

Ergodicity and Class-Ergodicity of Balanced Asymmetric Stochastic Chains

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Abstract—Unconditional consensus is the property of a consensus algorithm for multiple agents, to produce consensus irrespective of the particular time or state at which the agent states are initialized. Under a weak condition, so-called balanced asymmetry, on the sequence (A_n) of stochastic matrices in the agents states update algorithm, it is shown that (i) the set of accumulation points of states as n grows large is finite, (ii) the asymptotic unconditional occurrence of single consensus or multiple consensus is directly related to the property of absolute infinite flow of this sequence, as introduced by Touri and Nedić. The latter condition must be satisfied on each of the islands of the so-called unbounded interactions graph induced by (A_n) , as defined by Hendrickx et al. The property of balanced asymmetry is satisfied by many of the well known discrete time consensus models studied in the literature.

I. INTRODUCTION

Consensus problems in multi-agent systems have gained increasing attention in various research communities. Many of the consensus algorithms in the literature can be described by linear update equations:

$$X(n+1) = A_n X(n), \quad n \geq 0, \quad (1)$$

where $X(n)$ is the vector of states (the value of an unknown parameter or probability) and A_n for every $n \geq 0$ is a stochastic matrix, i.e., each row of A_n sums to 1. Distributed averaging algorithms were first introduced by DeGroot in [1]. Later, Chatterjee and Senata [2] considered the same class of consensus problems but with time-varying interaction rates. The authors found sufficient conditions for consensus via backward products of stochastic matrices. Results of [2] were generalized in [3], [4], [5], whereby more general conditions for consensus to occur were provided. Unlike [1], [2], in the model considered in [3], [4], [5], communication links between individuals are not necessarily bidirectional. Briefly stated, sufficient conditions for convergence in [3], [4], [5] are, non vanishing interaction rates, and continuously repeated connectivity of the integrated communication graph. As an alternative model, Vicsek et al. [6] considered a system of multiple agents moving in the plane with the same speed but different headings, where heading of agents are updated according to an averaging algorithm. Consensus was observed in simulations. Jadbabaie et al. in [7] analyzed a linearized version of the Vicsek model and provided conditions under which consensus occurs. The authors showed that consensus occurs exponentially fast if there exists an infinite

sequence of contiguous, nonempty, bounded, time-intervals $[n_i, n_{i+1})$, $i \geq 0$, starting at $n_0 = 0$, with the property that across each such interval, all agents are linked together (via a chain of neighbors). Following [7], many authors tried to generalize the consensus results by employing different techniques (see [8] and references therein). Hendrickx et al. in recent work [8] generalized previous work by exploring the so-called cut-balance property. The multiple consensus problem was also considered. However, in discrete time, a uniform positive lower bound for non zero interaction rates still seemed to be necessary, unlike the corresponding continuous time theorems.

Recently, Touri and Nedić [9], [10], [11], [12] have approached the consensus problem via the backward product of stochastic matrices as in [2]. For a class of random stochastic matrices, they have derived necessary and sufficient conditions for almost sure ergodicity. Existing results on consensus in discrete time distributed averaging algorithms are subsumed in the corollaries and deterministic counterpart of theorems stated in [10], [12].

In this paper, by introducing a property of stochastic chains, herein called balanced asymmetry, we derive necessary and sufficient conditions for unconditional consensus and multiple consensus to occur in a class of multi-agent systems with dynamics described in (1). Moreover, we show that if the balanced asymmetry property is satisfied, the set of accumulation points of states is finite. As will be shown, no uniform positive lower bound for non zero interaction rates or self-interaction rates is required. One may consider this work as the complementary of our previously published work [13]. In the current work, we mainly deal with convergence theorems, their proofs, and their relationship with existing convergence results, while in [13], we focus on applications of our theorems in known models in the literature.

This paper is organized as follows: Essential notions that are required to state the main results are defined and illustrated in Section II. Main results on unconditional consensus and multiple consensus are presented and proved in Section III. The contribution of the work is illustrated in Section IV, by providing examples of both exogenous and endogenous models. Concluding remarks end the paper in Section V.

A. Notation

Throughout this article, we adopt the following notation:

- S is the set of agents and $s = |S|$ is the number of agents.
- n stands for discrete time index.
- $X(n) = [X_1(n) \cdots X_s(n)]^T$, $n \geq 0$, is the state vector.

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- For every $n \geq 0$, $(1_n, 2_n, \dots, s_n)$ is a permutation of $\{1, 2, \dots, s\}$ such that agent i_n ($1 \leq i \leq s$) has the i th least state value among all agents at time n .
- $z_i(n) = X_{i_n}(n)$ is the i th least number among $X_1(n), \dots, X_s(n)$. Particularly, $z_1(n)$ and $z_s(n)$ are the state values of agents which have the least and the greatest state values at time n respectively.
- A_n , $n \geq 0$ is the matrix of interaction rates $a_{ij}(n)$, $1 \leq i, j \leq s$.
- At any instant $n \geq 0$, by removing the link from agent j to agent i , we mean changing the value of $a_{ij}(n)$ to zero, and $a_{ii}(n)$ to $a_{ii}(n) + a_{ij}(n)$.

II. ERGODICITY AND CLASS-ERGODICITY

Definition 1: Consider a multi-agent system whose states evolve according to (1). By unconditional consensus in system (1), we mean that no matter at what instant or at what values states are initialized, all $X_i(n)$'s, $i = 1, \dots, s$, converge to identical values as n goes to infinity.

We now define ergodicity according to [9]. Let (A_n) be a chain of stochastic matrices. For $k > l \geq 0$, denote $A(n, k) = A_{n-1}A_{n-2} \dots A_k$.

Definition 2: [9] A chain (A_n) of stochastic matrices is said to be *ergodic* if for every $k \geq 0$, $\lim_{n \rightarrow \infty} A(n, k)$ exists and is equal to a matrix with identical rows.

It is possible to show that occurrence of unconditional consensus in a multi-agent system is equivalent to ergodicity of the transition chain of the system. Beside consensus, there is another important notion, multiple consensus, that constitutes our focus in this work.

Definition 3: For a multi-agent system with dynamics described by (1), unconditional multiple consensus occurs if for every i , $1 \leq i \leq s$, $\lim_{n \rightarrow \infty} X_i(n)$ exists, no matter at what instant or at what values states are initialized.

To formulate multiple consensus as a property of chains of stochastic matrices, we introduce class-ergodicity, as follows.

Definition 4: A chain (A_n) of stochastic matrices is *class-ergodic* if for every $k \geq 0$, $\lim_{n \rightarrow \infty} A(n, k)$ exists and can be relabeled as a block diagonal matrix with each block having identical rows. By relabeling, we mean applying the same permutation to rows and columns of a square matrix.

Clearly, if (A_n) in (1) is class-ergodic, unconditional multiple consensus occurs. The converse is true also, by noting that the i th column of $A(n, k)$ is equal to $X(n)$ when X is initialized at time k by the initial value e_i denoting all components equal to zero, but the i th one equal to 1. Therefore, unconditional multiple consensus occurs in a system with dynamics described by (1) if and only if chain (A_n) is class-ergodic.

Thus, our interest in unconditional consensus and multiple consensus leads us to investigate conditions for ergodicity and class-ergodicity of stochastic chains. In the rest of this section, we provide essential notions that are employed to obtain our main results.

A. l_1 -approximation [10]

The following is an equivalent definition of l_1 -approximation as developed by Touri and Nedic in [10]:

Definition 5: [10] Chain (A_n) is said to be an l_1 -approximation of chain (B_n) if $\sum_{n=0}^{\infty} \|A_n - B_n\| < \infty$, where the norm refers to the *max norm*, i.e. the maximum of the absolute values of the matrix entries.

It is not difficult to show that l_1 -approximation is an equivalence relation in the set of chains of stochastic matrices. Importance of the l_1 -approximation notion comes from the following proposition that is a result of Lemma 1 in [10].

Proposition 1: Let (A_n) be l_1 -approximation of chain (B_n) . Then, (A_n) is class-ergodic if and only if (B_n) is.

B. Absolute Infinite Flow [11]

We now recall the definition of the absolute infinite flow property of a chain and briefly review the main result of [11].

Definition 6: A chain (A_n) of stochastic matrices has the absolute infinite flow property if

$$\sum_{n=0}^{\infty} \left(\sum_{i \in T(n+1)} \sum_{j \in \bar{T}(n)} a_{ij}(n) + \sum_{i \in \bar{T}(n+1)} \sum_{j \in T(n)} a_{ij}(n) \right) = \infty \quad (2)$$

where $T(0), T(1), \dots$ is an arbitrary sequence of subsets of $\{1, \dots, s\}$ with the same cardinality, and \bar{T}_i denotes the complement of T_i in $\{1, \dots, s\}$.

In [11], the authors showed that the absolute infinite flow property is necessary for ergodicity, and in case the chain is doubly stochastic, it is also sufficient.

C. Balanced Asymmetry

Definition 7: Consider a chain (A_n) of stochastic matrices. Chain (A_n) is said to be *balanced asymmetric* if there exists an $M \geq 1$ such that for any two non empty subsets S_1 and S_2 of $S = \{1, \dots, s\}$ with the same cardinality, we have

$$\sum_{i \in S_1} \sum_{j \in \bar{S}_2} a_{ij}(n) \leq M \sum_{i \in \bar{S}_1} \sum_{j \in S_2} a_{ij}(n), \quad \forall n \geq 0. \quad (3)$$

Due to importance of the balanced asymmetry property in this work, we illustrate this property by giving the following non trivial subclasses of balanced asymmetric chains:

1) *Chains of doubly stochastic matrices:* It can be shown that all chains of doubly stochastic matrices are balanced asymmetric with $M = 1$.

2) Chain possessing the following two properties:

self-confidence: There exists $\delta > 0$ such that $a_{ii}(n) \geq \delta$ for every $i = 1, \dots, s$ and $n \geq 0$.

cut-balance: [8] There exists $K \geq 1$, such that for every $E \subset \{1, \dots, s\}$

$$\sum_{i \in E} \sum_{j \in \bar{E}} a_{ij}(n) \leq K \sum_{i \in \bar{E}} \sum_{j \in E} a_{ij}(n), \quad \forall n \geq 0. \quad (4)$$

Indeed, (4) is equivalent to (3) when S_1 is identical to S_2 , while if $S_1 \neq S_2$, (3) is satisfied by recognizing that since $S_1 \cap \bar{S}_2$ and $\bar{S}_1 \cap S_2$ are non empty, a lower bound for both sums is δ , while $s - 1$ is an upper bound. Thus, set $M = \max\{K, (s - 1)/\delta\}$. Note that in [12], chains having cut-balance property are recalled as balanced chains.

Remark 1: Balanced asymmetry is a stronger condition than cut-balance, although the latter together with self-confidence becomes stronger than the former.

Remark 2: For those chains that are l_1 -approximation of balanced asymmetric chains, the absolute infinite flow property is equivalent to:

$$\sum_{n=0}^{\infty} \sum_{i \in T(n+1)} \sum_{j \in T(n)} a_{ij}(n) = \infty \quad (5)$$

for any sequence $T(n)$ of subsets of T as in Eq. (2). This can be easily seen by combining Eqs. (2) and (3).

D. Unbounded Interactions Graph [8]

The following definition is a discrete time version of the definition given in [8].

Definition 8: Let (A_n) be a stochastic chain representing interaction rates between s agents, where $S = \{1, \dots, s\}$ is the set of agents. We form a directed graph $G_A = \{S, E\}$ with $(i, j) \in E$ if and only if $\sum_{n=0}^{\infty} a_{ij}(n) = \infty$. G_A is called the *unbounded interactions graph* of A .

Recognizing that balanced asymmetry is a stronger condition than cut-balance, a proof quite similar to that of Theorem 2 (b) in [8], leads to the following proposition.

Proposition 2: Let (A_n) be stochastic chain with unbounded interactions graph G_A . If (A_n) is balanced asymmetric, then every weakly connected component of G_A is strongly connected.

According to Proposition 2, under balanced asymmetry of the update chain, its unbounded interactions graph can be partitioned into strongly connected components, herein called *islands*.

III. CONVERGENCE RESULTS

The following theorem guarantees existence of limits on the value space of agents (recall the definition of $z_i(n)$ from Part I-A).

Theorem 1: Consider a multi-agent system with dynamics described by (1). Assume that chain (A_n) is an l_1 -approximation of a balanced asymmetric chain. Then, $\lim_{n \rightarrow \infty} z_i(n)$ exists for every $i = 1, \dots, s$.

Sketch of proof: We first mention some general properties of z_i 's. According to the definition of $z_i(n)$ we have

$$z_1(n) \leq z_2(n) \leq \dots \leq z_s(n), \quad \forall n \geq 0 \quad (6)$$

Moreover, since states of agents are updated via a convex combination of their current states, $z_1(n)$ is a non-decreasing function of n , and $z_s(n)$ is a non-increasing function of n . Therefore, from (6) we have

$$z_1(0) \leq z_1(n) \leq z_2(n) \leq \dots \leq z_s(n) \leq z_s(0) \quad (7)$$

which means that functions z_i 's are bounded above as well as below, and so are X_i 's. As an immediate result, by defining $L \triangleq z_s(0) - z_1(0)$, we have $X_i(n) - X_j(n) \leq L$, $\forall n \geq 0$. Furthermore, one can obtain that $\lim_{n \rightarrow \infty} z_1(n)$ and $\lim_{n \rightarrow \infty} z_s(n)$ exist, since both functions $z_1(n)$ and $z_s(n)$ are monotonic and bounded.

Now, let (B_n) be a balanced asymmetric chain that is an l_1 -approximation of (A_n) . Let $A_n = B_n + M_n$, $\forall n \geq 0$. Assume that $\|M_n\| = m_n$, $\forall n \geq 0$, and

$$m'_n = \sum_{k=0}^{n-1} m_k, \quad n > 0, \quad (8)$$

and define $m'_0 = 0$. Note that m'_n remains bounded. Set $K = 2M$, and define function $S_r(n)$ for every r , $1 \leq r \leq s$ by

$$S_r(n) = \sum_{i=1}^r K^{-i} (z_i(n) + sm'_n L). \quad (9)$$

We recall that $L = z_s(0) - z_1(0)$. Since S_r is a linear combination of z_1, \dots, z_s , and m'_n is bounded, it is bounded. Our aim is to show that $S_r(n)$ is non-decreasing. If we do so, from boundedness and monotonic behavior of S_r we obtain that $\lim_{n \rightarrow \infty} S_r(n)$ exists for every $r = 1, \dots, s$. Moreover, defining $S_0 \equiv 0$, (9) implies

$$z_i(n) = K^i (S_i(n) - S_{i-1}(n)) - sm'_n L. \quad (10)$$

Thus, convergence of z_i 's is immediately implied from convergence of S_i , S_{i-1} , and m'_i . Hence, to complete the proof, we only need to show that S_r is non decreasing for every $r = 1, \dots, s$. In other words, we must show that for every $r = 1, \dots, s$ and every $n \geq 0$ we have $S_r(n+1) - S_r(n) \geq 0$. According to the definition of $z_i(n)$, we know that

$$S_r(n+1) = \sum_{i=1}^r K^{-i} (X_{i_{n+1}}(n+1) + sm'_{n+1} L). \quad (11)$$

On the other hand, according to (1) we have

$$\begin{aligned} X_{i_{n+1}}(n+1) &= \sum_{j=1}^s a_{i_{n+1}j} X_j(n) \\ &= \sum_{j=1}^s a_{i_{n+1}j_n} X_{j_n}(n) \\ &= \sum_{j=1}^s a_{i_{n+1}j_n} z_j(n) \end{aligned} \quad (12)$$

Note that the second equality above holds since $(1_n, 2_n, \dots, s_n)$ is a permutation of $\{1, 2, \dots, s\}$. We also have

$$sm'_{n+1} L = \sum_{j=1}^s m'_{n+1} L. \quad (13)$$

From (11), (12), and (13) we conclude

$$\begin{aligned} S_r(n+1) &= \sum_{i=1}^r K^{-i} \sum_{j=1}^s (a_{i_{n+1}j_n} z_j(n) + m'_{n+1} L) \\ &= \sum_{i=1}^r \sum_{j=1}^s K^{-i} (a_{i_{n+1}j_n} z_j(n) + m'_{n+1} L) \end{aligned} \quad (14)$$

Moreover,

$$\begin{aligned} S_r(n) &= \sum_{i=1}^r K^{-i} (z_i(n) + sm'_n L) \\ &= \sum_{i=1}^r K^{-i} (1 \cdot z_i(n) + sm'_n L) \\ &= \sum_{i=1}^r K^{-i} \left[\left(\sum_{j=1}^s a_{i_{n+1}j_n} \right) z_i(n) \right. \\ &\quad \left. + \left(\sum_{j=1}^s m'_n L \right) \right] \\ &= \sum_{i=1}^r \sum_{j=1}^s K^{-i} (a_{i_{n+1}j_n} z_i(n) + m'_n L) \end{aligned} \quad (15)$$

From (14) and (15) we conclude

$$\begin{aligned} S_r(n+1) - S_r(n) &= \sum_{i=1}^r \sum_{j=1}^s K^{-i} [a_{i_{n+1}j_n} (z_j(n) - z_i(n)) + m_n L] \end{aligned} \quad (16)$$

Clearly, we have

$$a_{i_{n+1}j_n} (z_j(n) - z_i(n)) + m_n L \geq b_{i_{n+1}j_n} (z_j(n) - z_i(n)) \quad (17)$$

Equations (16) and (17) together imply

$$\begin{aligned} S_r(n+1) - S_r(n) &\geq \sum_{i=1}^r \sum_{j=1}^s K^{-i} b_{i_{n+1}j_n} (z_j(n) - z_i(n)) \end{aligned} \quad (18)$$

We now note that for any $j > i$ we have

$$z_j(n) - z_i(n) = \sum_{k=i}^{j-1} (z_{k+1}(n) - z_k(n)) \quad (19)$$

Similarly, for $j < i$ we have

$$z_j(n) - z_i(n) = - \sum_{k=j}^{i-1} (z_{k+1}(n) - z_k(n)) \quad (20)$$

From (18), (19), (20), and the balanced asymmetry assumption on chain (B_n) , it is possible to conclude that

$$\begin{aligned} S_r(n+1) - S_r(n) &\geq \sum_{k=1}^{r-1} \left[K^{-s} \left(\sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} \right) \right. \\ &\quad \left. (z_{k+1}(n) - z_k(n)) \right] \\ &= K^{-s} \sum_{k=1}^{r-1} \left[\left(\sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} \right) \right. \\ &\quad \left. (z_{k+1}(n) - z_k(n)) \right] \\ &\geq 0 \end{aligned} \quad (21)$$

The details are omitted for lack of space. Therefore, $S_r(n)$ is a non decreasing function of n , and Theorem 1 is proved. \blacksquare

Although Theorem 1 characterizes the limiting behavior of agents' valuations if the transition chain is an l_1 -approximation of a balanced asymmetric chain, it guarantees neither consensus nor multiple consensus. In the next two parts, we explicitly address the issues of unconditional consensus (ergodicity) and unconditional multiple consensus (class-ergodicity).

A. Ergodicity

We are now ready to state our consensus result. The following theorem provides a necessary and sufficient condition for consensus to occur in a class of models with dynamics updated via (1).

Theorem 2: If chain (A_n) is l_1 -approximation of a balanced asymmetric chain, then (A_n) is ergodic if and only if it has the absolute infinite flow property.

Proof: The necessity of the absolute infinite flow property has been proved in [11]. Here we show that if chain (A_n) has the absolute infinite flow property and is an l_1 -approximation of a balanced asymmetric chain, then it is

ergodic, or equivalently, unconditional consensus occurs in system (1). With no loss of generality, we assume that states are initialized at $n = 0$ (Otherwise, if states are initialized at $n = n_0 \neq 0$, we remove the first n_0 term of (A_n) and obtain another chain which is still an l_1 -approximation of a balanced asymmetric chain and has the absolute infinite flow property, and proceed with the new chain).

Let (B_n) be a balanced asymmetric chain with bound M which is an l_1 -approximation of (A_n) . It is straightforward to verify that chain (A_n) has the absolute infinite flow property if and only if chain (B_n) does. The main part of the proof is common with the proof of Theorem 1. According to Theorem 1, we know that $\lim_{n \rightarrow \infty} z_i(n)$ exists for every $i = 1, \dots, s$. Let us define for every i , $1 \leq i \leq s$: $Z_i \triangleq \lim_{n \rightarrow \infty} z_i(n)$. We obtain from (6) that

$$Z_1 \leq Z_2 \leq \dots \leq Z_s \quad (22)$$

Since $z_1(n)$ and $z_s(n)$ are respectively the least and the greatest element of the set $\{X_1(n), X_2(n), \dots, X_s(n)\}$ for every $n \geq 0$, consensus occurs if and only if $Z_1 = Z_s$. Assume that this does not happen, or equivalently, $Z_1 < Z_s$. We wish to show that applying the absolute infinite flow property to (21) when $r = s$, leads to an unbounded $S_r(n)$ which is a contradiction. Since $Z_1 < Z_s$, from (22) we conclude that there exists p , $1 \leq p \leq s - 1$ such that $Z_p < Z_{p+1}$. Let $\epsilon \triangleq (Z_{p+1} - Z_p)/2 > 0$. Since Z_p and Z_{p+1} are the limit values of $z_p(n)$ and $z_{p+1}(n)$ respectively, there exists $N \geq 0$ such that

$$z_{p+1}(n) - z_p(n) > \epsilon, \quad \forall n \geq N \quad (23)$$

On the other hand, for balanced asymmetric chains, the absolute infinite flow property reduces to (5). From (5), we find for any state sets sequence $T(n)$ of the same cardinality:

$$\sum_{n=N}^{\infty} \sum_{i \in T(n+1)} \sum_{j \in T(n)} b_{ij}(n) = \infty \quad (24)$$

since $\sum_{n=0}^{N-1} \sum_{i \in T(n+1)} \sum_{j \in T(n)} b_{ij}(n)$ is finite. If in (24) we set:

$$T(n) = \{1_n, 2_n, \dots, r_n\} \quad (25)$$

we obtain

$$\sum_{n=N}^{\infty} \sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} = \infty \quad (26)$$

On the other hand, we note that according to Theorem 1, $\lim_{n \rightarrow \infty} S_r(n)$ exists for every $r = 1, \dots, s$. Therefore, we can write

$$\lim_{n \rightarrow \infty} S_r(n) - S_r(0) = \sum_{n=0}^{\infty} S_r(n+1) - S_r(n) \quad (27)$$

Equations (27) and (21) yield

$$\begin{aligned}
& \lim_{n \rightarrow \infty} S_r(n) - S_r(0) \\
& \geq \sum_{n=0}^{\infty} \left\{ K^{-s} \sum_{k=1}^{r-1} \left[\left(\sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} \right) \right. \right. \\
& \quad \left. \left. (z_{k+1}(n) - z_k(n)) \right] \right\} \\
& = K^{-s} \sum_{k=1}^{r-1} \left[\sum_{n=0}^{\infty} \left(\sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} \right) \right. \\
& \quad \left. (z_{k+1}(n) - z_k(n)) \right] \tag{28}
\end{aligned}$$

Setting $r = s$ we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} S_s(n) - S_s(0) \\
& \geq K^{-s} \sum_{k=1}^{s-1} \left[\sum_{n=0}^{\infty} \left(\sum_{i=k+1}^s \sum_{j=1}^k b_{i_{n+1}j_n} \right) \right. \\
& \quad \left. (z_{k+1}(n) - z_k(n)) \right] \tag{29}
\end{aligned}$$

From the above inequality, by removing most of positive terms from the RHS (namely by keeping only terms corresponding to $k = p$ and $n \geq N$), we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} S_s(n) - S_s(0) \\
& \geq K^{-s} \sum_{n=N}^{\infty} \left(\sum_{i=p+1}^s \sum_{j=1}^p b_{i_{n+1}j_n} \right) \\
& \quad (z_{p+1}(n) - z_p(n)) \tag{30}
\end{aligned}$$

Eqs. (23) and (30) imply

$$\begin{aligned}
& \lim_{n \rightarrow \infty} S_s(n) - S_s(0) \\
& \geq \epsilon \cdot K^{-s} \sum_{n=N}^{\infty} \sum_{i=p+1}^s \sum_{j=1}^p b_{i_{n+1}j_n} \tag{31}
\end{aligned}$$

From (26) we know that the RHS of inequality (31) is unbounded. Thus, the LHS is unbounded, and so is $S_s(n)$, which is a contradiction. This completes the proof. ■

B. Class-Ergodicity

In this section, the class-ergodicity problem that is equivalent to unconditional multiple consensus is analyzed. Our result about class-ergodicity is as follows.

Theorem 3: Let chain (A_n) be an l_1 -approximation of a balanced asymmetric chain. Then, (A_n) is class-ergodic if and only if the absolute infinite flow property holds over each island of the unbounded interactions graph induced by (A_n) .

Proof: We start the proof by forming a new chain (B_n) of the bounded interactions graph G_A by eliminating all interaction coefficients between each agent within an island and agents of other islands. From definition of G_A and its islands, it is immediately implied that (B_n) is an l_1 -approximation of (A_n) , and consequently, is an l_1 -approximation of a balanced asymmetric chain. According to Proposition 1, it suffices to prove that (B_n) is class-ergodic. The system with (B_n) as transition chain can be decomposed into subsystems corresponding to islands, as there is no communication between islands at all. It is straightforward to verify that each subchain of (B_n) corresponding to a subsystem is balanced asymmetric and possesses the absolute infinite flow property. Thus, Theorem 2 implies that each subchain is ergodic. Subsequently, (B_n) is class-ergodic.

We now prove the reverse. Thus we assume that (A_n) is class-ergodic and also is an l_1 -approximation of a balanced asymmetric chain, and prove that the absolute infinite flow property holds inside each island. Once again we form chain (B_n) from (A_n) by eliminating all interaction coefficients between agents of distinct islands. Since (B_n) is an l_1 -approximation of (A_n) , Proposition 1 implies that (B_n) is class-ergodic as well. Note that (B_n) is also an l_1 -approximation of a balanced asymmetric chain, as (A_n) is. It is sufficient now to show that the absolute infinite flow property holds inside islands of the bounded interactions graph induced by chain (B_n) . Define subchains of (B_n) corresponding to islands. We show that each island subchain is ergodic. Thus, consider an arbitrary initial state for each subsystem and by concatenating these states, form an initial vector $Y(0)$ for the original system:

$$Y(n+1) = B_n Y(n), \quad n \geq 0. \tag{32}$$

Since (B_n) is assumed class-ergodic, unconditional multiple consensus occurs in system (32). Let I be an arbitrary island. We wish to show that agents of I belong to the same consensus cluster. Assume on the contrary that, there exists an island I containing agents corresponding to distinct consensus clusters. We proceed with the exact same proof as Theorem 2, identifying this time Y with X in the theorem, and taking advantage of (31) by setting p as follows: since members of island I do not belong to the same cluster, I can be partitioned into non empty I_1 and \bar{I}_1 such that

$$\lim_{n \rightarrow \infty} Y_i(n) < \lim_{n \rightarrow \infty} Y_j(n), \quad \forall i \in I_1, j \in \bar{I}_1. \tag{33}$$

Recalling that (B_n) is an l_1 -approximation of a balanced asymmetric chain, the ordered limits $\{Z_k\}_{k=1, \dots, s}$ in Theorem 1 exist. Set p equal to the maximum index k such that

$$Z_k \leq \max\{ \lim_{n \rightarrow \infty} Y_i(n) | i \in I_1 \} \tag{34}$$

and follow step (27) to (31) in Theorem 2. Since, by definition of the island I :

$$\sum_{n=0}^{\infty} \sum_{i \in \bar{I}_1, j \in I_1} b_{ij}(n) = \infty, \tag{35}$$

the RHS of (31) goes to ∞ as in the proof of Theorem 2, which is a contradiction. Therefore all agents contained in every island end up in the same consensus cluster, i.e., consensus occurs inside each island. Since the initial state was arbitrary, we obtain that every subchain is ergodic. From ergodicity and balanced asymmetry of each subchain, we conclude that the absolute infinite flow property holds for each subchain, i.e., inside each island. ■

As a result of Theorem 3, the following result provides a sufficient condition for class-ergodicity of stochastic chains. Recall definitions of self-confidence and cut-balance from Part II-C. See [13] for the proof.

Theorem 4: If chain (A_n) is an l_1 -approximation of a self-confident and cut-balanced chain, it is also class-ergodic.

IV. DISCUSSION

Interaction rates a_{ij} 's in this paper can either be purely exogenous time-varying functions, as in [7], or endogenous functions evolving by variables within the model, as in [15]. In this section, by providing example systems of each type, we illustrate the contribution of our work.

A. An Exogenous Example

Consider dynamics (1) with chain (A_n) defined by:

$$A_n = \begin{bmatrix} 1/n & 1 - 1/n \\ 1 - 1/n & 1/n \end{bmatrix} \quad (36)$$

It is not difficult to see that chain (A_n) is balanced asymmetric and satisfies the absolute infinite flow property. Therefore, Theorem 2 implies that chain (A_n) is ergodic, and consequently, unconditional consensus occurs in algorithm (1). Note that the ergodicity (A_n) is not implied by the existing results in the literature. In particular, (A_n) as defined by (36), does not admit a positive lower bound on interaction coefficient, as required in [8], nor satisfies the weak feedback property, as required in [12].

B. Finite Range Interactions Models

In this class of models, at any time instant while updating states, agents rely more on the agents whose states are currently closer to theirs. Moreover, there exists a common finite radius of detection $R > 0$. To formulate this, we define a non negative decaying continuous function $f : R^{\geq 0} \rightarrow R^{\geq 0}$ with $f(0) > f(R) = 0$, and

$$a_{ij}(n) = \frac{f(\|x_i(n) - x_j(n)\|)}{\sum_{k=1}^s f(\|x_i(n) - x_k(n)\|)} \quad (37)$$

We now use Theorem 4 to show that $\lim_{n \rightarrow \infty} x_i(n)$ exists $\forall i$, irrespective of initial states.

Since f is a non negative decaying function, we obtain that $a_{ii} \geq 1/s$. Therefore, chain (A_n) has the self-confidence property, as required in Theorem 4. On the other hand, we can show that $a_{ij} \leq sa_{ji}$, $\forall i, j$. To see why, note that a_{ij} and a_{ji} in (37) have equal numerators, while both non zero denominators stay between $f(0)$ and $sf(0)$. Thus, chain (A_n) satisfies a property stronger than cut-balance, namely the so-called sub-symmetry property [14]. Theorem 4 now results in the existence of individual limits of states.

V. CONCLUDING REMARKS

In this paper, we considered a general distributed averaging algorithm in discrete time and investigated the limiting behavior of states. In the first step, by introducing the notion of balanced asymmetry, we derived a broad class of averaging algorithms, corresponding to balanced asymmetric chains, for which multiple consensus occurs if we are able to relabel agents at any instant. In other words, under the balanced asymmetry assumption, agents valuations can be associated with a finite number of accumulation points. Although limits are not identical necessarily, the result characterizes how states of agents behave as $n \rightarrow \infty$. The balanced asymmetry property is satisfied by many of known

models such as the JLM model [7] and the Krause model [15].

Inspired by [11], we then sought necessary and sufficient conditions for unconditional consensus (ergodicity) to occur in a wide class of algorithms. Unlike most of existing results in this area, our results do not involve any uniform positive lower bound assumption on non zero interaction rates or self-interactions. We also derived necessary and sufficient conditions for unconditional existence of individual limits in the same class of algorithms. An exogenous update chain model was proposed to illustrate the possibility of ergodicity without a uniform positive lower bound assumption on the interaction coefficients.

In future work, we shall investigate how well the results extend to systems in which the number of agents increases without bound.

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