W Technologies, Inc.*). Linear optical encodes with 100 nm resolution are adopted for position measurement.

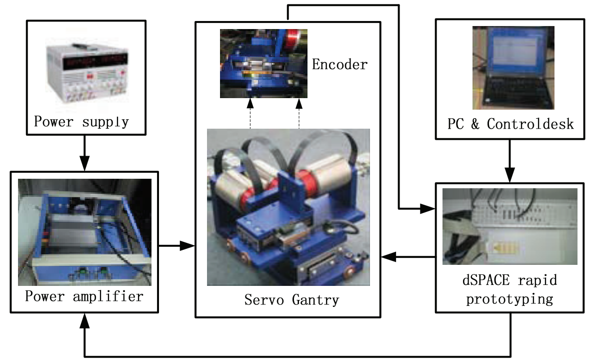

Fig. 2. The experiment setup of the VCM actuated servo gantry system.

The dynamical model of this servo gantry includes the dynamics of the current amplifier and the mechanical model of the system, where two Quanser® linear power amplifier modules with bandwidth $500Hz$ are employed to generate control currents. Hence the dynamics from control voltage to the amplifier current is assumed by a DC gain, i.e. $T_{ui}(s) = K_{ui}$. Also, we simplify the mechanical part of the system as a mass-spring-damper system as:

$$m_x\ddot{x} + c_x\dot{x} + k_x x = F_x - F_x^c \,\text{sgn}(\dot{x}) + \bar{\Delta} \,,$$

where m_x, c_x, k_x represents the moving mass, the equivalent damping coefficient, and the equivalent stiffness in the x axis respectively, and F_x is the driving force of the VCM, and $\bar{\Delta}$ represents unmodeled lumped nonlinear dynamics such as magnetic lag and eddy current loss, and other disturbances. Note that $F_x = K_F I_x$, where K_F is the force constant and I_x is the coil current. Therefore the transfer function from control voltage to position measurement can be written as:

$$G(s) = \frac{X(s)}{U(s)} = \frac{X(s)}{I(s)} \frac{I(s)}{U(s)} = \frac{K_F K_{ui}/m_x}{s^2 + (c_x/m_x)s + k_x/m_x} \qquad (21)$$

where the system coefficients can be identified by experimental measurements as shown in Table I. The discrete time representation of (21) is written as follows

	M_x kg	C_x $N \cdot s \cdot m^{-1}$	K_x $N \cdot m^{-1}$	K_F $N \cdot A^{-1}$	K_{ui} $A \cdot V^{-1}$
X-axis	0.213	3.666	17.089	5.87	0.5
Y-axis	0.647	11.135	51.909	11.74	0.5

TABLE I
SYSTEM COEFFICIENTS OF THE X-Y SERVO GANTRY.

$$G(z) = \frac{b_1 + z^{-1}b_0}{1 + z^{-1}a_1 + z^{-2}a_0} \,, \qquad (22)$$

where a_i, and b_i are readily obtained by the sampling rate.

A. SIMULATION RESULTS

Now we apply the proposed the control design for the VCM servo gantry (Figure 2). The aim is to track swept frequency reference trajectories. The linear part of system model (21) discussed admits the following discrete state-space representation:

$$A_{\rm o} = \begin{pmatrix} -a_1 & 1 \\ -a_0 & 0 \end{pmatrix}, \quad B_{\rm o} = \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}, \quad C_{\rm o} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where the nominal values of the coefficients are $a_1 = -f_{11} - f_{22}$, $a_0 = f_{11}f_{22} - f_{12}f_{21}$, and $b_1 = g_1$, $b_0 = f_{12}g_2 - f_{22}g_1$, with

$$F = e^{A_s T_{\rm s}} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

$$G = (e^{A_s T_{\rm s}} - I) A_s^{-1} B_s = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

and

$$A_s = \begin{pmatrix} -\dfrac{c_x}{m_x} & 1 \\ -\dfrac{k_x}{m_x} & 0 \end{pmatrix}, \quad B_s = \begin{pmatrix} 0 \\ \dfrac{K_F K_{ui}}{m_x} \end{pmatrix}$$

and the sampling time $T_{\rm s} = 1{\rm e}-4$ second, and the nominal values of the coefficients are $\frac{c_x}{m_x} = 17.21$, $\frac{k_x}{m_x} = 80.23$, $\frac{K_F K_{ui}}{m_x} = 13.78\,{\rm e}7$. The nominal values of c_x, k_x, and m_x are within 5%, 4%, and 2% different from their actual values respectively.

The variable frequency reference signal is generated by the following discrete exosystem

$$S(k) = \begin{pmatrix} \cos(\omega(k)t_{\rm s}) & \sin(\omega(k)t_{\rm s}) \\ -\sin(\omega(k)t_{\rm s}) & \cos(\omega(k)t_{\rm s}) \end{pmatrix}, \quad Q(k) = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

in which the time-varying terms are induced by $\omega_0 \leq \omega(k) \leq \omega_1$. The corresponding observer canonical form of the above system reads as:

$$S_{\rm o}(k) = \begin{pmatrix} -\alpha_1(k) & 1 \\ -\alpha_0(k) & 0 \end{pmatrix}, \quad Q_{\rm o} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \alpha_1(k) &= -s_{11}(k) - \frac{s_{12}(k)}{s_{12}(k-1)} s_{22}(k-1), \\ \alpha_0(k) &= -s_{12}(k+1)s_{21}(k) + \frac{s_{12}(k+1)}{s_{12}(k)} s_{11}(k)s_{22}(k). \end{aligned}$$

In this simulation, the angular frequency varies from 6π to 36π, with $\omega(t) = \omega_0 + \dot{\omega}\, t$, with $\omega_0 = 6\pi$ and $\dot{\omega} = 2\pi$.

By design, the internal model unit 1 reads as

$$\xi_1(k+1) = A_{\rm o}\, \xi_1(k) + B_{\rm o} u(k)$$

and the internal model 2 reads as

$$\begin{aligned} \xi_2(k+1) &= -q_0(k)\xi_2(k) - C_{\rm o}\xi_1(k) \\ u_{\rm im}(k) &= (p_0(k+1) - p_1(k)q_0(k))\xi_2(k) \\ &\quad + p_1(k)[-C_{\rm o}\xi_1(k)] \end{aligned}$$

where $p_1(k)$, $p_0(k)$ and $q_0(t)$ are calculated by solving the following algebraic Sylvester equation:

$$\begin{pmatrix} \underbrace{\begin{matrix} a_1 & 1 \\ a_0 & a_1 \\ 0 & a_0 \end{matrix}}_{\mathcal{O}_A} & \underbrace{\begin{matrix} b_1 & 0 \\ b_0 & b_1 \\ 0 & b_0 \end{matrix}}_{\mathcal{C}_B} \end{pmatrix} \begin{pmatrix} 1 \\ q_0(k) \\ p_1(k) \\ p_0(k) \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1(k) & 1 \\ \alpha_0(k) & \alpha_1(k) \\ 0 & \alpha_0(k) \end{pmatrix}}_{\mathcal{O}_S(t)} \begin{pmatrix} 1 \\ q_0(k) \end{pmatrix}. \tag{23}$$

The augmented system to be stabilized reads as:

$$\begin{aligned} x_{\rm o1}(k+1) &= -\alpha_1(k)x_{\rm o1}(k) + x_{\rm o2} + b_1 u_{\rm st}(k) \\ x_{\rm o2}(k+1) &= -\alpha_0(k)x_{\rm o1}(k) + b_0 u_{\rm st}(k) \\ y &= x_{\rm o1}. \end{aligned} \tag{24}$$

The reduced observer for state $x_{\rm o2}$ reads as

$$\begin{aligned} \hat{z}(k+1) &= -H\hat{z}(k) + (H\alpha_1(k) - \alpha_0(k) - HH)e(k) \\ &\quad + (b_0 - Hb_1)u_{\rm st}(k) \\ \hat{x}_{\rm o2}(k) &= \hat{z}(k) + He(k). \end{aligned} \tag{25}$$

It is seen that output injection gain can be chosen as a constant in $[-1, 1]$. In our case, $H = -0.01$ is selected for the design.

The feedback gain $K(\sigma)$ is designed based on the nominal values of a_i, b_i, $i = 0,\ 1$, which is conduced via solving the inequalities (19), where

$$A(\omega(k)) = \begin{pmatrix} -\alpha_1(\omega(k)) & 1 \\ -\alpha_0(\omega(k)) & 0 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

It is easy to verify that $A(\omega(k)) = \sum_{i=1}^{4} \sigma_i(k) A_i$.

As shown in Figure 3, the tracking performance of error is around $10\mu m$ (peak-peak) for the frequency-varying sinusoidal reference signals with the amplitude of $1000\mu m$ (peak-peak). An overall 1% tracking error is achieved with the proposed control. Meanwhile, the control effort $u(k)$ grows up according to the increase of the angular frequency $\omega(k)$ which agrees with the power requirement.

B. EXPERIMENTAL RESULTS

The same control algorithm is deployed on the VCM servo gantry described above, where a dSPACE® 1103 rapid prototyping system is used for controller implementation and real time control executions and the sampling time is chosen as $10k$ Hz. As depicted in Figure 4, the tracking performance is demonstrated in experiments, where a tracking error of $11\mu m$ in *RMS* value is achieved for the frequency-varying periodic reference with *RMS* value of $354\mu m$. Therefore we have an average tracking error less than 3.1%, which agrees well with the simulation results. Note that we use *RMS* value instead of peak-peak value to quantify the performance due to the existence of various disturbances and noises.

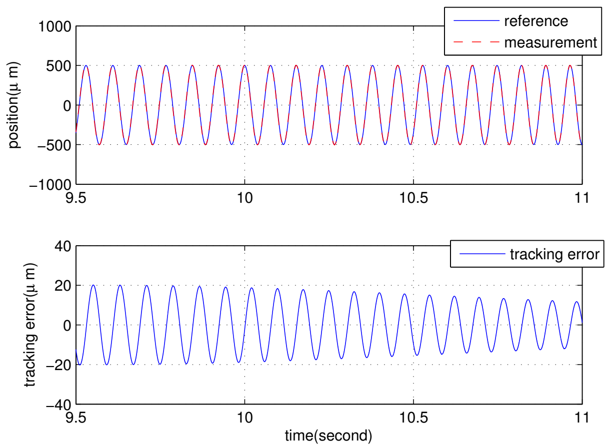

Fig. 3. Tracking swept frequency references in simulation.

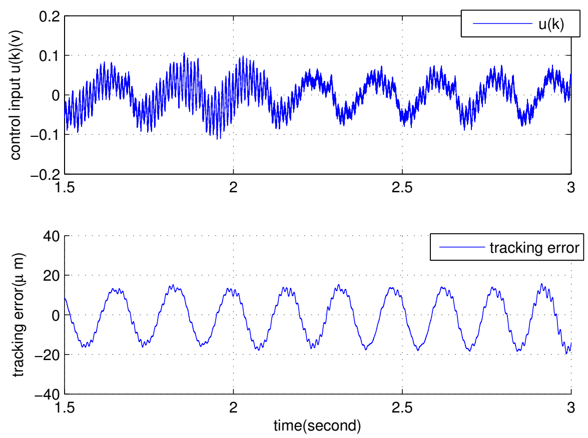

Fig. 4. Tracking swept frequency references in experiments.

VI. CONCLUSIONS

In this paper, we have proposed a discrete time-varying internal model-based tracking design for high precision tracking of swept frequency reference signals. Based on the recently proposed parallel structure of the time-varying compensator, we have developed asymptotic tracking conditions for the internal model unit, and low order robust stabilizer. In a discrete time setting, we have deployed the high precision tracking control approach on a voice coil motor actuated servo gantry system, where simulations and real-time experimental results are conducted and show good performance. The implementation of more complex references based on the proposed architecture is currently under investigation.

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