

Extension of the Krylov-Bogoliubov method and its application to the decay rate analysis of nonlinear control algorithms

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Abstract—An extension of the Krylov-Bogoliubov method and of the harmonic linearization method for approximating second order nonlinear systems with fast variation of damping factor and frequency of oscillation is revisited and a solution method involving a power series approach is proposed, which makes it easy for the extended asymptotic method to be applied to other damped nonlinear systems. Furthermore, the application of the method to the analysis of the decay rate of the nonlinear solution is presented. A performance measure for the analysis of the decay rate for nonlinear systems is proposed and exemplified on two nonlinear control algorithms, linear constrained control and constrained soft variable structure control with implicit Lyapunov functions.

I. INTRODUCTION

The dynamic properties of nonlinear systems can be studied using the theory of linear systems, especially of linear oscillators, by extending it to the asymptotic analysis of nonlinear solutions in the vicinity of linear ones [1]–[7]. One such example is the analysis of limit cycles, which are inherent nonlinear phenomena, in the vicinity of harmonic oscillations. Many asymptotic methods have been developed to describe a nonlinear oscillation in this way. Several of these methods, like the one studied here, are based on the asymptotic method of Krylov and Bogoliubov [1]. Popov [2] extended the method for damped nonlinear oscillations. The assumption for the extension was that the decay factor and the frequency of the nonlinear oscillation vary slowly. This assumption was relaxed in the extension of Zemskov [3], in which the nonlinear oscillation was approximated in the vicinity of an oscillation with time varying decay factor and frequency. This extension has been revisited here and used in the analysis of the decay rate of nonlinear control algorithms.

The decay rate of a system $\dot{x} = A(t)x$, with $A(t) \in \Omega \subseteq \mathbf{R}^{n \times n}$ is defined as the largest decay factor $\alpha > 0$ such that $\lim_{t \rightarrow \infty} e^{\alpha t} \|x(t)\| = 0$. For stable linear time invariant (LTI) systems this decay factor is the largest negative real part of the eigenvalues of the system matrix A . Using quadratic Lyapunov functions one can give a lower bound $\underline{\alpha}$ of the decay rate [8] as $\underline{\alpha} = \lambda_{\min}(Q)/\lambda_{\max}(P)$, with an arbitrary matrix $Q \succ 0$ and $A^T P + P A = -Q$, for which the following inequality $\dot{V}(x) \leq -2\underline{\alpha}V(x)$, $\forall x \in \mathbf{R}^n$ and thus $V(x) \leq V(0)e^{-2\underline{\alpha}t}$, $\forall x \in \mathbf{R}^n$ hold, which ensures that using $\lambda_{\min}(P)\|x(t)\|^2 = x^T \lambda_{\min}(P)Ix \leq x^T P x$, it follows that $\|x(t)\| \leq \sqrt{V(0)/\lambda_{\min}(P)}e^{-\underline{\alpha}t}$, $\forall x \in \mathbf{R}^n$, i.e. the decay rate of the system is at least $\underline{\alpha}$. This definition

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is also used for nonlinear control design, especially within the framework of linear matrix inequalities (LMI) based optimization.

For linear systems the decay factor is constant, for nonlinear systems it is time varying, with a variation and a rate of variation depending on the nonlinearity and on the parameters of the system. The definition of the decay rate does not take this fact into account. This implies that for nonlinear systems this definition of the decay rate is not necessarily a good quantification of the overall speed of the transient response, i.e., the same decay rate can be found with two different nonlinear control strategies for the same plant, but with different transient responses. For example, a fast decay factor far away from the equilibrium point contributes more to the overall speed response than one close to it. The instantaneous decay rate of the sought solution can be a better performance measure of the control strategy and by maximizing it, the system speed response could be improved.

The instantaneous decay rate of a nonlinear system can be approximated and analysed by means of asymptotic methods, see also [9] for a recent contribution. As an extension of the Krylov-Bogoliubov [1] asymptotic method, which in the first approximation is equivalent to the extended harmonic linearized solution, the method of Zemskov [3] can handle nonlinear oscillations with fast variation of the decay rate and frequency. But as with all asymptotic methods, the calculation of the sought solution becomes very complicated without the use of automated methods. One such method involving power series is proposed in this paper and its application to different nonlinear control laws is presented. Furthermore, a new performance measure involving the instantaneous decay rate is proposed.

This paper is organized as follows. In Section II two extensions of the asymptotic method of Krylov and Bogoliubov (KB method) are revisited and a power series solution method is proposed. In Section III the extension is applied to two nonlinear control strategies. In Section IV a new performance measure for the analysis of the decay rate of the solution based on performance diagrams is proposed and two nonlinear control laws for a given plant are compared in the parameter plane. Finally, Section V concludes the paper.

II. ASYMPTOTIC METHODS FOR NONLINEAR SYSTEMS

A. Extension of the KB method for nonlinear systems with slow variation of damping coefficient and frequency

As a generalization of the KB method, the Popov extension (KBP) [2] approximates the solution of a second order

nonlinear differential equation

$$\ddot{x} + 2\hat{c}_1\dot{x} + \hat{c}_0x = \varepsilon f(x, \dot{x}), \quad (1)$$

for slowly varying damping factor and frequency of the nonlinear oscillation $x(t)$. The asymptotic solution, i.e. the approximating solution which becomes exact for $\varepsilon = 0$, has the form

$$x(t, \varepsilon) = a \sin \psi + \varepsilon u_1(a, \psi) + \dots + \varepsilon^m u_m(\psi, a), \quad (2)$$

with $a = a(t)$, $\psi = \psi(t)$ and

$$\dot{a} = \xi_0 a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots + \varepsilon^m A_m(a) = a\xi(a), \quad (3)$$

$$\dot{\psi} = \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots + \varepsilon^m B_m(a) = \omega(a), \quad (4)$$

and differs from the solution of the KB method by the term $\xi_0 a$ in eq. (3). The unknown functions u_1, \dots, u_m , A_1, \dots, A_m and B_1, \dots, B_m are sought, such that the asymptotic solution (2) satisfies eq. (1) with an accuracy up to the order ε^{m+1} . This extension allows the asymptotical approximation of fast decaying nonlinear oscillations in the vicinity of a linear damped oscillation with the constant damping factor $\xi_0 = -\hat{c}_1$ and frequency $\omega_0 = \sqrt{\hat{c}_2^2 - \xi_0^2}$.

The asymptotic solution in the KBP method is constructed by twice differentiating eq. (2) and inserting the first and the second time derivatives in eq. (1). By equating the factors in equal powers of ε and through equating the coefficients of the trigonometric functions $\sin \psi$ and $\cos \psi$, as well as the free terms, one obtains $3 \times m$ (partial) differential equations in u_1, \dots, u_m , A_1, \dots, A_m and B_1, \dots, B_m .

The first order approximation gives [2]

$$\xi(a) = \xi_0 + \varepsilon \frac{h_1(a)}{2a\omega_0}, \quad \omega(a)^2 = \omega_0^2 - \varepsilon \frac{g_1(a)}{a}, \quad (5)$$

with the slowly variable functions of the amplitude a ,

$$\frac{h_1(a)}{a} = \frac{1}{\pi a} \int_0^{2\pi} f(a \sin \psi, a\omega_0 \cos \psi + a\xi_0 \sin \psi) \cos \psi d\psi, \quad (6)$$

$$\frac{g_1(a)}{a} = \frac{1}{\pi a} \int_0^{2\pi} f(a \sin \psi, a\omega_0 \cos \psi + a\xi_0 \sin \psi) \sin \psi d\psi. \quad (7)$$

In this case one can use for eq. (6) and (7) the describing functions resulting from the harmonic linearization method, as it is shown in the next section.

B. Extended harmonic linearization method for damped oscillations

The harmonic linearization of a nonlinear function $y = F(x, sx)$ within a nonlinear differential equation of the form $N(s)x + Z(s)F(x, sx) = 0$, $s \equiv \frac{d}{dt}$, which fulfills the assumptions of the harmonic linearization method [2], is for $x = A \sin \psi$ and $sx = A\Omega \cos \psi$ given by

$$F(x, sx) = q(A, \Omega)x + \frac{q'(A, \Omega)}{\Omega}sx, \quad (8)$$

with the coefficients

$$q(A, \Omega) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi) \sin \psi d\psi, \quad (9)$$

$$q'(A, \Omega) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi) \cos \psi d\psi.$$

They are calculated by assuming the equivalence condition, that the first harmonic of the Fourier series expansion of the nonlinear oscillation $y = F(x, sx)$ is basically its only significant part and is equivalent to the oscillation produced in the case of a linear dependency of y on x and sx . Here A denotes a slowly varying amplitude in contrast to a fast varying amplitude a .

With fast varying amplitude, but slow varying damping factor ξ as in the KBP method, it follows that $sx = a\xi \sin \psi + a\omega \cos \psi$. The Fourier series expansion of the nonlinear damped oscillation $y = F(x, sx)$ has now *damped harmonics* with slowly varying frequencies and damping factors. The first *damped harmonic* is given by

$$F(x, sx) = \left(q(a, \xi, \omega) - \frac{\xi}{\omega} q'(a, \xi, \omega) \right) x + \frac{q'(a, \xi, \omega)}{\omega} sx.$$

Here the coefficients $q(a, \xi, \omega)$ and $q'(a, \xi, \omega)$ depend generally also on $\xi(a)$ and $\omega(a)$. Usually, the nonlinearity $F(x)$ depends only on x . In this case, one can use $q(a, \xi, \omega) = q(a)$ and $q'(a, \xi, \omega) = q'(a)$. If the nonlinearity $F(x, sx)$ depends also on sx , but the term $|\xi/\omega|$ is small, one can use eq. (9), i.e. $q(a, \xi, \omega) = q(a, \omega)$ and $q'(a, \xi, \omega) = q'(a, \omega)$. Otherwise, the coefficients $q(a, \xi, \omega)$ and $q'(a, \xi, \omega)$ must be recalculated considering that $|\xi| \ll \omega$.

The harmonic linearized characteristic equation of (1) is

$$s^2 + \left(2\hat{c}_1 + \frac{q'(a, \xi, \omega)}{\omega} \right) s + \hat{c}_0 + q(a, \xi, \omega) - \frac{\xi}{\omega} q'(a, \xi, \omega) = 0. \quad (10)$$

Eq. (10) can be regarded as the (extended) harmonic balance equation, which indicates the existence of a vanishing oscillatory motion.

C. Extension of the KBP Method for nonlinear systems with fast variation of damping coefficient and frequency

In the case of variable-structure systems and optimal systems, where the damping factor and frequency vary strongly, this method has been extended in [3] to meet the dynamic properties of such systems. The asymptotic solution has the same form as in eq. (2), but the rate of variation of the damping factor $\xi(a)$ and frequency $\omega(a)$, i.e. $\dot{\xi}(a)$ and $\dot{\omega}(a)$, are not constrained. The extension allows the asymptotic approximation of the solution of the nonlinear differential equation

$$\ddot{x} + 2\hat{c}_1(a)\dot{x} + \hat{c}_0(a)x = \varepsilon f(x, \dot{x}), \quad (11)$$

in the vicinity of a nonlinear oscillation $x(t, \varepsilon=0) = a \sin \psi$, with $\dot{a} = a\xi_{(*)}$ and $\dot{\psi} = \omega_{(*)}$ and time varying damping factor $\xi_{(*)}$ and frequency $\omega_{(*)}$, which can have any rate of variation.

The asymptotic construction of the solution of the nonlinear differential equation (11) involves similar equations to (3) and (4) but with $\xi_{(*)}$ and $\omega_{(*)}$ instead of ξ_0 and ω_0 . The search of the functions u_1, \dots, u_m , A_1, \dots, A_m and B_1, \dots, B_m is made with the same procedure by twice differentiating the asymptotic solution and substituting the time derivatives in eq. (11). In the first order approximation the factors $\xi_{(*)}$ and $\omega_{(*)}$, as well as the functions $u_1(a, \psi)$, $A_1(a)$ and $B_1(a)$ are

calculated by equating the factors in equal powers of ε . For $\varepsilon = 0$ the resulting equations are

$$\omega_{(*)}^2 = \omega_{(0)}^2 + (\xi_{(*)} - \xi_{(0)})^2 + \dot{\xi}_{(*)}, \quad (12)$$

$$\xi_{(*)} = \xi_{(0)} - \frac{\dot{\omega}_{(*)}}{2\omega_{(*)}}, \quad (13)$$

from which $\xi_{(*)}(a)$ and $\omega_{(*)}(a)$ can be calculated through successive approximations as in [3]. Here $\xi_{(0)}$ and $\omega_{(0)}$ are the slowly variable damping factor and frequency that can be calculated with the KMP method. In the (0)-th approximation, for slowly variable damping factor and frequency, i.e. $\dot{\xi}_{(0)} \approx 0$ and $\dot{\omega}_{(0)} \approx 0$, it follows that $\xi_{(0)} = \xi$ and $\omega_{(0)} = \omega$, with ξ and ω calculated with the KBP method, i.e. from eq. (5). In the (1)-st approximation, i.e. $\dot{\xi}_{(0)} \approx 0$ and $\dot{\omega}_{(0)} \approx 0$, it follows that $\xi_{(1)} = \xi_{(0)} - \dot{\omega}_{(0)}/2\omega_{(0)}$ and $\omega_{(1)}^2 = \omega_{(0)}^2 + \dot{\xi}_{(0)}$. In the (k+1)-th approximation, it follows that $\xi_{(k+1)} = \xi_{(0)} - \frac{\dot{\omega}_{(*)}^{(k)}}{2\omega_{(*)}^{(k)}}$ and $\omega_{(k+1)}^2 = \omega_{(0)}^2 + (\xi_{(k)} - \xi_{(0)})^2 + \dot{\xi}_{(k)}$.

Eq. (12) and (13) show that the application of the extended method of harmonic linearization as well as the KBP method are constrained by more than the small variation of $\dot{\xi}$ and $\dot{\omega}$, but by the therefor sufficient conditions [3] that $\dot{\omega}_{(*)}/\omega_{(*)} \approx 0$ and $\dot{\xi}_{(*)} \approx -\dot{\omega}_{(*)}^2/4\omega_{(*)}^2$. Eq. (13) shows also that the damping factor of the nonlinear generating oscillation $\xi_{(*)}$ can be positive, thus generating an unstable oscillation, if the magnitude of the rate of variation of the instantaneous frequency $\dot{\omega}_{(*)}$ becomes too large compared to the formal coefficient $\xi_{(0)}$ and the instantaneous frequency $\omega_{(*)}$.

With the use of the instantaneous damping factor $\xi_{(*)}$ and frequency $\omega_{(*)}$, the functions $u_1(a, \psi)$, $A_1(a)$ and $B_1(a)$ can be calculated from the (partial) differential equations resulting from equating the coefficients of ε , as with the KBP method in Section II-A. Without neglecting $\dot{\omega}_{(*)}$ and $\dot{\xi}_{(*)}$ and substituting $A_1(a) = a\Phi_1(a)$, the following differential equations in $\Phi_1(a)$ and $B_1(a)$ are obtained

$$\begin{aligned} d_{11}\Phi_1 - d_{12}B_1 &= -a\xi_{(*)} \frac{d\Phi_1}{da} + \frac{g_1(a, \xi_{(*)}, \omega_{(*)})}{a}, \\ d_{21}\Phi_1 + d_{22}B_1 &= -a\xi_{(*)} \frac{dB_1}{da} + \frac{h_1(a, \xi_{(*)}, \omega_{(*)})}{a}, \end{aligned} \quad (14)$$

with $d_{11} = d_{22} = -\dot{\omega}_{(*)}/\omega_{(*)}$, and $d_{12} = d_{21} = 2\omega_{(*)}$. These equations can also be solved by successive approximations. In the (0)-th approximation, i.e. $dA_{1,(0)}/da \approx 0$ and $dB_{1,(0)}/da \approx 0$, $A_{1,(0)}$ and $B_{1,(0)}$ are slowly variable functions of the amplitude a and can be calculated using the coefficients $q(a, \xi_{(*)}, \omega_{(*)}) = g_1(a, \xi_{(*)}, \omega_{(*)})/a$ and $q'(a, \xi_{(*)}, \omega_{(*)}) = h_1(a, \xi_{(*)}, \omega_{(*)})/a$, as in Section II-B. Their derivatives with respect to a are then introduced on the right hand side of eq. (14) to find $A_{1,(1)}$ and $B_{1,(1)}$.

Finally, the function $u_1(a, \psi)$, which does not contain any first harmonic term, can be also calculated, see [3]. But if eq. (11) fulfills the conditions of the harmonic linearized method, then the higher harmonics of x , i.e. $u_1(a, \psi)$, can be neglected.

In the first order approximation the solution of eq. (11) is $x(t, \varepsilon) = a \sin \psi + \varepsilon u_1(a, \psi)$, with $\dot{a} = \xi_{(*)}a + \varepsilon A_1(a)$ and

$\dot{\psi} = \omega_{(*)} + \varepsilon B_1(a)$. By means of the harmonic linearization of the nonlinearity $F(x, \dot{x}) = \varepsilon f(x, \dot{x})$ the resulting characteristic equation of (11) is

$$s^2 + \left(-2\xi_{(*)} - \frac{q'(a)}{\omega_{(*)}}\right)s + \xi_{(*)}^2 + \omega_{(*)}^2 - q(a) + \frac{\xi_{(*)}}{\omega_{(*)}}q'(a) = 0. \quad (15)$$

with $q(a) = q(a, \xi_{(*)}, \omega_{(*)})$ and $q'(a) = q'(a, \xi_{(*)}, \omega_{(*)})$. The same result is achieved with the extended KBP method using $\xi_{(*)}$, $\omega_{(*)}$, $A_{1,(0)}$ and $B_{1,(0)}$.

In the next section a method involving power series is proposed, which can be easily applied to find the functions $\xi_{(*)}(a)$, $\omega_{(*)}(a)$, as well as $A_{1,(*)}(a)$ and $B_{1,(*)}(a)$ for any second order damped nonlinear system.

D. Power series solution procedure

For investigating the properties of the nonlinear equation (11), it suffices to know the variation of the damping factor $\xi_f(a) = \xi_{(*)} + \varepsilon A_{1,(*)}$ and frequency $\omega_f(a) = \omega_{(*)} + \varepsilon B_{1,(*)}$ of the nonlinear oscillation $x(t, \varepsilon)$, which involves solving eq. (12), (13) and (14). A possible solution procedure is considering the unknown functions $\xi_{(k)}(a)$, $\omega_{(k)}(a)$, $A_{1,(k)}(a)$, $B_{1,(k)}(a)$ in power series form, i.e.

$$\begin{aligned} \xi_{(k)}(a) &= \sum_{l \geq 0} c_{1l} a^l, & \gamma_{(k)}(a) &= \omega_{(k)}^2(a) = \sum_{l \geq 0} c_{2l} a^l, \\ A_{1,(k)}(a) &= \sum_{l \geq 0} c_{3l} a^l, & B_{1,(k)}(a) &= \sum_{l \geq 0} c_{4l} a^l. \end{aligned} \quad (16)$$

By substituting eq. (16) in eq. (12) and (13), one obtains

$$\begin{aligned} \gamma_{(k+1)} &= \omega_{(k+1)}^2 = \gamma_{(0)} + (\xi_{(k)} - \xi_{(0)})^2 + a\xi_{(k)} \frac{d\xi_{(k)}}{da}, \\ \xi_{(k+1)} &= \xi_{(0)} - \frac{a\xi_{(k)}}{4\gamma_{(k)}} \frac{d\gamma_{(k)}}{da} \approx \xi_{(0)} \left(1 - \frac{a}{4\gamma_{(k)}} \frac{d\gamma_{(k)}}{da}\right). \end{aligned} \quad (17)$$

For the (0)-th order approximation, the coefficients c_{1l}, \dots, c_{4l} can be calculated using the solution of the characteristic equation of eq. (11) for $\varepsilon = 0$, i.e. $s^2 + 2b(a)s + c(a)^2 = 0$, with $s = \xi_{(0)}(a) \pm j\omega_{(0)}(a)$. This characteristic equation can be the harmonic linearized expression of a nonlinear differential equation. The formal damping factor $\xi_{(0)}$ and frequency $\omega_{(0)}$ can have any rate of variation. This solution procedure reduces the computation of $\xi_{(*)}$ and $\omega_{(*)}$ to differentiating (fractional) polynomial functions. A similar result is obtained by substituting eq. (16) in eq. (14).

E. Generality of the method

The method can be applied to any nonlinear differential equation, that can be formulated as in eq. (11). For $\varepsilon = 0$ the equation can be seen furthermore as a harmonic linearized nonlinear differential equation, calculated using the KBP method from Section II-A. In both methods, $\varepsilon f(x, \dot{x})$ represents a *weak* nonlinearity, which will change the form of the linear (harmonic linearized) solution for $\varepsilon = 0$, adding small quantities of the order of ε . For example, if the nonlinearity corresponds to higher harmonics of the solution x and the linear part of the equation has lowpass character, then $\varepsilon f(x, \dot{x})$ represents a *weak* nonlinearity.

Both methods can be applied only if x represents a nonlinear oscillation. For non-oscillating systems there are

a number of other approaches [5,7] of approximating the nonlinear solution, the describing function approach has been considered also in the case of exponential and mixed inputs in [4]. The Popov extension of the asymptotic method can handle asymmetric oscillations, with which one can approximate non-oscillating systems [2].

III. APPLICATION TO SINGLE INPUT LINEAR PLANTS WITH ACTOR SATURATION

For a LTI-system with a single input and actor saturation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\hat{a}_0 & -\hat{a}_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \mathbf{x} \in \mathbf{R}^n, u \in \mathbf{R}, |u| \leq 1, \quad (18)$$

the problem of maximizing the decay rate can be defined as maximizing $-\dot{V}(\mathbf{x})$ at each $\mathbf{x} \in \mathbf{R}^n$, where $V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$ is a Lyapunov function for the system. The saturated linear feedback with optimal decay rate is a bang-bang type control, $u = -\text{sgn}(\mathbf{b}^T \mathbf{P} \mathbf{x})$ [10].

The control laws which are studied here are linear, i.e. high gain linear feedback and nonlinear, i.e. soft variable structure control with implicit Lyapunov functions. Each nonlinear feedback system is formulated within the asymptotic method for fast varying damping coefficient and frequency and performance diagrams in the parameter plane are constructed to compare the control laws. The power series approach employed in this paper makes the application of the asymptotic method easily possible and thus allows application to other nonlinear control schemes and/or nonlinear plants. A new measure is then proposed to compare the control laws. This measure is based on the performance diagrams and involves their gradient with respect to control law parameters.

A. Saturated linear feedback

The closed loop system with a linear plant and a saturated input from eq. (18) with a linear feedback $u = -(k_1 x + k_2 \dot{x})$, can be described by eq. (1) with $2\hat{c}_1 = \hat{a}_1 + k_2$, $\hat{c}_0 = \hat{a}_0 + k_1$ and $\varepsilon f(x, \dot{x}) = \text{sat}(u) - u$. Because u can be seen as the first *damped harmonic* of $\text{sat}(u)$, the difference $\text{sat}(u) - u$ contains only higher order *damped harmonics* and due to the lowpass character of the linear part, $\varepsilon f(x, \dot{x})$ can be seen as a *weak* nonlinearity.

The first order solution with the KBP method is

$$\begin{aligned} x &= a \sin \psi, \quad \dot{x} = a \xi \sin \psi + a \omega \cos \psi, \\ \xi &= -[\hat{c}_1 + (1 - q(a_u))k_2/2], \quad \omega^2 = \hat{c}_0 + (1 - q(a_u))k_1 - \xi^2, \\ u &= a_u \sin(\psi + \arctan((k_2 \omega)/(k_1 + k_2 \xi))), \\ a_u &= a \sqrt{(k_1 + k_2 \xi)^2 + (k_2 \omega)^2}. \end{aligned}$$

$q(a_u)$ is the describing function of the saturation function. In the case of a nonsaturating feedback the solution is given for $\varepsilon = 0$.

The extended harmonic linearized characteristic equation, calculated by replacing the nonlinearity $\varepsilon f(x, \dot{x}) = \mathbf{k}^T \mathbf{x} - \text{sat}(\mathbf{k}^T \mathbf{x})$ with $\varepsilon f(x, \dot{x}) = (1 - q(a_u)) \mathbf{k}^T \mathbf{x}$, is

$$s^2 + [2\hat{c}_1 + k_2(1 - q(a_u))]s + \hat{c}_0 + k_1(1 - q(a_u)) = 0.$$

For the normalized saturation function one can use the describing function of the harmonic linearization method, i.e.

$q(a_u) = 2/\pi \left(\arcsin(1/a_u) + 1/a_u \sqrt{1 - 1/a_u^2} \right)$. However, because of its non-polynomial form, this function is not easy to use within the extension of the KBP method, hence one alternative is to approximate it by cubic splines¹, i.e.

$$q(v_u) = \begin{cases} \sum_{l=0}^3 \rho_{1l} \cdot v_u^l, & v_u \in [0, 0.5] \\ \sum_{l=0}^3 \rho_{2l} \cdot (v_u - 0.5)^l, & v_u \in [0.5, 1] \\ 1, & v_u \in [1, \infty) \end{cases}$$

where $v_u = a_u^{-1}$ and a_u is the amplitude of the nonlinear oscillation of the control law. The coefficients ρ can be found using numeric methods.² This allows the formulation of the unknown functions $\xi_{(*)}$, $\omega_{(*)}$ in power series form, which has a big advantage in calculating higher order approximations using eq. (17), with $\xi_{(0)} = \xi$ and $\omega_{(0)} = \omega$.

The differential equations for the amplitude and frequency of the nonlinear oscillation $x = a \sin \psi$ within the extension of the KBP method are then given by $\dot{a} = a \xi_{(k)}(v_u)$ and $\dot{\psi} = \omega_{(k)}(v_u)$. With the saturated linear feedback no other functions $A_{1,(*)}$ or $B_{1,(*)}$ are needed because the only nonlinearity in the feedback system is the saturation function and the application of the asymptotic method can be seen as an improvement of the solution of the KBP method, i.e. of $\xi_{(0)}$ and $\omega_{(0)}$. In the case of variable structure feedback systems, the functions mentioned above are indeed necessary.

Example 1: For a linear plant $\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{b} u$ with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.6025 & -0.1708 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |u| \leq 1, \quad (19)$$

the linear feedback law $\mathbf{k}^T = [0.3899 \quad 1.4625]$ constrained to a virtual input saturation of $|u| \leq \beta (= 50)$ maximizes a lower bound of the decay rate $\alpha = 0.2731$ of a quadratic Lyapunov function. Fig. 1 (top) shows the approximation of the transient response by means of the extended asymptotic method $k = 1$. Fig. 1 (bottom) shows the L_2 norm of the relative error of the approximation, $\|\hat{x} - x\|_2 / \|x\|_2$, as a function of β . A higher β means a stronger nonlinearity. $\beta = 1$ corresponds to a nonsaturating linear feedback.

B. Saturated soft variable structure control with implicit and nested Lyapunov functions (saturated soft VSC)

One type of soft variable structure control is a linear state feedback with continuously varying coefficients determined by a parameter $p(\mathbf{x}) \in (0, 1)$, which depends on the current states. Design constraints have been formulated to ensure that $p(\mathbf{x})$ is an implicit Lyapunov function of the system, see [11], which is defined by the selection strategy $g(\mathbf{x}, p) = \mathbf{x}^T \mathbf{R}(p) \mathbf{x} - 1 = 0$.

A continuous family of linear state controllers $u = -\mathbf{k}^T(p) \mathbf{x}$ is then used, each leading to a stable control loop associated with ellipsoidal Lyapunov regions, $\mathcal{G}(p)$, whose

¹This function is continuously differentiable in $v_u = 1$ and can therefore be used with the asymptotic method. For a higher degree of accuracy, higher order splines are needed in order to construct a describing function that is continuously differentiable at a higher order in $v_u = 1$.

² $\rho = \begin{bmatrix} 0.256 & -0.3587 & 1.333 & 0 \\ -1.5894 & 0.0254 & 1.1667 & 0.6090 \end{bmatrix}$

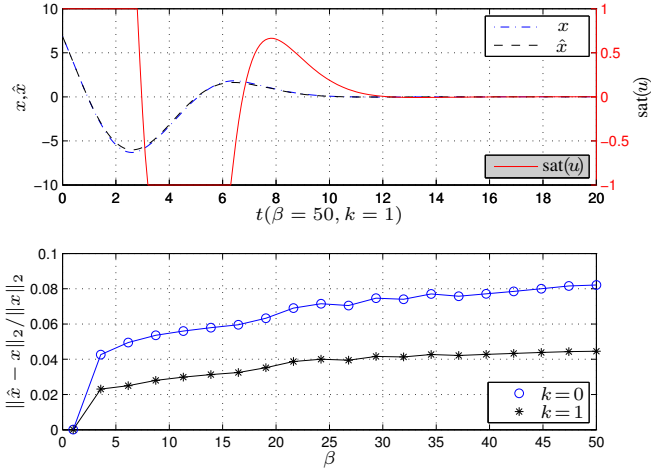


Fig. 1. Approximation \hat{x} of the transient response x from eq. (19) with the linear control for $\beta = 50$ (top) and L_2 -norm of the relative approximation error as a function of β (bottom).

sizes decrease as $p(x)$ decreases. The control parameters are designed in such a way, that with a smaller distance to the equilibrium state, i.e. as $p(x)$ decreases, the magnitude of the control law increases, but remains within the input bounds. In the closed loop system, the instantaneous eigenvalues *move* along rays towards negative infinity. The variation of the parameter $p(x)$ is linked to the borders of the nested Lyapunov regions, $\partial\mathcal{G}(p)$, where each border is assigned a unique value of $p(x)$, for the outermost border $p = 1$, at the equilibrium point $p = 0$. With this type of parameter varying control, no sliding mode can occur. See [12] for the formulation of the control design as LMI based optimization problem.

The saturated control law $u = -\text{sat}(k^T(p)x)$ [13] accounts for the saturation nonlinearity of the input. By maximizing the decay rate of the Lyapunov function, the saturated feedback law converges also to a bang-bang type feedback, but the optimal decay rate differs from that of the saturated linear feedback.

The closed loop system with a single input second order linear plant from eq. (18) and saturated soft VSC feedback $u = -\text{sat}(k^T(p)x)$ can be expressed as in eq. (11) with $2\hat{c}_1(a) = \hat{a}_1 + k_2(p) = -2\xi_p$, $\hat{c}_0(a) = \hat{a}_0 + k_1(p)$ and $\varepsilon f(x, \dot{x}) = \text{sat}(u) - u$. The harmonic linearized characteristic equation is constructed in two steps. In the first step the closed loop system with the nonsaturating feedback is harmonically linearized, i.e. $\varepsilon = 0$, and $\xi_{(*)}$, $\omega_{(*)}$ are calculated. In the second step the saturating feedback is considered and the functions $A_{1,(*)}$ and $B_{1,(*)}$ are calculated.

In the first step, the resulting second order differential equation $\ddot{x} - 2\xi_p\dot{x} + (\xi_p^2 + \omega_p^2)x = 0$, with $\xi_p = \xi_1/p$, $\omega_p = \omega_1/p$ and ξ_1 , ω_1 as the damping factor and frequency with the first controller, i.e. $p = 1$, has parameter varying coefficients. The corresponding harmonic linearized equation is $s^2 - 2\xi_{(*)}s + (\xi_{(*)}^2 + \omega_{(*)}^2) = 0$, where $\xi_{(*)}$ and $\omega_{(*)}$ can be calculated from eq. (12) and (13) with $\xi_{(0)} = \xi_p$, $\omega_{(0)} = \omega_p$, where $p, \dot{p}, \dots, p^{(k)}$ are parameters determined from the selection strategy $g(x, p) = 0$. See [6] for a similar approach

to harmonic linearization of linear time-varying systems. In the second step $A_{1,(*)}$ and $B_{1,(*)}$ are calculated from eq. (14) with $\hat{k}_1(a) = \xi_{(*)}^2(a) + \omega_{(*)}^2(a) - \hat{a}_0$ and $\hat{k}_2(a) = -2\xi_{(*)}(a) - \hat{a}_1$.

The extended harmonic linearized equation of the closed loop system with the saturated soft VSC feedback is

$$s^2 - 2\xi_f(a)s + \xi_f^2(a) + \omega_f^2(a) = 0,$$

with $\xi_f(a) = \xi_{(*)} + \varepsilon\Phi_{1,0}(a)$, $\omega_f(a) = \omega_{(*)} + \varepsilon B_{1,0}(a)$ and

$$\varepsilon\Phi_{1,0}(a) = \frac{d_{22}q + d_{12}q'}{d_{11}^2 + d_{12}^2}, \quad \varepsilon B_{1,0}(a) = -\frac{d_{21}q - d_{11}q'}{d_{21}^2 + d_{22}^2},$$

where $q = g_1(a, \xi_{(*)}, \omega_{(*)})/a = (\hat{k}_1(a) + \hat{k}_2(a)\xi_{(*)})(1 - q(a_u))$, $q' = h_1(a, \xi_{(*)}, \omega_{(*)})/a = \hat{k}_2(a)\omega_{(*)}(1 - q(a_u))$.

Example 2: Fig. 2 shows for the same example as in eq. (19) the approximation of the transient response of the linear plant with input saturation and soft variable structure control law with the variation parameter $p(x) \in [p_m, 1]$. For $p_m = 1$ the control law is a saturated linear feedback.

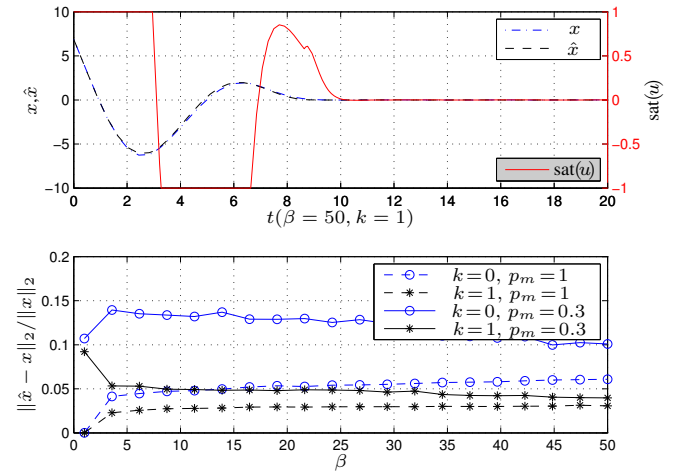


Fig. 2. Approximation \hat{x} of the transient response x from eq. (19) with the soft VSC control for $\beta = 50$ (top) and L_2 -norm of the relative approximation error as a function of β (bottom).

IV. ANALYSIS OF THE DECAY RATE

Both control laws have been optimized using the decay rate of quadratic Lyapunov functions. Formulated by means of LMIs, the optimization problems include also stability conditions and the same domain of initial values for the sought solutions $|x_0| \in \mathcal{X}_0$, see [12] for further details on the LMI constraints. Due to solver constraints, a virtual bound on the control law must be defined, here the same virtual control output bound $|u| \leq \beta = 2$ has been used. This resulted in the same Lyapunov function based decay rate for both nonlinear control strategies, $\alpha = 0.1932$. The difference in the transient response of the solution is shown in Fig. 3. For a higher bound β this difference becomes smaller, but each time the same Lyapunov function based decay rate is obtained. In each case, the maximum convergence rate can be obtained for $\beta \rightarrow \infty$, where they become bang-bang type control laws. However, the transient responses will be different.

The difference can also be seen in the performance diagrams in Fig. 4. The figure shows the contour lines and

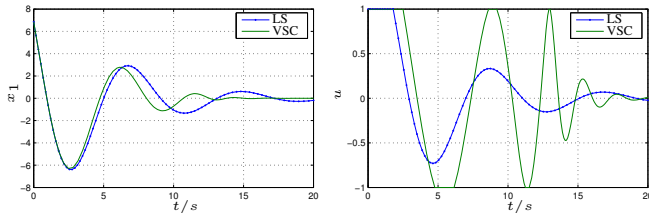


Fig. 3. Transient response for saturated linear feedback (LS) and soft variable structure control (VSC) ($\beta = 2$).

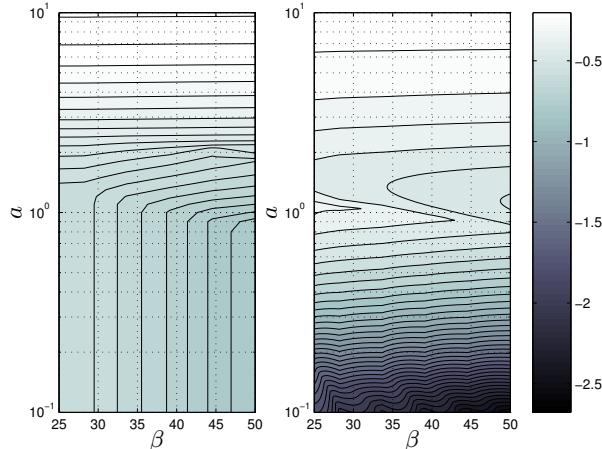


Fig. 4. Performance diagrams $\xi(a)$ ($= \text{const.}$) for the saturated linear feedback (left) and saturated soft VSC feedback (right).

the variation of the damping factors $\xi(a)$ as functions of the oscillation amplitude a and virtual input constraint β . As it can be seen in this figure, for a given β the variation of the damping factor $\xi(a)$ resulting from the variation of a is different with each control strategy. With the saturated linear feedback (Fig. 4, left) the instantaneous decay factor $\xi(a)$ remains constant if a becomes smaller than a certain value, in which region the closed loop is simply linear (unsaturated case). Furthermore, the smaller the angle between the gradient of the damping factor (orthogonal to the level sets) and its variation rate generated by the control law (here vertical lines), the faster becomes the transient response.

Without solving the optimization problem, one can analyse the difference also for $\beta \rightarrow \infty$ by means of the discussed asymptotic method. Thus, in the convergence optimal case, the Lyapunov function based decay rate (smallest decay factor), i.e. $\max \xi(a)$, will be in the case of the saturated linear feedback not higher than the constant decay factor from the unsaturated case. With the saturated soft VSC feedback the decay rate will be higher, due to the variation of the instantaneous damping from the unsaturated case.

With other (nonlinear) plants and nonlinear control laws, the contour lines of the damping factor and frequency will be different, but the performance diagrams offer a measure to analyse which control strategy reaches a faster transient response. Also, for other nonlinear control strategies, like the implicit polynomial strategies [14], the performance diagrams could bring more insight into the system behaviour. The determination of a quasi optimal control law may be formulated as an optimization problem where a control law

must be chosen from an admissible set of control laws, such that the instantaneous variation of the decay rate $\partial \xi / \partial \mathbf{k}^T$ is minimized. See also [15] for an example of an application of the Hamilton-Jacobi theory for a similar optimization of a nonlinear plant. This would give not only an alternative optimization method for variable structure control laws, but also a possibility to design such control laws for nonlinear plants.

V. CONCLUSION

A solution method has been presented for finding the required functions in the asymptotic method of [3] for nonlinear systems with fast varying damping factor and frequency, which is easily adaptable to other nonlinear systems. The discussed asymptotic method does not depend on the objective function of the control law optimization and this allows the comparison of different control laws by means of performance diagrams. As a measure in choosing the best decay rate or in comparing two different control laws, the gradients of the functions ξ and ω with respect to the control parameters have been proposed, which can be formulated as a series solution.

REFERENCES

- [1] N. Krylov and N. Bogoliubov, *Introduction to nonlinear mechanics*. Princeton: Princeton University Press, 1947.
- [2] P. Popow and J. Paltow, *Näherungsmethoden zur Untersuchung nicht-linearer Regelungssysteme*. Leipzig: Akademische Verlagsgesellschaft, 1963.
- [3] V. Zemskow and N. Zemskova, "Asymptotic method of investigation of dynamic performance of nonlinear systems with fast variation of damping coefficient and frequency," *Automation and Remote control*, vol. 39, no. 12, pp. 1874–1881, 1978.
- [4] A. Gelb and W. Vander Velde, *Multiple-Input Describing Functions and Nonlinear System Design*. McGraw-Hill, 1968.
- [5] I. Murty, "A unified Krylov-Bogoliubov method for solving second-order non-linear systems," *International Journal of Non-Linear Mechanics*, vol. 6, no. 1, pp. 45–53, 1971.
- [6] R. Bittel and D. Siljak, "An application of the Krylov-Bogoliubov method to linear time-varying systems," *International Journal of Control*, vol. 11, no. 3, pp. 423–429, 1970.
- [7] A. Shamsul, "A modified and compact form of Krylov-Bogoliubov-Mitropolskii unified method for solving an nth order non-linear differential equation," *International Journal of Non-Linear Mechanics*, vol. 39, no. 8, pp. 1343–1357, 2004.
- [8] J.-J. Slotine and W. Li, *Applied nonlinear control*. Englewood Cliffs: Prentice Hall, 1991.
- [9] I. Boiko, "On frequency-domain criterion of finite-time convergence of second-order sliding mode algorithms," *Automatica*, vol. 47, pp. 1969–1973, 2011.
- [10] T. Hu and Z. Lin, *Control Systems with actor saturation. Analysis and design*. Boston: Birkhäuser, 2001.
- [11] J. Adamy, *Strukturvariable Regelungen mittels impliziter Lyapunov-Funktionen*, ser. Fortschr.-Ber. VDI Reihe 8. Düsseldorf: VDI Verlag, 1991, no. 271.
- [12] D. Yankulova, "Entwurf strukturvariabler Regelungen mittels linearer Matrixungleichungen," Ph.D. dissertation, Technische Universität Darmstadt, 2010.
- [13] H. Lens, J. Adamy, and D. Domont-Yankulova, "A fast nonlinear control method for linear systems with input saturation," *Automatica*, vol. 47, pp. 857–860, 2011.
- [14] B. Jasiewicz, *Über weiche strukturvariable Regelung mittels impliziter Lyapunov-Funktionen - von der impliziten zur expliziten Regelung*, ser. Fortschr.-Ber. VDI Reihe 8. Düsseldorf: VDI Verlag, 2010, no. 1169.
- [15] S. Chin and C. Pengilly, "The application of harmonic linearization to optimal control problems," *International Journal of Control*, vol. 12, no. 6, pp. 999–1007, 1970.