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Eigenvalue Assignment for Componentwise Ultimate Bound Minimisation in LTI Discrete-Time Systems

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Abstract— **We address optimal eigenvalue assignment in order to obtain minimum ultimate bounds on every component of the state of a linear time-invariant (LTI) discrete-time system in the presence of non-vanishing disturbances with known constant bounds. As opposed to some continuous-time cases where ultimate bounds can be made arbitrarily small by applying feedback with sufficiently high gain so that the closed-loop eigenvalues are sufficiently fast, the ultimate bound of a discrete-time system with an additive bounded disturbance can never be made smaller than some set that depends on the disturbance bound, even if all closed-loop eigenvalues are set at zero (the fastest possible in discrete-time). In this context, our contribution is twofold: (a) we single out cases where feedback that may not assign all closed-loop eigenvalues at zero achieves the minimum possible ultimate bound for some component of the system state, and (b) by employing an existing componentwise ultimate bound computation formula, we find a class of systems for which assigning all closed-loop eigenvalues at zero indeed yields minimum ultimate bounds. An intermediate result—and our third contribution—in the derivation of (b) is the obtention of the Jordan decomposition that minimises the componentwise ultimate bound formula employed.**

I. INTRODUCTION

In the presence of bounded disturbances that do not vanish as the state approaches an equilibrium point, asymptotic stability is not possible but, under certain conditions, the *ultimate boundedness* of the system's trajectories can be guaranteed [6]. A guaranteed ultimate bound on the system's trajectories can be effectively interpreted as a measure of "attenuation" of the effect of disturbances. Thus, the assignment of a prespecified ultimate bound by feedback control and/or the estimation of tight ultimate bounds are problems of interest in control system design that have many applications e.g., in systems involving quantisation [12], unknown disturbance signals [10], etc.

Estimates of ultimate bounds can be obtained through the use of level sets of suitable Lyapunov functions (see, e.g., Section 9.2 of [6]). This Lyapunov approach can be applied, in principle, to general nonlinear systems and it is thus very powerful. However, in the linear case the standard Lyapunov approach that employs quadratic Lyapunov functions may yield conservative bounds since the structure of the system and also possibly of the perturbation (whose norm typically needs to be bounded for the analysis) are not specifically taken into account. In [7], [8] and [1] a new method for componentwise ultimate bound computation was proposed. The method exploits the system geometry through its analysis in modal coordinates after transformation into its Jordan canonical form. This method has the advantage of not requiring the computation of a Lyapunov function for the system or bounding the norm of the perturbation vector. In addition, the aforementioned references presented examples where the bounds obtained via the componentwise approach were much tighter than those obtained via standard Lyapunov analysis employing quadratic functions. The componentwise ultimate-bound approach has been developed both for continuous- and discrete-time systems and has been successfully applied to the analysis of sampled-data systems involving quantisation [2], to the development of robust control design methods [9] and has recently been extended to switched systems [4].

Of direct relevance to the present paper is the result in [9] that shows that, for continuous-time systems, arbitrarily small ultimate bounds can be guaranteed by assigning closed-loop eigenvalues with arbitrarily large negative real part when disturbances are "matched" (that is, disturbances in the span of the system's control input matrix). For discrete-time systems it may then be natural to conjecture that assigning all closed-loop eigenvalues at zero, the "fastest" possible in discrete time, would achieve the smallest possible ultimate bound (when estimated via the componentwise formula of [8]). The current paper analyses this conjecture for systems with a single control input and a single disturbance input.

The first contribution of the paper is to show that the above conjecture is not true in general. Specifically, we establish that if the transfer function between the control input and the i-th state component has relative degree 1 and is minimum phase, then the *minimum possible* ultimate bound (not just its estimate) for the i-th component of the state can be achieved by assigning one closed-loop eigenvalue at zero and the remaining ones coinciding with the zeros of the aforementioned transfer function.

The second contribution is the characterisation of a class of second-order discrete-time systems for which we prove that all components of the ultimate bound of [8] are simultaneously minimised when all closed-loop eigenvalues are assigned at zero. An intermediate result—given for systems of arbitrary order, not only second order—is to establish an optimal choice of generalised eigenvectors for the Jordan

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canonical form of the closed-loop evolution matrix for which all components of the ultimate bound of [8] are simultaneously minimised. This intermediate result constitutes our third contribution and extends previous work in [3], which showed that, in general, there is room for obtaining tighter ultimate bound estimates by adequately optimising over the available free parameters of the Jordan decomposition employed to obtain the bounds.

Notation. $\mathbb{R}$ and $\mathbb{R}_{+0}$ denote the sets of real and nonnegative real numbers, respectively. $|M|$ and $\mathbb{R}\mathrm{e}(M)$ denote the *elementwise* magnitude and real part, respectively, of a matrix or vector M, and j is the imaginary unit ($j^2 = -1$). If M is a square matrix, then $\boldsymbol{\sigma}(M)$ denotes its spectrum, i.e., the set of its eigenvalues. The i-th row of a matrix M is denoted by $M_{(i,:)}$. If $x(t)$ is a vector-valued function, then $\limsup_{t\to\infty} x(t)$ denotes the vector obtained by taking $\limsup_{t\to\infty}$ of each component of $x(t)$. If $x, y \in \mathbb{R}^n$, '$x \preceq y$' denotes the set of componentwise inequalities $x_i \leq y_i$, $i = 1, \ldots, n$, between the components of x and y.

II. ULTIMATE BOUND MINIMISATION: FROM CONTINUOUS TO DISCRETE-TIME

In this section we present the componentwise ultimate bound computation formulae of [8], which constitute the basis for our development in the forthcoming sections. We also revisit the result of [9] for continuous-time systems, where it is shown that the ultimate bound can be made arbitrarily small by assigning all closed-loop system eigenvalues with arbitrarily large negative real part. Finally, we give a counterexample that shows that assigning all eigenvalues at zero for discrete-time systems does not minimise the ultimate bound formula in general.

Consider a single-input LTI continuous or discrete-time system affected by a single disturbance input

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t), \tag{1a}$$

$$x(k+1) = Ax(k) + Bu(k) + Hw(k), \tag{1b}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ is the control input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $H \in \mathbb{R}^{n\times 1}$ are constant matrices. The disturbance variable w is bounded as $|w(\cdot)| \leq \mathbf{w}$, where $\mathbf{w} \in \mathbb{R}_{+0}$ is a nonnegative number. Throughout the paper we will assume that the pair (A, B) is controllable.

We consider the state feedback $u(\cdot) = Fx(\cdot)$ so that the closed-loop trajectories of (1), namely

$$\dot{x}(t) = A^{cl}x(t) + Hw(t), \tag{2a}$$

$$x(k+1) = A^{cl}x(k) + Hw(k), \tag{2b}$$

with $A^{cl} = A + BF$, have an ultimate bound as small as possible.

Formulae for the computation of ultimate bounds for the systems (2) are presented in Theorem 2.1 below.

Theorem 2.1 ([8][1]): Consider LTI systems (2) where the disturbance variable w is bounded as $|w(\cdot)| \leq \mathbf{w}$, for some nonnegative number $\mathbf{w} \in \mathbb{R}_{+0}$. Suppose the matrix A^{cl} is stable (i.e., all its eigenvalues have negative real part in the continuous-time case, or have magnitude less than one in the discrete-time case) and let $\Lambda = V^{-1}A^{cl}V$ be its Jordan canonical form. Then, the states of the systems are ultimately bounded as

$$\limsup_{t\to\infty} |x(t)| \preceq |V| \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| |V^{-1}H|\mathbf{w} \triangleq \mathbf{b}^c \tag{3a}$$

$$\limsup_{k\to\infty} |x(k)| \preceq |V| \left(I - |\Lambda|\right)^{-1} |V^{-1}H|\mathbf{w} \triangleq \mathbf{b} \tag{3b}$$

∘

For continuous-time systems with "matched" perturbations, ultimate bounds may be arbitrarily reduced by high-gain control, as shown in the next result.

Theorem 2.2 ([9]): Consider system (1a) with matched perturbations, i.e., with $H = \kappa B$ for some number $\kappa \in \mathbb{R}, \kappa \neq 0$. Let $c_1, \ldots, c_n$ be any set of n distinct complex numbers with negative real part and such that they appear in conjugate pairs. For each $\mu > 0$, let F be a corresponding state feedback gain that causes (see, e.g., [11]) $A^{cl} = A + BF = V\Lambda V^{-1}$, with $\Lambda = \mu\,\mathrm{diag}(c_1, \ldots, c_n)$. Then the ultimate bound $\mathbf{b}^c$ defined in (3a) satisfies

$$\lim_{\mu\to\infty} \mathbf{b}^c = 0.$$

∘

Remark 2.3: As opposed to the continuous-time case, in discrete-time the ultimate bound on a state vector component can never be smaller than the effect of the perturbation on that component in one time step, i.e., the ultimate bound on the state vector's i-th component can never be smaller than $|h_i|\mathbf{w}$, where h_i denotes the i-th component of H. This follows from direct analysis of (2b). ∘

From the continuous-time result in Theorem 2.2, a reasonable conjecture for discrete-time systems is that the minimum possible componentwise ultimate bound $\mathbf{b}$ in (3b), whose value is $|H|\mathbf{w}$ (see Remark 2.3), would be obtained by selecting the state feedback gain F so that all eigenvalues of $A^{cl} = A + BF$ are placed at zero, the discrete-time counterpart of "high-gain" feedback control. However, the following example shows that this conjecture is not generally true for LTI discrete-time systems.

Example 2.1: For the system (1b) with

$$A = \begin{bmatrix} -3.3 & 1 \\ 0.1 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad H = 0.2\,B, \quad \mathbf{w} = 1,$$

consider the assignment of two equal real eigenvalues and, accordingly, one eigenvector and one generalised eigenvector, as follows [11]:

$$\Lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad V = \begin{bmatrix} V^1 & \underbrace{V^1\alpha_1 - (\lambda I - A)^{-1}V^1}_{V^2} \end{bmatrix} \tag{4}$$

where $\lambda \in (-1, 1)$ is the eigenvalue, $V^1 \triangleq (\lambda I - A)^{-1}B$ is the eigenvector, and $\alpha_1 \in \mathbb{R}$ is a free coefficient that determines the direction of the generalised eigenvector V^2.

Substituting (4) into (3b) and optimising first with respect to α_1 (see Theorem 4.1 below) and then λ, we find that the minimum value of the first component of the ultimate bound, $\mathbf{b}_1$, is achieved at $\lambda = 0.7$, while the minimum of the second component, $\mathbf{b}_2$, is obtained at $\lambda = 0$, as seen in Fig. 1. ∘

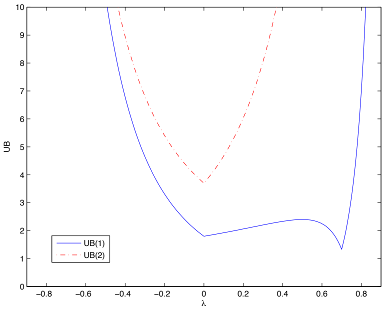

Fig. 1. Example where the ultimate bound is not minimum at $\lambda = 0$.

Example 2.1 shows that the assignment of all eigenvalues at zero is not, in general, the best choice in order to minimise ultimate bounds for discrete-time systems. This fact invalidates the previous conjecture, stated after Theorem 2.2. Hence, it is of interest to investigate and characterise the cases, if any, for which assigning all eigenvalues at zero is indeed optimal.

In general, to have a comprehensive overview on ultimate bound optimisation, we should investigate assignment of both distinct and equal eigenvalues. This is done, respectively, in Sections III and IV below. The comparison between the minimum ultimate bounds corresponding to each of these two assignment options and (for systems of order larger than two) a combination of equal/distinct closed-loop eigenvalues, yields the best eigenstructure choice as far as minimisation of the ultimate bound (3b) is concerned. Although not considered in this paper, the case of (some) closed-loop eigenvalues coinciding with (some) open-loop eigenvalues, when these are within the stability region, should also be taken into account to complete the analysis.

III. DISTINCT EIGENVALUE ASSIGNMENT

Given an n-th order single-input system, the closed-loop eigenvector matrix V associated with distinct closed-loop eigenvalues $\lambda_1, \ldots, \lambda_n \notin \boldsymbol{\sigma}(A)$, such that the corresponding eigenvalue matrix is $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, has the form

$$V = \begin{bmatrix} V^1 & \ldots & V^n \end{bmatrix} = \begin{bmatrix} v_1^1 & \ldots & v_1^n \\ \vdots & & \vdots \\ v_n^1 & \ldots & v_n^n \end{bmatrix} \tag{5a}$$

$$= \begin{bmatrix} (\lambda_1 I - A)^{-1}B & \ldots & (\lambda_n I - A)^{-1}B \end{bmatrix} \tag{5b}$$

where

$$v_i^k = \frac{[\operatorname{adj}(\lambda_k I - A)]_{(i,:)}}{\det(\lambda_k I - A)} B = \frac{\mathbf{P}_i(\lambda_k)}{\det(\lambda_k I - A)} \tag{6}$$

and

$$\mathbf{P}_i(\lambda) \triangleq [\operatorname{adj}(\lambda I - A)]_{(i,:)} B \tag{7}$$

is a polynomial of order at most $n-1$. Note that the matrix VD obtained by multiplication of (5) by any diagonal matrix D with nonzero diagonal entries is also an eigenvector matrix associated with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Since for diagonal Λ the ultimate bound given by the formula (3b) is invariant under right multiplication of V by any such diagonal matrix D (see [3]), then we can select D to be the identity matrix and consider V of the form (5) without loss of generality.

Substituting (5) and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ into (3b) yields that the i-th component of the ultimate bound is given by

$$\begin{aligned} \mathbf{b}_i &= |V_{(i,:)}|(I - |\Lambda|)^{-1}|V^{-1}H|\mathbf{w} \\ &= \frac{|v_i^1||z_1|}{1-|\lambda_1|} + \cdots + \frac{|v_i^i||z_i|}{1-|\lambda_i|} + \cdots + \frac{|v_i^n||z_n|}{1-|\lambda_n|}, \end{aligned} \tag{8}$$

where we have defined $V^{-1}H\mathbf{w} \triangleq \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}^T$. The fact $|V^{-1}H\mathbf{w}| = |V^{-1}H|\mathbf{w}$ was used in the derivation of (8), which follows since $\mathbf{w}$ is a nonnegative scalar.

The location of the roots of the polynomials $\mathbf{P}_i(\lambda)$ defined in (7) can determine whether or not placing all eigenvalues at zero gives the smallest ultimate bounds for all components. Theorem 3.1 below extracts a special case for which the assignment of all eigenvalues at zero may be not optimal.

Theorem 3.1: Let $\mathbf{b}_i$ denote the i-th component of the ultimate bound vector $\mathbf{b}$ in (3b), as given in (8). Suppose b_i, the i-th component of the input vector B, is nonzero. Then the polynomial $\mathbf{P}_i(\lambda)$ defined in (7) has degree $n-1$. Let $\sigma_1, \ldots, \sigma_{n-1}$ be its roots. If $|\sigma_k| < 1$ and $\sigma_k \neq 0$ for $k = 1, \ldots, n-1$, then assigning the closed-loop eigenvalues

$$\lambda_k = \begin{cases} \sigma_k & \text{for } k = 1, \ldots, n-1 \\ 0 & \text{for } k = n \end{cases} \tag{9}$$

results in

$$\mathbf{b}_i = |h_i|\mathbf{w} \tag{10}$$

where h_i is the i-th component of the disturbance vector H.

Proof: Since $\operatorname{adj}(\lambda I - A) = I\lambda^{n-1} + R_2\lambda^{n-2} + \cdots + R_n$, for some matrices $R_i \in \mathbb{R}^{n\times n}$, $i = 2, \ldots, n$, then the leading term of the polynomial $\mathbf{P}_i(\lambda)$ defined in (7) is $b_i\lambda^{n-1}$. Thus, the assumption $b_i \neq 0$ implies that $\mathbf{P}_i(\lambda)$ has order $n-1$. By the assignment (9), it follows from (6) that

$$v_i^k = \begin{cases} 0 & \text{for } k = 1, \ldots, n-1, \\ -\left[A^{-1}\right]_{(i,:)} B & \text{for } k = n, \end{cases} \tag{11}$$

and hence, substituting the above into (8), yields

$$\mathbf{b}_i = \frac{|v_i^n||z_n|}{1-|\lambda_n|} = |v_i^n||z_n| = \left| -\left[A^{-1}\right]_{(i,:)} B \right| |z_n|. \tag{12}$$

Let $u^1, \ldots, u^n$ be the columns of V^{-1}, and let u_n^k be the n-th component of u^k, for $k = 1, \ldots, n$. Since $VV^{-1} = I$, and considering (11), the multiplication of the i-th row of V and matrix V^{-1} gives

$$v_i^n u_n^1 = 0, \quad \ldots, \quad v_i^n u_n^i = 1, \quad \ldots, \quad v_i^n u_n^n = 0. \tag{13}$$

Substituting (13) into (12) yields

$$\begin{aligned} \mathbf{b}_i &= |v_i^n| \left| \begin{bmatrix} u_n^1 & \ldots & u_n^i & \ldots & u_n^n \end{bmatrix} H \right| \mathbf{w} \\ &= \left| \begin{bmatrix} |v_i^n| u_n^1 & \ldots & |v_i^n| u_n^i & \ldots & |v_i^n| u_n^n \end{bmatrix} H \right| \mathbf{w} \\ &= \left| \begin{bmatrix} 0 & \ldots & \pm 1 & \ldots & 0 \end{bmatrix} H \right| \mathbf{w} \quad = \quad |h_i|\mathbf{w} \end{aligned}$$

and the result then follows. ■

As discussed in Remark 2.3, the value (10) is the lowest possible value for the i-th component of the ultimate bound vector for the discrete-time system (1b). Theorem 3.1 then states that provided $b_i \neq 0$ and the roots of the polynomials $\mathbf{P}_i(\lambda)$ defined in (7) are distinct, nonzero and inside the unit circle, then the assignment (9) achieves this lowest possible ultimate bound.

Remark 3.2: The polynomial $\mathbf{P}_i(\lambda)$ defined in (7) equals the numerator of the transfer function from the control input to the i-th component of the state vector. Consequently, and since the leading term of $\mathbf{P}_i(\lambda)$ is $b_i\lambda^{n-1}$, then the conditions that $b_i \neq 0$ and all roots of $\mathbf{P}_i(\lambda)$ inside the unit circle in Theorem 3.1 can be equivalently stated as "the transfer function between the input and the i-th component of the state has relative degree 1 and is minimum phase". Hence, non-minimum phase zeros impose additional penalties to the minimisation of ultimate bounds, in correspondence with the well-known fundamental obstacle imposed by unstable zero dynamics to "high-gain" feedback performance [5]. ○

Corollary 3.3: Under the conditions of Theorem 3.1, let $u = Fx$ be the state feedback control that assigns the eigenvalues (9) to the system (1b), that is, F is such that $A + BF = V\Lambda V^{-1}$. Then the i-th row of the closed-loop matrix $A + BF$ has all its elements equal to zero and $F = -A_{(i,:)}/b_i$, where b_i is the i-th component of the control input matrix B.

Proof: Since $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then it follows from (9) and (11) that the i-th row of $V\Lambda$ is zero. Thus the i-th row of $A + BF = V\Lambda V^{-1}$ is also zero. That is,

$$A_{(i,:)} + b_i F = 0,$$

and hence $F = -A_{(i,:)}/b_i$ and the result is proved. ■

Example 3.1: Consider a randomly generated third order system with matrices

$$A = \begin{bmatrix} 0.3575 & -0.2155 & 0.4121 \\ 0.5155 & 0.3110 & -0.9363 \\ 0.4863 & -0.6576 & -0.4462 \end{bmatrix}, B = \begin{bmatrix} -0.9077 \\ -0.8057 \\ 0.6469 \end{bmatrix}, H = \begin{bmatrix} 0.3897 \\ 0.3658 \\ 0.9004 \end{bmatrix}$$

and disturbance bound $\mathbf{w} = 1$. The roots of the second order polynomials $[\operatorname{adj}(\sigma I - A)]_{(k,:)} B$, for $k = 1, 2, 3$, are

$$\begin{array}{ll} k{=}1: & \{1.2535, -0.9037\} \\ k{=}2: & \{-1.9588, 0.5376\} \\ k{=}3: & \{0.2659 \pm 0.8229j\} \end{array}$$

Thus, noticing that the first and the second sets above contain roots outside the unit circle, the only ultimate bound that can be minimised to its minimum possible value is $\mathbf{b}_3$.

According to (9), assigning the eigenvalues matrix $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 = 0.2658{+}0.8229j$, $\lambda_2 = 0.2658{-}0.8229j$, $\lambda_3 = 0$, results in

$$\mathbf{b} = \begin{bmatrix} 13.3301 & 12.9097 & 0.9004 \end{bmatrix}^T = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & |h_3|\mathbf{w} \end{bmatrix}^T.$$

○

If the conditions of Theorem 3.1 are not satisfied, the minimum ultimate bound should be determined by comparing the smallest values obtained via distinct, equal, or a combination of both, eigenvalue assignment. As a step towards this end, we next investigate the eigenstructure assignment corresponding to all equal eigenvalues.

IV. EQUAL EIGENVALUE ASSIGNMENT

Given a single input system of order n, for the assignment of n equal eigenvalues corresponding to the Jordan block

$$\Lambda = \begin{bmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & & \dots & \lambda \end{bmatrix}, \quad \lambda \in (-1, 1), \quad \lambda \notin \boldsymbol{\sigma}(A) \quad (14)$$

the generalised eigenvector matrix is

$$V = \begin{bmatrix} V^1 & \dots & V^n \end{bmatrix} = P\alpha \quad (15)$$

where

$$P = \begin{bmatrix} P^1 & \dots & P^n \end{bmatrix} \quad (16a)$$

$$P^i = (-1)^{i-1}(\lambda I - A)^{-i}B \quad \text{for } i = 1, \dots, n \quad (16b)$$

$$\alpha = \begin{bmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ 0 & \alpha_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_1 \\ 0 & \dots & 0 & \alpha_0 \end{bmatrix}, \quad \alpha_0 \triangleq 1, \quad (16c)$$

and the coefficients $\alpha_1, \dots, \alpha_{n-1}$ can be arbitrarily chosen. The ultimate bound components using the transformation matrix (15) have the form

$$\begin{aligned} \mathbf{b}_i &= |V_{(i,:)}|(I - |\Lambda|)^{-1}|V^{-1}H|\mathbf{w} \\ &= |P_{(i,:)}\alpha|(I - |\Lambda|)^{-1}|\alpha^{-1}P^{-1}H|\mathbf{w} \\ &= |P_{(i,:)}\alpha|(I - |\Lambda|)^{-1}|\beta Z| \end{aligned} \quad (17)$$

where

$$Z \triangleq P^{-1}H\mathbf{w} = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}^T \quad (18)$$

$$\beta = \alpha^{-1} = \begin{bmatrix} 1 & \beta_1 & \dots & \beta_{n-1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta_1 \\ 0 & \dots & 0 & 1 \end{bmatrix}, \quad (19)$$

$$\beta_k \triangleq -\sum_{\ell=1}^{k} \alpha_\ell \beta_{k-\ell}, \quad \text{with } \beta_0 \triangleq 1. \quad (20)$$

Direct computation of $\mathbf{b}_i$ employing (17)–(19) yields

$$\mathbf{b}_i = \sum_{k=1}^{n} \sum_{s=k}^{n} \gamma(k, s, n, 0), \text{ where} \quad (21)$$

$$\gamma(k, s, n, j) \triangleq \frac{\left|\sum_{r=1}^{k} P_i^r \alpha_{k-r}\right| \left|\sum_{\ell=s}^{n} \beta_{\ell-s} z_{\ell+j}\right|}{(1 - |\lambda|)^{s-k+1}}, \quad (22)$$

where P_i^r denotes the i-th component of the column vector P^r. From (22), it can be shown that γ satisfies the recursion

$$\gamma(k, s-1, n-1, j+1) = \gamma(k, s, n, j) \cdot (1 - |\lambda|). \quad (23)$$

In order to minimise the ultimate bound, in addition to the optimal location of the closed-loop eigenvalue λ, the optimal values for the $n{-}1$ coefficients $\alpha_1, \dots, \alpha_{n-1}$ should be determined. This is because, although the feedback gain that assigns all closed-loop eigenvalues at λ only depends on λ and not on $\alpha_1, \dots, \alpha_{n-1}$, different matrices $V = P\alpha$ as in (15)–(16c) may yield different ultimate bounds when computed using the formula (3b). Note also that the ultimate bound computed as in (3b) is invariant under scaling of V, i.e., replacing V by κV for any real number $\kappa \neq 0$ gives the same bound. This justifies the selection of $\alpha_0 = 1$ in (16c) without loss of generality. The optimal choice of $\alpha_1, \dots, \alpha_{n-1}$ is revealed in the following theorem.

Theorem 4.1: Consider the i-th ultimate bound component (17) obtained via the eigenvalue/eigenvector assignment (14)–(16) and assume that $\lambda \notin \boldsymbol{\sigma}(A)$. Let $z_1, \dots, z_n$ be defined as in (18). If $z_n \neq 0$, define $\mathrm{N} \triangleq n$. If $z_n = 0$, let N be the least index for which $z_j = 0$ for $j = \mathrm{N}+1, \dots, n$. Then, an optimal choice of the coefficients $\alpha_1, \dots, \alpha_{n-1}$ in order to have minimum ultimate bounds is

$$\alpha_k = \begin{cases} \frac{z_{\mathrm{N}-k}}{z_{\mathrm{N}}} & \text{for } k = 1, \dots, \mathrm{N}-1, \\ 0 & \text{for } k = \mathrm{N}, \dots, n-1, \end{cases} \tag{24}$$

for which the ultimate bound equals

$$\mathbf{b}_i = \sum_{k=1}^{\mathrm{N}} \frac{\left|\sum_{r=1}^{k} P_i^r z_{\mathrm{N}-k+r}\right|}{(1-|\lambda|)^{\mathrm{N}-k+1}}, \quad i = 1, \dots, n. \tag{25}$$

Proof: We proceed by induction on the system order, n. First, consider the case $n = 2$. Direct computation of (3b) from (21)–(22) with $n = 2$ yields

$$\mathbf{b}_i = \frac{|P_i^1||z_2\,\alpha_1 - z_1|}{1-|\lambda|} + \frac{|P_i^1\,\alpha_1 + P_i^2|\,|z_2|}{1-|\lambda|} + \frac{|P_i^1|\,|z_2|}{(1-|\lambda|)^2} \triangleq \mu_i(\alpha_1) \tag{26}$$

for $i = 1, 2$, where z_k and P_i^k, for $i, k = 1, 2$, depend on λ but not on α_1, and are well-defined whenever $\lambda \notin \boldsymbol{\sigma}(A)$. For each fixed value of $\lambda \in (-1, 1)$ so that $\lambda \notin \boldsymbol{\sigma}(A)$ and for which both $z_2 \neq 0$ and $P_i^1 \neq 0$, the function $\mu_i(\alpha_1)$ defined in (26) is continuous and its graph is a line with negative slope from $-\infty$ to $m_i \triangleq \min\{z_1/z_2, -P_i^2/P_i^1\}$, it is constant between m_i and $M_i \triangleq \max\{z_1/z_2, -P_i^2/P_i^1\}$, and has positive slope from M_i to ∞. Thus, any $\alpha_1 \in [m_i, M_i]$ is a minimiser of $\mathbf{b}_i$ and it is possible to select

$$\alpha_1 = \frac{z_1}{z_2}, \tag{27}$$

for which $\mathbf{b}_i$ is minimum for $i = 1, 2$. If λ is such that either $z_2 = 0$ or $P_i^1 = 0$, then the expression (26) is independent of α_1 and, hence, any choice of α_1 is equally optimal. Therefore, (24) is true for $n = 2$.

We next consider a system of arbitrary order n. The i-th component of the ultimate bound is given by (21)–(22), which we write as

$$\begin{aligned} \mathbf{b}_i &= \sum_{k=1}^{n-1}\sum_{s=k}^{n-1} \gamma(k, s, n, 0) + \sum_{k=1}^{n} \gamma(k, n, n, 0) \\ &= \sum_{k=1}^{n-1} \left[\sum_{s=k}^{n-1} \frac{\gamma(k, s-1, n-1, 1)}{1-|\lambda|}\right] + \sum_{k=1}^{n} \gamma(k, n, n, 0), \end{aligned} \tag{28}$$

where we have used the recursion formula (23). We have

$$\begin{aligned} &\sum_{s=k}^{n-1} \gamma(k, s-1, n-1, 1) = \\ &= \sum_{s=k}^{n-1} \gamma(k,s,n-1,1) + \gamma(k,k-1,n-1,1) - \gamma(k,n-1,n-1,1) \end{aligned} \tag{29}$$

Substituting (29) into (28) and using (23) again, we arrive at

$$\mathbf{b}_i = \underbrace{\sum_{k=1}^{n-1}\sum_{s=k}^{n-1} \frac{\gamma(k, s, n-1, 1)}{1-|\lambda|}}_{\triangleq R(n-1)} + \underbrace{\sum_{k=1}^{n} \gamma(k, k, n, 0)}_{\triangleq S(n)} \tag{30}$$

The expression $R(n-1)$ in (30) does not depend on α_{n-1}. By the induction hypothesis that the result holds for a system of order $n-1$, it follows that the expression [recall (21)–(22)]

$$\sum_{k=1}^{n-1}\sum_{s=k}^{n-1} \gamma(k, s, n-1, 0) \tag{31}$$

is minimised by selecting

$$\alpha_k = \begin{cases} \frac{z_{\mathrm{N}'-k}}{z_{\mathrm{N}'}} & \text{for } k = 1, \dots, \mathrm{N}'-1 \\ 0 & \text{for } k = \mathrm{N}', \dots, n-2, \end{cases} \tag{32}$$

where N' denotes the least index for which $z_j = 0$ for $j = \mathrm{N}'+1, \dots, n-1$. By (22), this fact implies that $R(n-1)$ in (30) is minimised by the selection

$$\alpha_k = \begin{cases} \frac{z_{\mathrm{N}-k}}{z_{\mathrm{N}}} & \text{for } k = 1, \dots, \mathrm{N}-1, \\ 0 & \text{for } k = \mathrm{N}, \dots, n-2, \end{cases} \tag{33}$$

which coincides with (24) for $k = 1, \dots, n-2$.

Using (22), the quantity $S(n)$ as defined in (30) can be written as

$$\begin{aligned} S(n) &= \sum_{k=1}^{n} \frac{\left|\sum_{r=1}^{k} P_i^r \alpha_{k-r}\right| \left|\sum_{\ell=k}^{n} \beta_{\ell-k} z_\ell\right|}{1-|\lambda|} \\ &= \sum_{k=1}^{\mathrm{N}} \frac{\left|\sum_{r=1}^{k} P_i^r \alpha_{k-r}\right| \left|\sum_{\ell=k}^{\mathrm{N}} \beta_{\ell-k} z_\ell\right|}{1-|\lambda|} = S(\mathrm{N}), \end{aligned} \tag{34}$$

where we have used (24) and the definition of N. Substitution of (33) into (34), and cancelling out the $|z_{\mathrm{N}}|$ factor, yields

$$S(n) = \sum_{k=1}^{\mathrm{N}} \frac{\left|\sum_{r=1}^{k} P_i^r z_{\mathrm{N}-k+r}\right| \left|\sum_{\ell=k}^{\mathrm{N}} \beta_{\ell-k} \alpha_{\mathrm{N}-\ell}\right|}{1-|\lambda|}. \tag{35}$$

Since $\beta = \alpha^{-1}$, it follows from (16c), (19), (20) and (24) that

$$\sum_{\ell=k}^{\mathrm{N}} \beta_{\ell-k} \alpha_{\mathrm{N}-\ell} = \begin{cases} 1 & \text{if } k = \mathrm{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Therefore, the expression for $S(n)$ reduces to

$$S(n) = \frac{\left|\sum_{r=1}^{\mathrm{N}} P_i^r z_r\right|}{1-|\lambda|} = \frac{|h_i|\mathbf{w}}{1-|\lambda|}, \tag{37}$$

where we have used (16), (18) and the definition of N. From (14) and the fact that $|\lambda| < 1$, it follows that the matrix $(I - |\Lambda|)^{-1}$ can be decomposed as

$$(I - |\Lambda|)^{-1} = \frac{I}{1 - |\lambda|} + M(\lambda), \tag{38}$$

where $M(\lambda)$ is a strictly upper triangular matrix (i.e. with zeros on the main diagonal) whose nonzero entries have the form $(1 - |\lambda|)^{-k}$ for $k \geq 2$. By (3b) and (38), we can write

$$\mathbf{b}_i = \frac{|V|_{(i,:)}|V^{-1}H|\mathbf{w}}{1 - |\lambda|} + |V|_{(i,:)}M(\lambda)|V^{-1}H|\mathbf{w} \tag{39}$$

Comparison of (39) and (30), recalling (22) and (35), yields

$$S(n) = \frac{|V|_{(i,:)}|V^{-1}H|\mathbf{w}}{1 - |\lambda|}, \tag{40}$$

$$R(n-1) = |V|_{(i,:)}M(\lambda)|V^{-1}H|\mathbf{w}. \tag{41}$$

For every V invertible, we have

$$\begin{aligned} &|H| = |VV^{-1}H| \preceq |V||V^{-1}H| \text{ whence} \\ &|h_i| \leq |V|_{(i,:)}|V^{-1}H| \end{aligned} \tag{42}$$

and using (42) in (40), it follows that

$$S(n) \geq \frac{|h_i|\mathbf{w}}{1 - |\lambda|} \tag{43}$$

for every V and hence for every selection of $\alpha_1, \ldots, \alpha_{n-1}$. From this fact and (37) we conclude that the selection (24) minimises $S(n)$. Thus, (24) minimises both $R(n-1)$ (by the induction hypothesis) and $S(n)$, and hence $\mathbf{b}_i$ for $i = 1, \ldots, n$. The proof by induction is now complete. Finally, (25) follows after substituting (24) into (21)–(22) and using (36). ■

Theorem 4.1 establishes that for systems having all eigenvalues equal and a single disturbance input, there is an optimal choice of generalised eigenvectors in the Jordan canonical transformation associated with the given repeated eigenvalue that achieves the smallest possible ultimate bound using the componentwise formula. This result extends previous work in [3] which showed that, in general, there is room for obtaining tighter ultimate bound estimates by adequately optimising in the available free parameters of the Jordan decomposition employed to obtain the bounds.

Remark 4.2: The work in [3] considered systems with as many disturbance inputs as states, that is, $H = I$ in (1b). In this case, it is in general not possible to find common free parameters in the Jordan transformation that simultaneously minimise the ultimate bound for *all* state components. As we have seen in Theorem 4.1, for a single disturbance input there is a common choice that minimises all components simultaneously. ○

To complete the argument following Theorem 3.1 and for the situation that the conditions in Theorem 3.1 are not satisfied, the minimum ultimate bounds obtained from distinct, equal, or a combination of both, eigenvalue assignment should be computed and compared. To illustrate, an example of a second order system is studied below.

Example 4.1: For a second order system with matrices
$A = \left[\begin{smallmatrix} -0.1396 & 0.8098 \\ -0.6304 & 0.9595 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ -0.1223 \end{smallmatrix}\right]$, $H = \left[\begin{smallmatrix} 1 \\ -0.7778 \end{smallmatrix}\right]$,
the roots of the polynomials $\mathbf{P}_1(\lambda)$ and $\mathbf{P}_2(\lambda)$ defined in (7) are 1.0585 and -5.2955, respectively, which are outside the stability region. Hence, according to Theorem 3.1, none of the ultimate bound components can be minimised to its minimum possible value. Taking $\mathbf{w} = 1$ and searching over distinct eigenvalues results in $\mathbf{b}_1^{\min} = 4.481$ with $\{\lambda_1, \lambda_2\} = \{0.8644, -0.1525\}$ and $\mathbf{b}_2^{\min} = 4.963$ with $\{\lambda_1, \lambda_2\} = \{-0.5593, 0.4576\}$. Next, setting equal eigenvalues and looking for the minimum ultimate bound yields $\lambda = 0$ as the best eigenvalue and $\mathbf{b}^{\min} = [3.0039 \ 2.0035]$. By comparing the minimum values achieved by distinct and equal eigenvalue assignment, it can be seen that $\lambda = 0$ gives the smallest ultimate bounds for both components. ○

In the following section we investigate minimum ultimate bound assignment for second order systems for which both entries of the input vector B are nonzero but the condition in Theorem 3.1 regarding the roots of the polynomials (7) are not satisfied for either component of the ultimate bound.

V. MINIMUM ULTIMATE BOUND ASSIGNMENT FOR A CLASS OF SECOND ORDER SYSTEMS

We will show in this section that for second order systems for which the conditions of Theorem 3.1 do not hold for either state, that is, both polynomials $\mathbf{P}_1(\lambda)$ and $\mathbf{P}_2(\lambda)$ defined in (7) have order one (this requires both entries of the input vector B to be nonzero) but their roots are outside the unit circle, and provided the root of a third polynomial associated with the disturbance vector H is outside the unit circle, then the assignment of both eigenvalues at zero achieves the best ultimate bound vector using the formula (3b).

Consider the system (1b) with matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}. \tag{44}$$

For the above system with $b_1 \neq 0$ and $b_2 \neq 0$, we have

$$\mathbf{P}_1(\lambda) \triangleq b_1\lambda - a_{22}b_1 + a_{12}b_2 = b_1(\lambda - c) \tag{45}$$

$$\mathbf{P}_2(\lambda) \triangleq b_2\lambda - a_{11}b_2 + a_{21}b_1 = b_2(\lambda - d) \tag{46}$$

with roots

$$c \triangleq a_{22} - a_{12}\frac{b_2}{b_1}, \quad d \triangleq a_{11} - a_{21}\frac{b_1}{b_2}. \tag{47}$$

Direct calculations show that $c \neq d$ if and only if (A, B) is controllable (which we have assumed). Suppose

$$|c| \geq 1 \quad |d| \geq 1 \tag{48}$$

and hence the result in Theorem 3.1 does not hold.

We require a third polynomial involving the disturbance matrix H. Suppose $h_1 \neq 0$, $h_2 \neq 0$ and define

$$\mathbf{K}(\lambda) \triangleq (b_2h_1 - b_1h_2)\lambda - (b_2h_1d - b_1h_2c). \tag{49}$$

We will first consider the case where B and H are not aligned, that is,

$$B^{\perp}H \triangleq \begin{bmatrix} -b_2 & b_1 \end{bmatrix} H = b_2h_1 - b_1h_2 \neq 0. \tag{50}$$

When (50) holds, (49) has a well-defined root

$$r \triangleq \frac{b_2 h_1 d - b_1 h_2 c}{b_2 h_1 - b_1 h_2}. \tag{51}$$

It can be easily checked that $h_1 \neq 0$, $h_2 \neq 0$ and $c \neq d$ ensure $r \neq c$ and $r \neq d$. (The analysis for $h_1 = 0$ or $h_2 = 0$ is simpler and not included due to space limitations.)

The main result of this section is the following.

Theorem 5.1: Consider the second order system (1b), (44), with the definitions (45)–(47), (49) and (51), and so that (48) holds. Then, the minimum ultimate bound computed via the formula (3b), denoted $\mathbf{b}^e$, is achieved by assigning both closed-loop eigenvalues at $\lambda = 0$ in the following cases:

(i) condition (50) holds and r defined in (51) satisfies $|r| \geq 1$. In this case

$$\mathbf{b}^e = \begin{bmatrix} |h_1| + |h_1| \frac{|rc|}{|r-c|} \\ |h_2| + |h_2| \frac{|rd|}{|r-d|} \end{bmatrix} \mathbf{w}. \tag{52}$$

(ii) condition (50) does not hold. In this case $H = \kappa B$, for some $\kappa \in \mathbb{R}, \kappa \neq 0$, and

$$\mathbf{b}^e = \begin{bmatrix} |\kappa\, b_1|(1 + |c|) \\ |\kappa\, b_2|(1 + |d|) \end{bmatrix} \mathbf{w}. \tag{53}$$

The remainder of this section contains the proof of Theorem 5.1. The proof of Theorem 5.1(i) follows from Lemmas 5.2, 5.3 and 5.4, which are given below and address the assignment of equal eigenvalues, distinct real eigenvalues and complex conjugate eigenvalues, respectively. The proof of Theorem 5.1(ii) is given by Lemma 5.5.

Lemma 5.2 (Equal Eigenvalues): Consider the second order system (1b), (44), with the definitions (45)–(47), (49) and (51). If conditions (48) and (50) hold, then the minimum ultimate bound vector obtained by assigning two equal eigenvalues is achieved at $\lambda = 0$ and given by $\mathbf{b}^e$ in (52).

Proof: (Outline.) We consider the first component of $\mathbf{b}^e$ (an identical analysis can be made for the second component). Direct computation of $\mathbf{b}_1^e$ employing (25) in Theorem 4.1 and using (45)–(47), (49)–(51), yields

$$\mathbf{b}_1^e(\lambda) = \frac{|h_1|\mathbf{w}}{1 - |\lambda|} + \frac{|h_1|\mathbf{w}}{(1 - |\lambda|)^2} \frac{|\lambda - r||\lambda - c|}{|r - c|}. \tag{54}$$

The result that $\lambda = 0$ is the minimum of (54) for $|c| \geq 1$ can be obtained by considering different cases for the location of r relative to c. For each of these cases different intervals for λ have to be taken so that the absolute values can be removed and the derivative of (54) with respect to λ computed to analyse the critical points and monotonicity properties of $\mathbf{b}_1^e$. The result (52) is then obtained by direct substitution of $\lambda = 0$ in (54) and the analogous expression for $\mathbf{b}_2^e(\lambda)$. ■

Lemma 5.3 (Distinct Real Eigenvalues): Consider the second order system (1b), (44), with the definitions (45)–(47), (49) and (51). If conditions (48) and (50) hold, then any ultimate bound vector achieved via the assignment of two distinct real eigenvalues, denoted as $\mathbf{b}^{dr}$, satisfies

$$\mathbf{b}^{dr} > \mathbf{b}^e \tag{55}$$

where $\mathbf{b}^e$ is given in (52).

Proof: (Outline.) Direct calculation using (5)–(8) for $n = 2$ and (45)–(47), (49)–(51), yields

$$\mathbf{b}^{dr}(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{|h_1|\mathbf{w}|\lambda_1 - c||\lambda_2 - r|}{|r-c|(1-|\lambda_1|)|\lambda_1 - \lambda_2|} + \frac{|h_1|\mathbf{w}|\lambda_2 - c||\lambda_1 - r|}{|r-c|(1-|\lambda_2|)|\lambda_1 - \lambda_2|} \\ \frac{|h_2|\mathbf{w}|\lambda_1 - d||\lambda_2 - r|}{|r-d|(1-|\lambda_1|)|\lambda_1 - \lambda_2|} + \frac{|h_2|\mathbf{w}|\lambda_2 - d||\lambda_1 - r|}{|r-d|(1-|\lambda_2|)|\lambda_1 - \lambda_2|} \end{bmatrix} \tag{56}$$

where λ_1 and λ_2, the eigenvalues to be assigned, are real numbers of magnitude smaller than one.

The ultimate bound minimisation is performed in two steps since (56) is a function of two variables, λ_1 and λ_2. First, $\mathbf{b}^{dr}(\lambda_1, \lambda_2)$ is differentiated with respect to λ_1 to obtain the optimal $\lambda_1 = \lambda_1^{\text{opt}}$ as a function of λ_2 and then, $\lambda_1^{\text{opt}}(\lambda_2)$ is substituted into $\mathbf{b}^{dr}(\lambda_1, \lambda_2)$ to differentiate with respect to λ_2 and find the optimal minimiser $\lambda_2 = \lambda_2^*$. Finally, the resulting minimum ultimate bound $\mathbf{b}^{dr}(\lambda_1^*, \lambda_2^*)$, where $\lambda_1^* = \lambda_1^{\text{opt}}(\lambda_2^*)$, is compared with the minimum ultimate bound obtained by equal eigenvalues assignment, (52).

In order to simplify the differentiation of (56), one should consider all the different cases for which the absolute values can be taken out from the expression. To this purpose, the two basic cases are when r defined in (51) is inside or outside the unit circle. Due to space limitations, we only present the result for $|r| \geq 1$ for the first component of (56) (the result for $|r| < 1$ and the second component of (56) can be obtained similarly). After differentiation as described above, the critical points and their corresponding minimum ultimate bound are obtained as

$$\{\lambda_1^*, \lambda_2^*\} = \left\{ \frac{c+1-\text{sign}(c)\sqrt{c^2-1}}{2}, \frac{c-1-\text{sign}\sqrt{c^2-1}}{2} \right\},$$

$$\mathbf{b}_1^{dr}(\lambda_1^*, \lambda_2^*) = \frac{\text{sign}(cr)|h_1|\mathbf{w}\left(2cr - 1 + \text{sign}(c)2r\sqrt{c^2-1}\right)}{|r-c|}. \tag{57}$$

Note that (57) subsumes all four cases of $|c| > 1$ and $|r| > 1$. (The cases $|c| = 1$ or $|r| = 1$ are treated separately and not included due to space limitations.) The result (55) is obtained by straightforward comparison between (57) and (the first component of) (52) for the mentioned four cases. ■

Lemma 5.4 (Complex Conjugate Eigenvalues): Consider the second order system (1b), (44), with the definitions (45)–(47), (49) and (51). If conditions (48) and (50) hold, and r defined in (51) satisfies $|r| \geq 1$, then any ultimate bound vector achieved via the assignment of complex conjugate eigenvalues, denoted as $\mathbf{b}^{cc}$, satisfies

$$\mathbf{b}^{cc} > \mathbf{b}^e \tag{58}$$

where $\mathbf{b}^e$ is given in (52).

Proof: (Outline.) Let $\lambda_1 = \rho \mathrm{e}^{j\theta}$ and $\lambda_2 = \rho \mathrm{e}^{-j\theta}$, for $\rho \in (0, 1)$ and $\theta \in (0, 2\pi)$. Substituting these expressions in (56) we have

$$\mathbf{b}^{cc}(\rho, \theta) = \begin{bmatrix} |h_1| \frac{|\rho \mathrm{e}^{j\theta} - c||\rho \mathrm{e}^{-j\theta} - r|}{|r-c|(1-\rho)\rho|\sin\theta|} \\ |h_2| \frac{|\rho \mathrm{e}^{j\theta} - d||\rho \mathrm{e}^{-j\theta} - r|}{|r-d|(1-\rho)\rho|\sin\theta|} \end{bmatrix} \mathbf{w} \tag{59}$$

Consider the first component of (59) (an analogous derivation holds for the second component). We introduce the function

$$\Psi(\theta, \rho) \doteq \frac{|\frac{\rho}{c} e^{j\theta} - 1||\frac{\rho}{r} e^{j\theta} - 1|}{(1-\rho)\rho|\sin\theta|},$$

with which the first component in (59) is written as

$$|h_1|\mathbf{w}\tfrac{|\rho e^{j\theta}-c||\rho e^{-j\theta}-r|}{|r-c|(1-\rho)\rho|\sin\theta|} = |h_1|\mathbf{w}\Psi(\theta,\rho)\tfrac{|rc|}{|r-c|}. \tag{60}$$

To show the result we will find a lower bound of $\Psi(\theta,\rho)$ for all $\theta \in (0,2\pi)$ and $\rho \in (0,1)$. We separate the analysis in two cases: (a) $cr > 0$, and (b) $cr < 0$.

a) Case $cr > 0$: Write

$$\Psi^2(\theta,\rho) = \underbrace{\left[\frac{(1+\frac{\rho^2}{c^2}-2\frac{\rho}{c}\cos\theta)}{(1-\rho)\rho|\sin\theta|}\right]}_{\doteq\psi_c}\underbrace{\left[\frac{(1+\frac{\rho^2}{r^2}-2\frac{\rho}{r}\cos\theta)}{(1-\rho)\rho|\sin\theta|}\right]}_{\doteq\psi_r}. \tag{61}$$

It may be verified that ψ_c and ψ_r are minimised over θ at $\theta_c^* = \arccos(\frac{2\rho c}{c^2+\rho^2})$ and $\theta_r^* = \arccos(\frac{2\rho r}{r^2+\rho^2})$, with minimum values $\psi_c(\theta_c^*,\rho) = \frac{1-\rho^2/c^2}{\rho(1-\rho)}$ and $\psi_r(\theta_r^*,\rho) = \frac{1-\rho^2/r^2}{\rho(1-\rho)}$. In turn, $\psi_c(\theta_c^*,\rho)$ and $\psi_r(\theta_r^*,\rho)$ are minimised over ρ at

$$\rho_c^* = \left(c^2-|c|\sqrt{c^2-1}\right) \quad \text{and} \quad \rho_r^* = \left(r^2-|r|\sqrt{r^2-1}\right), \tag{62}$$

where $\rho_c^*, \rho_r^* \in (0,1)$ for $|c| > 1$ and $|r| > 1$, and $\rho_c^* = 1$ or $\rho_r^* = 1$ correspond to infima if $|c| = 1$ or $|r| = 1$. The corresponding minimum/infimum values are

$$\psi_c(\theta_c^*,\rho_c^*) = 2\left(1+\sqrt{1-1/c^2}\right), \quad \psi_r(\theta_r^*,\rho_r^*) = 2\left(1+\sqrt{1-1/r^2}\right). \tag{63}$$

From (63) we have that $\psi_c(\theta_c^*,\rho_c^*) \geq 2$ and $\psi_r(\theta_r^*,\rho_r^*) \geq 2$, which in turn imply from (61) that $\Psi(\theta,\rho) > 2$ [$\Psi(\theta,\rho) = 2$ is excluded because it corresponds to infima attained on the boundary of the stability region] for all $(c \geq 1, r \geq 1)$ or $(c \leq -1, r \leq -1)$. Since in either of these cases the inequality $\frac{|rc|}{|r-c|} \geq 1$ holds, we have that $\Psi(\theta,\rho)\frac{|rc|}{|r-c|} > \frac{2|rc|}{|r-c|} \geq 1 + \frac{|rc|}{|r-c|}$, from which, on comparing (59) and (52), follows that (58) holds for $|c| \geq 1$ and $|r| \geq 1$ with $cr > 0$.

b) Case $cr < 0$: Without loss of generality assume $c > 0$ and $r < 0$, and let $\bar{r} = -r > 0$. In this case and we write $\Psi^2(\theta,\rho)$ from (61) as

$$\Psi^2(\theta,\rho) = \left[\frac{(1+\frac{\rho^2}{c^2}-2\frac{\rho}{c}\cos\theta)(1+\frac{\rho^2}{\bar{r}^2}+2\frac{\rho}{\bar{r}}\cos\theta))}{\sin^2\theta}\right]\left[\frac{1}{(1-\rho)^2\rho^2}\right].$$

It may be shown that the first factor on the RHS of this equation is minimised at $\theta^* = \arccos\left[\frac{\rho(c-\bar{r})}{c\bar{r}-\rho^2}\right]$ and yields

$$\Psi^2(\theta^*,\rho) = \frac{1+\frac{\rho^2}{c\bar{r}}}{(1-\rho)^2\rho^2} > \frac{1}{(1-\rho)^2\rho^2} \geq 16, \tag{64}$$

for $\rho \in (0,1)$, $c \geq 1, \bar{r} \geq 1$. Since, in this case, $\frac{|rc|}{|r-c|} = \frac{|\bar{r}c|}{|\bar{r}+c|} \geq \frac{1}{2}$, we have from (64) that $\Psi(\theta,\rho)\frac{|rc|}{|r-c|} > \frac{4|rc|}{|r-c|} > 1 + \frac{|rc|}{|r-c|}$, from which (58) follows, completing the proof. ■ The proof of Theorem 5.1(i) thus follows by combining Lemmas 5.2, 5.3 and 5.4. The proof of Theorem 5.1(ii) is given next as Lemma 5.5.

Lemma 5.5: Consider the second order system (1b), (44) with the definitions (45)–(47), (49) and (51), and suppose that (48) holds. If $H = \kappa B$, with $\kappa \in \mathbb{R}, \kappa \neq 0$, then the minimum ultimate bound computed via the formula (3b) is given by (53) and achieved by assigning both closed-loop eigenvalues at $\lambda = 0$.

Proof: Proceeding as in the proofs of Lemmas 5.2, 5.3 and 5.4 taking $H = \kappa B$, yields that (53) is the minimum achievable ultimate bound when assigning equal eigenvalues, $\mathbf{b}^{dr} = 2(|c| + \sqrt{c^2-1})|\kappa B|\mathbf{w}$ when assigning distinct real eigenvalues and $\mathbf{b}^{cc} = 4|c||\kappa B|\mathbf{w}$ when assigning complex conjugate eigenvalues. (These results can also be obtained formally by taking the limit as $r \to \infty$ in the corresponding expressions for H and B not aligned.) It is easy to check that $\mathbf{b}^{dr} > \mathbf{b}^e$ and $\mathbf{b}^{cc} > \mathbf{b}^e$, thus proving the result. ■

VI. CONCLUSIONS

This paper has analysed minimum ultimate bounds for a LTI discrete-time system with non-vanishing disturbances. We have shown that in discrete-time systems "high-gain feedback" does not necessarily lead to small ultimate bounds to matched input disturbances, as is the case for continuous-time systems, where fast closed-loop eigenvalues can arbitrarily reduce these bounds. We have presented conditions under which not placing all eigenvalues at the origin leads to the lowest possible ultimate bound for one component of the system state. These conditions require that the transfer function from the control input to that state have relative degree one and be minimum phase, which points to a well-known fundamental obstacle imposed by open-loop non-minimum phase zeros to high-gain feedback performance. As a counterpart result, we have characterised a class of discrete-time systems for which assigning all closed-loop eigenvalues at zero does indeed attain the minimum ultimate bounds, and have provided a general construction for the Jordan decomposition that minimises the componentwise ultimate bound formula employed. Follow-up work directions include the extension of the present results to a more general characterisation of structural limitations to ultimate bounds, and applications in ultimate bound minimisation by feedback in switched discrete-time systems with control inputs.

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