2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

# $\mathcal{H}_\infty$ control for singular stochastic systems

Asma Barbata, Michel Zasadzinski, Harouna Souley Ali and Hassani Messaoud

*Abstract*— **In this paper, we deal with the bounded real lemma for stochastic singular systems with multiplicative noises. Based on the adaptation of Itô calculus, the admissibility for this class of systems is defined and the bounded real lemma is derived using the mean square exponential stability. This lemma is then used to synthetize a $\mathcal{H}_\infty$ output feedback controller for the considered class of systems that achieves a given level of disturbance attenuation. The design is based on the solution of linear matrix inequalities coupled with an algebraic constraint.**

*Index Terms*— **Stochastic singular systems, Multiplicative noise, Itô calculus, Admissibility, Mean square exponential stability, Bounded real lemma, $\mathcal{H}_\infty$ output feedback controller, Disturbance attenuation, Linear matrix inequalities.**

## I. INTRODUCTION

The Itô stochastic systems have been investigated widely during the last decades as they get many importance in system theory ; they are also called systems with multiplicative noises because these stochastic systems are affected by multiplicative noises in the state equation. It is assumed that the noises are zero mean Brownian motions. The advantage of stochastic differential equation (SDE) is that they contain a random term which represents the randomness within the systems to model. Thus, the studied systems are composed by two parts : the drift one which corresponds to the dominant action of the system and the diffusion one representing randomness along the dominant curve. The stochastic modelling has then got a great role during the last years in engineering and sciences [1], [2], [3], [4]. Many phenomena such as population evolution, earthquakes, network models and the movement of particles in a gas can be described with this class of models (see for example [3]). In addition, modeling systems with stochastic differential equations is more realistic when the deterministic description is not satisfactory. The $\mathcal{H}_\infty$ control and observation for stochastic systems have been treated in [5], [6], [7], [8], [9], [10]

On the other hand, and always in order to better model systems, many attention has been focused on the singular systems, also called descriptor ones since many years. They present many advantages as they permit more general representation than the classical state space form : in fact more than the classical differential equations that model physical systems, they take into account non dynamical constraint and impulsive elements in their representation [11], [12], [13], [14], [15], [16]. This class of systems are described by a differential-algebraic equation (DAE).

So, combining the Itô stochastic representation and descriptor form permit to consider a large class of realistic systems which are modeled with a stochastic differential algebraic equation (SDAE) [17], [18]. But in our knowledge, there are not many works about these systems especially in the control domain. Then in this paper, our goal is to develop a bounded real lemma for this class of systems and to use it in order to design a $\mathcal{H}_\infty$ output feedback controller which ensures the mean-square exponential stability (MSES) and a disturbance attenuation level. The advantage of the $\mathcal{H}_\infty$ criterion is that the disturbances is considered to be with bounded energy only ; there are no statistical other requirements on the signal.

Our paper is organized as follows. In section II, the Itô formula is adapted to SDAE. This can be done with an analysis of the solution of the SDAE which split the state into two parts : a dynamical one and an algebraic one. This approach allows us to define the admissibility associated to a SDAE, i.e. when the algebraic constraints are free of noise and can be solved with an unique solution using the dynamical variables. In addition, the equilibrium point of the SDAE should be MSES.

The bounded real lemma for stochastic singular systems is derived in section III. This lemma guarantees that the SDAE is admissible with a disturbance attenuation condition.

A $\mathcal{H}_\infty$ controller design method is given in section IV. This represents the main result of this paper with the bounded real lemma proposed in section III. This section is decomposed into three parts. These two results are a novelty in our knowledge. Due to additive terms generated by the Itô calculus, the bounded real lemma and the $\mathcal{H}_\infty$ controller design can not be derived using the versions of the deterministic bounded real lemma given in the literature. The closed loop formula are given in section IV-A. In section IV-B, we show that our results can be directly extended to the case where the measurement equation depends on the control inputs. The $\mathcal{H}_\infty$ controller is given in section IV-C. This controller guarantees that the singular closed loop system is admissible with a given disturbance attenuation level. The design of the $\mathcal{H}_\infty$ controller is based on the solution of linear matrix inequalities (LMI) coupled with an algebraic constraint.

**Notations.** Throughout the paper, $\mathbf{E}$ represents expectation operator with respect to some probability measure $\mathcal{P}$. $L_2\left(\Omega, \mathbb{R}^k\right)$ is the space of square-integrable $\mathbb{R}^k$-valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\Omega$ is the

A. Barbata, M. Zasadzinski and H. Souley Ali are with the Centre de Recherche en Automatique de Nancy (CRAN-UMR 7039), Université de Lorraine, IUT de Longwy, 54400 Cosnes et Romain, France (e-mails: asma.barbata@univ-lorraine.fr,michel.zasadzinski@univ-lorraine.fr,harouna.souley@univ-lorraine.fr). A. Barbata and H. Messaoud are with the Ecole Nationale d'Ingénieurs de Monastir, avenue Ibn El Jazzar, 5019 Monastir, Tunisie (e-mail: Hassani.Messaoud@enim.rnu.tn).

978-3-952-41734-8/©2013 EUCA

sample space, $\mathcal{F}$ is a $\sigma$-algebra of subsets of the sample space called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $(\mathcal{F}_t)_{t\geqslant 0}$ denotes an increasing family of $\sigma$-algebras $(\mathcal{F}_t) \in \mathcal{F}$. We denote by $\widehat{L}_2\left([0,\infty);\mathbb{R}^k\right)$ the space of non-anticipatory square-integrable stochastic process $f(.) = (f(t))_{t\in[0,\infty)}$ in $\mathbb{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying

$$\|f\|^2_{\widehat{L}_2} = \mathbf{E}\left\{\int_0^\infty \|f(t)\|^2 \,\mathrm{d}\, t\right\} < \infty$$

where $\|.\|$ is the well-known Euclidean norm. $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest eigenvalues of a symmetric square matrix, respectively. For a symmetric matrix $A$, $A > 0$ means that the matrix $A$ is positive definite. Symbols $<$, $\leqslant$ and $\geqslant$ for matrices are defined similarly. $\mathrm{Im}(A)$ is the column space of matrix $A$. A matrix $Z = A^\perp$ is an orthogonal complement of a matrix $A$ if $AZ = 0$ and $\mathrm{rank}([\,A^T \;\; Z\,])$ is maximal. In a matrix, the notation "$\star$" is used for the blocks induced by symmetry.

## II. PRELIMINARIES ON STOCHASTIC SINGULAR SYSTEMS

Let us consider the following SDAE

$$E \,\mathrm{d}\, x = f(x) \,\mathrm{d}\, t + g(x) \,\mathrm{d}\, w \tag{1}$$

where $x \in \mathbb{R}^n$ is the semi-state vector and $w \in \mathbb{R}^d$ is a multi-dimensional independent Brownian motion with $f(0) = 0$ and $g(0) = 0$. The matrix $E \in \mathbb{R}^{n\times n}$ is assumed to be singular, i.e. $\mathrm{rank}(E) = \overline{r} < n$.

The function $f(x)$ is Lebesgue integrable and the function $g(x)$ is Lebesgue square-integrable as it is needed for Itô calculus [3], [4].

To guarantee the existence and the uniqueness of the solution $x$ of the SDE (1), the functions $f(x)$ and $g(x)$ satisfy the following relations $\forall\, x \in \mathbb{R}^n$ and $\forall\, \overline{x} \in \mathbb{R}^n$ (see [3], [4], [18])

$$\|f(x)\|^2 + \|g(x)\|^2 \leqslant k_1(1 + \|x\|^2), \tag{2a}$$

$$\max(\|f(x) - f(\overline{x})\|\,, \|g(x) - g(\overline{x})\|) \leqslant k_2 \,\|x - \overline{x}\|\,, \tag{2b}$$

where $k_1$ and $k_2$ are given strictly positive reals.

Using the approach given in [18], the semi-state $x$ can be splitted into a "dynamic" part $x_u = E^+Ex$ and an "algebraic" part $x_v = (I_n - E^+E)x$ where $E^+$ is any reflexive generalized inverse of $E$ satisfying $E = EE^+E$ and $E^+ = E^+EE^+$ [19]. So we have $x = x_u + x_v$, $Ex_u = Ex$ and $Ex_v = 0$. Then, premultiplying the two sides of equation (1) by $\begin{bmatrix} EE^+ \\ I_n - EE^+ \end{bmatrix}$ leads to the following system

$$E \,\mathrm{d}\, x_u = EE^+ f(x_u + x_v) \,\mathrm{d}\, t + EE^+ g(x_u + x_v) \,\mathrm{d}\, w,$$
$$0 = (I_n - EE^+) f(x_u + x_v),$$
$$0 = (I_n - EE^+) g(x_u + x_v)$$

which is equivalent to the SDAE (1). Using the projector property of the reflexive generalized inverse [19] $(E^+Ex_u = \; x_u)$, we obtain the following equivalent form

$$\mathrm{d}\, x_u \;=\; E^+ f(x_u + x_v) \,\mathrm{d}\, t + E^+ g(x_u + x_v) \,\mathrm{d}\, w, \tag{3a}$$

$$0 \;=\; (I_n - EE^+) f(x_u + x_v), \tag{3b}$$

$$0 \;=\; (I_n - EE^+) g(x_u + x_v). \tag{3c}$$

Let $V(x) = x^T E^T P x$ with $E^T P = P^T E \geqslant 0$ be a Lyapunov function. Since $Ex = Ex_u$, we have $V(x) = V(x_u)$. Then applying the Itô formula on equation (3a) gives

$$\mathrm{d}\, V(x_u) = \mathfrak{L}(V(x_u)) \,\mathrm{d}\, t + \mathfrak{B}(V(x_u)) \,\mathrm{d}\, w,$$
$$\mathfrak{L}(V(x_u)) = \frac{\partial V}{\partial x_u}(x_u) E^+ f(x_u + x_v)$$
$$+ \frac{1}{2}\,\mathrm{tr}(g^T(x_u + x_v)(E^+)^T \frac{\partial^2 V}{\partial x_u^2}(x_u) E^+ g(x_u + x_v)),$$
$$\mathfrak{B}(V(x_u)) = \frac{\partial V}{\partial x_u}(x_u) E^+ g(x_u + x_v).$$

Since $x_u^T P^T E = x_u^T E^T P = x^T E^T P = x^T P^T E$, the previous relations are equivalent to

$$\mathrm{d}\, V(x) = \mathfrak{L}(V(x)) \,\mathrm{d}\, t + \mathfrak{B}(V(x)) \,\mathrm{d}\, w, \tag{4}$$

$$\mathfrak{L}(V(x)) = f^T(x) P x + x^T P^T f(x) + \mathrm{tr}(g^T(x)(E^+)^T E^T P E^+ g(x)), \tag{5}$$

$$\mathfrak{B}(V(x)) = 2 x^T P^T g(x). \tag{6}$$

if and only if the conditions (3a), (3b) and (3c) are satisfied.

In order to avoid that the solution of the SDAE (1) is directly affected by the noise $w$, this noise does not appear in the algebraic constraints, i.e. the condition (3c) must hold.

The SDAE (1) is said to be of index 1 if [18]

- the noise does not appear in the algebraic constraints, i.e. the condition (3c) must hold,
- the algebraic constraints are uniquely solvable for the algebraic variables, i.e. the equation (3b) has an unique solution $x_v$ in function of $x_u$. This can be done if and only if $\det\left(E + (I_n - EE^+)\dfrac{\mathrm{d}\, f(x)}{\mathrm{d}\, x}\right) \neq 0$ [18].

The MSES of the SDAE (1) is equivalent to the MSES of the SDE (3a) and is defined as follows [3], [20].

*Definition 1:* The equilibrium of SDE (3a) is said to be MSES if

$$\limsup_{t\to+\infty} \frac{1}{t} \ln(\mathbf{E}(\|x(t, t_0, x_0)\|^2)) < 0. \tag{7}$$

Relation (7) stands that there exist $M > 0$ and $\alpha > 0$ such that

$$\mathbf{E}\big(\|x(t, t_0, x_0)\|^2\big) \leqslant M \,\|x_0\|^2 \, e^{-\alpha(t - t_0)}$$

for all $x_0 \in \mathbb{R}^n$ and $t \geqslant t_0 \geqslant 0$.

The following lemma can be used to study the stability of a SDE [3], [20] for $t_0 = 0$.

*Lemma 1:* ([3]) Assume that there exist a Lyapunov function $V(x_u)$ which is twice continuously differentiable on $x_u$, and $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that

$$c_1 \,\|x_u(t)\|^2 \leqslant V(x_u) \leqslant c_2 \,\|x_u(t)\|^2\,, \tag{8}$$

$$\mathfrak{L}(V(x_u)) \leqslant -c_3 V(x_u) \quad \forall\, x \in \mathbb{R}^n, \tag{9}$$

then the equilibrium point of the SDE (3a) is mean-square exponentially stable, i.e.

$$\mathbf{E}\left\{\|x_u(t)\|^2\right\} \leqslant \frac{c_2}{c_1}\,\|x_{u_0}\|^2 e^{-c_3 t} \quad \forall\, t \geqslant 0, \quad \forall\, x_{u_0} \in \mathbb{R}^n. \tag{10}$$

Now, using the above properties, we can define the admissibility of a SDAE.

*Definition 2:* The SDAE (1) is said to be admissible if

- the SDAE (1) is of index 1,
- the equilibrium point of the SDAE (1) is MSES.

# III. BOUNDED REAL LEMMA FOR STOCHASTIC SINGULAR SYSTEMS

In the sequel of this paper, we consider a singular stochastic linear system given by

$$E\,\mathrm{d}\,x = (A_t x + B_t v)\,\mathrm{d}\,t + \sum_{i=1}^{d}(A_{w_i}x + B_{w_i}v)\,\mathrm{d}\,w_i \quad (11a)$$

$$z = C_z x + D_{zv}v \quad (11b)$$

where $x \in \mathbb{R}^n$ is the semi-state vector, $z \in \mathbb{R}^\ell$ is the controlled output, $v \in \mathbb{R}^q$ is the perturbation vector with bounded energy and $w \in \mathbb{R}^d$ is a multi-dimensional independent Brownian motion where $w_i$ is the $i^{\text{th}}$ component of $w$. The matrix $E \in \mathbb{R}^{n\times n}$ is assumed to be singular, i.e. $\operatorname{rank}(E) = \bar{r} < n$.

The bounded real lemma for the stochastic singular system (11) is given by the following theorem.

*Theorem 1:* The stochastic singular system (11) is admissible and satisfies the following performance inequality

$$\|z\|^2_{\widehat{L}_2} \leqslant \gamma^2 \|v\|^2_{\widehat{L}_2}, \quad \forall\, v \in \widehat{L}_2, v \neq 0, x_0 = 0 \quad (12)$$

where $\gamma > 0$ is the disturbance attenuation factor if there exist a matrix $P$ and two scalars $\mu_1 > 0$ and $\mu_2 > 0$ such that the following conditions

$$\operatorname{Im}\Big(\begin{bmatrix} A_{w_1} & B_{w_1} & \dots & A_{w_d} & B_{w_d}\end{bmatrix}\Big) \subseteq \operatorname{Im}(E) \quad (13)$$

$$E^T P = P^T E \geqslant 0 \quad (14)$$

$$\Psi = \begin{bmatrix} (1,1) & \star & C_z^T \\ (2,1) & (2,2) & D_{zv}^T \\ C_z & D_{zv} & -I_\ell \end{bmatrix} < 0 \quad (15)$$

hold where

$$(1,1) = A_t^T P + P^T A_t + \sum_{i=1}^{d} A_{w_i}^T (E^+)^T E^T P E^+ A_{w_i} + \mu_1 E^T P + \mu_2 I_n,$$

$$(2,1) = B_t^T P + \sum_{i=1}^{d} B_w^T (E^+)^T P^T E E^+ A_{w_i},$$

$$(2,2) = \sum_{i=1}^{d} B_{w_i}^T (E^+)^T E^T P E^+ B_{w_i} - \gamma^2 I_q.$$

The performance index $\gamma$ in (12) corresponds to the $\mathcal{H}_\infty$ criterion for stochastic systems defined in [5].

*Proof:* Part 1 : index 1.

In this part, we consider the SDAE (11a) with $v = 0$. Using the notations introduced in SDAE (1), we have $f(x) = A_t x$ and $g(x) = [\ A_{w_1}x \ \dots\ A_{w_d}x\ ]$. The noise does not appear in the algebraic constraints if condition (3c) holds. This is equivalent to condition $\operatorname{Im}(g(x)) \subseteq \operatorname{Im}(E)$. This latest condition is satisfied $\forall\, x \in \mathbb{R}^n$ if relation (13) is verified. In the case where $v \neq 0$, $g(x)$ is replaced by $g(x, v) = [\ A_{w_1}x + B_{w_1}v \ \dots\ A_{w_d}x + B_{w_d}v\ ]$ and condition (13) guarantees that the algebraic part of the SDAE (11a) is not affected by the noise.

If condition (13) holds, the equation (3b) has an unique solution $x_v$ in function of $x_u$ if and only if $\det\Big(E + (I_n - EE^+)\dfrac{\mathrm{d}\,f(x)}{\mathrm{d}\,x}\Big) \neq 0$ [18]. Since $f(x) = A_t x$, we have

$$E + (I_n - EE^+)\frac{\mathrm{d}\,f(x)}{\mathrm{d}\,x} = E + (I_n - EE^+)A_t$$

and the above condition on the determinant becomes

$$\det(E + (I_n - EE^+)A_t) \neq 0.$$

In [21], it is shown that $\det(E + (I_n - EE^+)A_t) \neq 0$ is equivalent to the fact that the DAE

$$E\dot{x} = A_t x$$

is regular (i.e $\det(sE - A) \not\equiv 0$ [11]) and does not have impulsive modes. Using lemma 2 in [13], this DAE is regular and does not have impulsive modes if there exists a matrix $P$ such that the conditions (13) and

$$A_t^T P + P A_t < 0$$

are satisfied. The above inequality holds if the LMI (15) is satisfied since we have

$$A_t^T P + P A_t \leqslant -\sum_{i=1}^{d} A_{w_i}^T (E^+)^T E^T P E^+ A_{w_i} - \mu_1 E^T P - \mu_2 I_n < 0.$$

Then, the SDAE (11a) is of index 1 if the conditions (13), (14) and (15) are satisfied.

Part 2 : disturbance attenuation and MSES.

Now, we consider that the SDAE (11a) is of index 1 (see the part 1). Then the conditions (3b) and (3c) hold and we can use the relations (4), (5) and (6) to apply the Itô formula on SDAE (11a) with the Lyapunov function $V(x) = x^T E^T P x$ where the matrix $P$ satisfies the relation (14). We obtain

$$\mathrm{d}\,V(x) = \mathfrak{L}(V(x))\,\mathrm{d}\,t + \mathfrak{B}(V(x))\,\mathrm{d}\,w, \quad (16)$$

$$\mathfrak{L}(V(x)) = x^T\Big(A_t^T P + P^T A_t + \sum_{i=1}^{d} A_{w_i}^T (E^+)^T E^T P E^+ A_{w_i}\Big)x + 2x^T P^T B_t v + \sum_{i=1}^{d} v^T B_{w_i}^T (E^+)^T E^T P E^+ B_{w_i} v + 2\sum_{i=1}^{d} x^T A_{w_i}^T (E^+)^T E^T P E^+ B_w v, \quad (17)$$

$$\mathfrak{B}(V(x)) = 2x^T P^T \Big[A_{w_1}x + B_{w_1}v \quad \dots \quad A_{w_d}x + B_{w_d}v\Big]. \quad (18)$$

Since for non-anticipatory stochastic process, the expectation and integral operator can commute [5], the performance index (12) can be rewritten as follows

$$J_{zv} = \int_0^{+\infty} \mathbf{E}\Big(z^T(t)z(t) - \gamma^2 v^T(t)v(t) + \mathrm{d}\,V(x(t))\Big)\,\mathrm{d}\,t - \mathbf{E}(V(x(t)))_{t\to+\infty} + \mathbf{E}(V(x(t)))_{t=0}. \quad (19)$$

Since $x_0 = 0$, $\mathbf{E}(\mathfrak{B}(V(x))) = 0$ and $\mathbf{E}(V(x(t)))_{t\to+\infty} \geqslant 0$, the relation (19) leads to the following inequality

$$J_{zv} \leqslant \int_0^{+\infty} \mathbf{E}\Big(z^T(t)z(t) - \gamma^2 v^T(t)v(t) + \mathfrak{L}(V(x(t)))\Big)\,\mathrm{d}\,t. \quad (20)$$

The inequality (12) holds if $J_{zv} \leqslant 0$, i.e. if

$$z^T z - \gamma^2 v^T v + \mathfrak{L}(V(x)) + \mu_1 V(x) + \mu_2 x^T x = \begin{bmatrix} x^T & v^T \end{bmatrix} \underbrace{\begin{bmatrix} (1,1) & \star \\ (2,1) & (2,2) \end{bmatrix}}_{\Theta} \begin{bmatrix} x \\ v \end{bmatrix} \leqslant 0 \quad (21)$$

where

$$(1,1) = A_t^T P + P^T A_t + \sum_{i=1}^{d} A_{w_i}^T (E^+)^T E^T P E^+ A_{w_i} + C_z^T C_z + \mu_1 E^T P + \mu_2 I_n,$$

$$(2,1) = B_t^T P + \sum_{i=1}^{d} B_w^T (E^+)^T P^T E E^+ A_{w_i} + D_{zv}^T C_z,$$

$$(2,2) = \sum_{i=1}^{d} B_{w_i}^T (E^+)^T E^T P E^+ B_{w_i} + D_{zv}^T D_{zv} - \gamma^2 I_q.$$

The inequality (21) is satisfied if

$$\Theta < 0. \tag{22}$$

Applying the Schur lemma on inequality (22) gives the LMI (15).

To prove the MSES, we consider that $v = 0$ in the SDAE (11a) and $\mathfrak{L}(V(x))$ becomes (see (17) with $v = 0$)

$$\mathfrak{L}(V(x)) = x^T \left( A_t^T P + P^T A_t + \sum_{i=1}^{d} A_{w_i}^T (E^+)^T E^T P E^+ A_{w_i} \right) x. \tag{23}$$

Since $\mathfrak{L}(V(x)) \leqslant -\mu_1 V(x)$ where $\mathfrak{L}(V(x))$ is given in (23) if the LMI (22) is satisfied, then condition (9) in lemma 1 is satisfied with $c_3 = \mu_1$.

Let $\mathcal{S}_u = \{x_u\}$ be the set of all vectors $x_u$ defined in section II. Notice that $A_t^T P + P^T A_t < 0$ since the LMI (22) is satisfied, then using property M6 in [22] (p. 23-24) with $\mu(A) = 0.5\lambda_{\max}(A + A^T)$ [22] (p. 26), we obtain

$$\begin{aligned} -0.5\lambda_{\max}(A_t^T P + P^T A_t) &\leqslant \mathrm{Re}(\lambda_i(P^T A_t)) \\ &\leqslant 0.5\lambda_{\max}(A_t^T P + P^T A_t) < 0 \end{aligned}$$

where $\mathrm{Re}(x)$ is the real part of $x$ and $i = 1, \ldots, n$. It follows that all the eigenvalues of matrix $P$ are non zero and $\det(P) \neq 0$. Then we have $\mathrm{rank}(E^T P) = \mathrm{rank}(E) = \dim(\mathcal{S}_u)$ since $x_u = E^+ E x$ and $\mathrm{rank}(E^+ E) = \mathrm{rank}(E)$ for all reflexive generalized inverses [19]. Then the condition (8) in lemma 1 is satisfied with $c_1 = \lambda_{\min}(E^T P)$ and $c_2 = \lambda_{\max}(E^T P)$, where $\lambda_{\min}(E^T P)$ is the smallest non zero eigenvalue of matrix $E^T P$.

Then the SDAE (11a) is MSES and the proof is ended. ■

In the sequel, we will use the bounded real lemma given in theorem 1 to design a $\mathcal{H}_\infty$ controller for a stochastic singular system such that the closed loop system is admissible and satisfies a $\mathcal{H}_\infty$ specification. Notice that, in our knowledge, the developments presented in the next section constitute a new approach for stochastic singular systems.

## IV. $\mathcal{H}_\infty$ CONTROLLER

### A. Closed loop relations

Let us consider the following stochastic descriptor system

$$\begin{aligned} E \,\mathrm{d}x &= (A_t x + B_t v + G_t u)\,\mathrm{d}t \\ &\quad + \sum_{i=1}^{d} (A_{w_i} x + B_{w_i} v)\,\mathrm{d}w_{x_i} && \text{(24a)} \\ z &= C_z x + D_{zv} v + D_{z_u} u && \text{(24b)} \\ y &= C_y x + D_{yv} v && \text{(24c)} \end{aligned}$$

where $x \in \mathbb{R}^n$ is the semi-state vector, $z \in \mathbb{R}^\ell$ is the controlled output, $y \in \mathbb{R}^p$ is the measurement vector, $u \in \mathbb{R}^m$ is the control input, $v \in \mathbb{R}^q$ is the perturbation vector with bounded energy and $w \in \mathbb{R}^d$ is a multi-dimensional independent Brownian motion where $w_i$ is the $i^{\text{th}}$ component of $w$. The matrix $E \in \mathbb{R}^{n \times n}$ is assumed to be singular, i.e. $\mathrm{rank}(E) = \overline{r} < n$.

A full-order dynamic output feedback controller $K(s)$ is given by

$$\begin{aligned} E_k \,\mathrm{d}x_k &= (A_k x_k + B_k y)\,\mathrm{d}t && \text{(25a)} \\ u &= C_k x_k + D_k y && \text{(25b)} \end{aligned}$$

where $s$ is the Laplace variable and $x_k \in \mathbb{R}^n$ and $E_k \in \mathbb{R}^{n \times n}$ with $\mathrm{rank}\, E_k = k \leqslant n$, i.e. the matrix $E_k$ is singular if $k < n$ and regular if $k = n$.

Inserting the control law given by (25) into the system (24) leads to the following closed loop system

$$E_c \,\mathrm{d}x_c = (A_t^c x_c + B_t^c v)\,\mathrm{d}t + \sum_{i=1}^{d} (A_{w_i}^c x_c + B_{w_i}^c v)\,\mathrm{d}w_i) \tag{26a}$$

$$z = C^c x_c + D^c v \tag{26b}$$

where

$$E_c = \begin{bmatrix} E & 0 \\ 0 & E_k \end{bmatrix}, A_t^c = \begin{bmatrix} A_t + G_t D_k C_y & G_t C_k \\ B_k C_y & A_k \end{bmatrix}, x_c = \begin{bmatrix} x \\ x_k \end{bmatrix},$$

$$B_t^c = \begin{bmatrix} B_t + G_t D_k D_{yv} \\ B_k D_{yv} \end{bmatrix}, A_{w_i}^c = \begin{bmatrix} A_{w_i} & 0 \\ 0 & 0 \end{bmatrix}, B_{w_i}^c = \begin{bmatrix} B_{w_i} \\ 0 \end{bmatrix},$$

$$C^c = \begin{bmatrix} C_z + D_{zu} D_k C_y & D_{zu} C_k \end{bmatrix}, D^c = D_{zv} + D_{zu} D_k D_{yv}.$$

The above relationships can be rewritten as follows

$$\begin{aligned} \begin{bmatrix} A_t^c & B_t^c \\ C^c & D^c \end{bmatrix} &= \left[\begin{array}{cc|c} A_t & 0 & B_t \\ 0 & 0 & 0 \\ \hline C_z & 0 & D_{zv} \end{array}\right] + \left[\begin{array}{cc} 0 & G_t \\ I_n & 0 \\ \hline 0 & D_{zu} \end{array}\right] \\ &\quad \times \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \left[\begin{array}{cc|c} 0 & I_n & 0 \\ C_y & 0 & D_{yv} \end{array}\right] \\ &= \begin{bmatrix} \mathcal{A}_t & \mathcal{B}_t \\ \mathcal{C}_z & D_{zv} \end{bmatrix} + \begin{bmatrix} \mathcal{G}_t \\ \mathcal{D}_{zu} \end{bmatrix} \Phi \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yv} \end{bmatrix}. \end{aligned} \tag{27}$$

### B. Case where $D_{yu} \neq 0$

If the measurement equation (24c) contains a feedthrough term $D_{yu} \neq 0$

$$y = C_y x + D_{yv} v + D_{yu} u \tag{28}$$

the controller $K(s)$ in (25) is replaced by a controller $\overline{K}(s)$ given by

$$\begin{aligned} E_k \,\mathrm{d}x_k &= (\overline{A}_k x_k + \overline{B}_k y)\,\mathrm{d}t && \text{(29a)} \\ u &= \overline{C}_k x_k + \overline{D}_k y && \text{(29b)} \end{aligned}$$

and the closed loop transfer functions obtained either with (24a), (24b), (24c) and (25) or with (24a), (24b), (28) and (29) are identical if and only if $\overline{K}(s) = K(s)(I_p + D_{yu} K(s))^{-1}$ and we obtain

$$\begin{aligned} \overline{A}_k &= A_k - B_k (I_p + D_{yu} D_k)^{-1} D_{yu} C_k, \\ \overline{B}_k &= B_k (I_p + D_{yu} D_k)^{-1}, \\ \overline{C}_k &= C_k - D_k (I_p + D_{yu} D_k)^{-1} D_{yu} D_k, \\ \overline{D}_k &= D_k (I_p + D_{yu} D_k)^{-1} \end{aligned}$$

and $\det(I_p + D_{yu}D_k) \neq 0$ is the well-posedness condition [23].

### C. Synthesis of the $\mathcal{H}_\infty$ controller

*Theorem 2:* Let $E_k \in \mathbb{R}^{n\times n}$ be a matrix with $\operatorname{rank}(E_k) = k \leqslant n$. For the stochastic singular system (24), the system (25) is a $\mathcal{H}_\infty$ controller that achieves the admissibility of the SDAE (26) and the criterion

$$\|z_c\|^2_{\widehat{L}_2} \leqslant \gamma^2\|v\|^2_{\widehat{L}_2}, \quad \forall\, v \in \widehat{L}_2, w \neq 0, x_0 = 0 \tag{30}$$

if the there exist matrices $S$, $N_1$, $N_2$, $\overline{S}$ and $Z$, and two scalars $\mu_1 > 0$ and $\mu_2 > 0$ such that the following conditions

$$\begin{bmatrix} E^TS & E^TN_1 \\ E_k^TN_2 & E_k^T\overline{S} \end{bmatrix} = \begin{bmatrix} S^TE & N_2^TE_k \\ N_1^TE & \overline{S}^TE_k \end{bmatrix} \geqslant 0 \tag{31}$$

$$\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{N}_1^T \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{d} B_{w_i}^T \overline{\overline{E}} B_{w_i} - \gamma^2 I_q & D_{zv}^T \\ D_{zv} & -I_\ell \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & \mathcal{N}_1 \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} \mathcal{N}_2^T & 0 & 0 \\ 0 & \mathcal{N}_3^T & 0 \\ 0 & 0 & I_\ell \end{bmatrix} \times \begin{bmatrix} (1,1)_{(33)} & \star & C_z^T \\ B_t^TS + \sum_{i=1}^{d} B_{w_i}^T\overline{\overline{E}}A_{w_i} & \sum_{i=1}^{d} B_{w_i}^T\overline{\overline{E}}B_{w_i} - \gamma^2 I_q & D_{zv}^T \\ C_z & D_{zv} & -I_\ell \end{bmatrix} \times \begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & \mathcal{N}_3 & 0 \\ 0 & 0 & I_\ell \end{bmatrix} < 0 \tag{33}$$

$$EZ = \begin{bmatrix} A_{w_1} & B_{w_1} & \dots & A_{w_d} & B_{w_d} \end{bmatrix} \tag{34}$$

are satisfied, where

$$(1,1)_{(33)} = A_t^TS + S^TA_t + \sum_{i=1}^{d} A_{w_i}^T \overline{\overline{E}} A_{w_i} + \mu_1E^TS + \mu_2I_n$$

and where $\overline{\overline{E}} = \overline{\overline{E}}^T = (E^+)^TE^TSE^+$, $\mathcal{N}_1 = {D_{zu}^T}^{\perp}$, $\mathcal{N}_2 = C_y^\perp$ and $\mathcal{N}_3 = D_{yv}^\perp$. Once these matrices found, the controller matrices $A_k$, $B_k$, $C_k$ and $D_k$ are obtained by solving the following LMI

$$\Psi_c < 0 \tag{35}$$

where $P_c = \left[\begin{smallmatrix} S & N_1 \\ N_2 & \overline{S} \end{smallmatrix}\right]$, $\Phi = \left[\begin{smallmatrix} A_k & B_k \\ C_k & D_k \end{smallmatrix}\right]$ and $\Psi_c$ is given in (37).

*Proof:* Consider the Lyapunov function $V_c(x_c) = x_c^TE_c^TP_cx_c$ satisfying $E_c^TP_c = P_c^TE_c \geqslant 0$. The matrix $P_c$ can be written as follows

$$P_c = \begin{bmatrix} S & N_1 \\ N_2 & \overline{S} \end{bmatrix} \tag{36}$$

where $S \in \mathbb{R}^{n\times n}$ and, using (27), the LMI (14) is equivalent to LMI (31).

Using (27), the LMI (15) applied to the closed loop system (26) can be rewritten as follows

$$\Psi_c = \underbrace{\begin{bmatrix} I_{2n} & 0 \\ 0 & 0 \\ 0 & \mathcal{D}_{zu} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} P_c^T\mathcal{G}_t\Phi & P_c^T\mathcal{G}_t\Phi \\ \Phi & \Phi \end{bmatrix}}_{\Omega} \underbrace{\begin{bmatrix} \mathcal{C}_y & 0 & 0 \\ 0 & \mathcal{D}_{yv} & 0 \end{bmatrix}}_{\Xi}$$
$$+ \begin{bmatrix} \mathcal{C}_y^T & 0 \\ 0 & \mathcal{D}_{yv}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi^T\mathcal{G}_t^TP_c & \Phi^T \\ \Phi^T\mathcal{G}_t^TP_c & \Phi^T \end{bmatrix} \begin{bmatrix} I_{2n} & 0 & 0 \\ 0 & 0 & \mathcal{D}_{zu}^T \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} (1,1)_c & \star & \mathcal{C}_z^T \\ (2,1)_c & (2,2)_c & D_{zv}^T \\ \mathcal{C}_z & D_{zv} & -I_\ell \end{bmatrix}}_{\Gamma} < 0 \tag{37}$$

where (we have used $E_c^TP_c = P_c^TE_c$)

$$\begin{aligned} (1,1)_c &= \mathcal{A}_t^TP_c + P_c^T\mathcal{A}_t + \sum_{i=1}^{d} {A_{w_i}^c}^T\,\overline{E}A_{w_i}^c + \mu_1E_c^TP_c + \mu_2I_{2n} \\ &= \begin{bmatrix} A_t^TS + S^TA_t & A_t^TN_1 \\ N_1^TA_t & 0 \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} A_{w_i}^T\overline{\overline{E}}A_{w_i} & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \mu_1E^TS + \mu_2I_n & \mu_1E^TN_1 \\ \mu_1E_k^TN_2 & \mu_1E_k^T\overline{S} + \mu_2I_n \end{bmatrix}, \end{aligned} \tag{38}$$

$$\begin{aligned} (2,1)_c &= \mathcal{B}_t^TP_c + \sum_{i=1}^{d} {B_{w_i}^c}^T\,\overline{E}A_{w_i}^c \\ &= \begin{bmatrix} B_t^TS & B_t^TN_1 \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} B_{w_i}^T\overline{\overline{E}}A_{w_i} & 0 \end{bmatrix}, \end{aligned} \tag{39}$$

$$(2,2)_c = \sum_{i=1}^{d} {B_{w_i}^c}^T\,\overline{E}B_{w_i}^c - \gamma^2I_q = \sum_{i=1}^{d} B_{w_i}^T\overline{\overline{E}}B_{w_i} - \gamma^2I_q, \tag{40}$$

$$\begin{aligned} \overline{E} = \overline{E}^T &= (E_c^+)^TE_c^TP_cE_c^+ \\ &= \begin{bmatrix} (E^+)^TE^TSE^+ & (E^+)^TE^TN_1E_k^+ \\ (E_k^+)^TE_k^TN_2E^+ & (E_k^+)^TE_k^T\overline{S}E_k^+ \end{bmatrix}, \end{aligned} \tag{41}$$

since $\left[\begin{smallmatrix} E & 0 \\ 0 & E_k \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} E^+ & 0 \\ 0 & E_k^+ \end{smallmatrix}\right]$, with $\overline{\overline{E}} = \overline{\overline{E}}^T = (E^+)^TE^TSE^+$.

The inequality (37) has a solution given by $P_c$ and $\Phi$ if there exists $P_c$ such that the two following LMI [24], [25]

$$\mathcal{N}_\Delta^T\Gamma\mathcal{N}_\Delta < 0 \tag{42}$$

$$\mathcal{N}_\Xi^T\Gamma\mathcal{N}_\Xi < 0 \tag{43}$$

hold where $\mathcal{N}_\Delta = {\Delta^T}^\perp$ and $\mathcal{N}_\Xi = \Xi^\perp$. Using (27), $\mathcal{N}_\Delta$ and $\mathcal{N}_\Xi$ are given by

$$\mathcal{N}_\Delta = \begin{bmatrix} 0_{2n\times q} & 0 \\ I_q & 0 \\ 0 & \mathcal{N}_1 \end{bmatrix}, \quad \mathcal{N}_\Xi = \begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & 0 & 0_{n\times\ell} \\ 0 & \mathcal{N}_3 & 0 \\ 0 & 0 & I_\ell \end{bmatrix} \tag{44}$$

where

$$\mathcal{N}_1 = {D_{zu}^T}^{\perp}, \qquad \mathcal{N}_2 = C_y^\perp \qquad \text{and} \qquad \mathcal{N}_3 = D_{yv}^\perp.$$

The LMI (42) is equivalent to the LMI (32) and the LMI (43) is equivalent to the LMI (33).

Using (27), the condition (13) applied to the closed loop system (26) is equivalent to the existence of a matrix $Z_c$ such that

$$E_c Z_c = \left[\begin{smallmatrix} E & 0 \\ 0 & E_k \end{smallmatrix}\right] \left[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A_{w_1} & 0 & B_{w_1} & \dots & A_{w_d} & 0 & B_{w_d} \\ 0 & 0 & 0 & 0\dots & 0 & 0 & 0 \end{smallmatrix}\right]. \quad (45)$$

The condition (45) is equivalent to the condition (34). ■

*Remark 1:* In theorem 2, we can choose any matrix $E_k$ such that the LMI (31) is satisfied. If the matrix $E_k = I_n$ verifies the condition (31), we have $k = n$ and (25) corresponds to a full order nonsingular controller. ■

*Remark 2:* In the case of deterministic singular systems, the $\mathcal{H}_\infty$ controller requires to solve an LMI in $S$ and an LMI in $R$ where $R \in \mathbb{R}^{n\times n}$ is a sub-matrix of $P_c^{-1}$ defined as follows (see [13], [14], [16], [26])

$$P_c^{-1} = \begin{bmatrix} R & M_1 \\ M_1^T & \overline{R} \end{bmatrix}.$$

To obtain an LMI in $R$, these authors pre-multiply $\Gamma$ by the matrix $\left[\begin{smallmatrix} P_c^{-T} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_\ell \end{smallmatrix}\right]$ and post-multiply $\Gamma$ by the matrix $\left[\begin{smallmatrix} P_c^{-1} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_\ell \end{smallmatrix}\right]$ (see the relation at the end of the proof of theorem 4.2 in [24] and relation (37) in this paper). Since, in the deterministic case, there is no term of type $\sum_{i=1}^{d} A_{w_i}^{c\,T} \overline{E} A_{w_i}^c$ where $\overline{E} = (E_c^+)^T E_c^T P_c E_c^+$, there is not a nonlinearity of the type $R^T S R$ in the obtained inequality.

We can not use this approach in the stochastic case due to the term $\sum_{i=1}^{d} A_{w_i}^{c\,T} \overline{E} A_{w_i}^c$ in matrix $\Gamma$ introduced by the Itô calculus (see the term $\mathrm{tr}(g^T(x)(E^+)^T E^T P E^+ g(x))$ in (5) where $P$ is replaced by $P_c$). This is an important specificity of the stochastic case compared to the deterministic one. ■

## V. CONCLUSION

In this paper, we have proposed an $\mathcal{H}_\infty$ controller for singular stochastic systems with multiplicative noises. We have first derive a stochastic version of the bounded real lemma for this class of systems. This bounded real lemma was based on the derivation of the Itô formula for singular stochastic systems. This has been done by splitting the state space into two subspaces : the first for the dynamics variables $x_u$ and the second for the algebraic variables $x_v$. This bounded real lemma is based on the solution of two LMI coupled with an equality constraint in order to check if the considered stochastic singular system is admissible (i.e. is of index one (regularity, absence of impulsive behavior and algebraic constraints without noises) and mean square exponentially stable. In a second time, we have applied the proposed bounded real lemma to derive an output feedback $\mathcal{H}_\infty$ controller which achieves both the admissibility of the closed loop system and a given level of disturbance attenuation. We have show that, to design this $\mathcal{H}_\infty$ controller, the existence conditions are formulated by means of three LMI coupled with an algebraic constraint. If these existence conditions are satisfied, the controller matrices can be obtained by solving a LMI.

## REFERENCES

[1] R. Has'minskii, *Stochastic Stability of Differential Equations*. Aplhen aan den Rijn, The Netherlands : Sijthoff and Noordhoff, 1980.

[2] X. Mao, "Stochastic stabilization and destabilization," *Syst. & Contr. Letters*, vol. 23, pp. 279–290, 1994.

[3] ——, *Stochastic Differential Equations & Applications*. London : Horwood, 1997.

[4] B. Øksendal, *Stochastic Differential Equations : an Introduction with Applications*, 6th ed. New York : Springer-Verlag, 2003.

[5] D. Hinrichsen and A. Pritchard, "Stochastic $\mathcal{H}_\infty$," *SIAM J. Contr. Opt.*, vol. 36, pp. 1504–1538, 1998.

[6] S. Xu and T. Chen, "Reduced-order $\mathcal{H}_\infty$ filtering for stochastic systems," *IEEE Trans. Sign. Proc.*, vol. 50, pp. 2998–3007, 2002.

[7] V. Dragan, A. Halanay, and A. Stoica, "A small gain theorem for linear stochastic systems," *Syst. & Contr. Letters*, vol. 30, pp. 243–251, 1997.

[8] S. Halabi, H. Souley Ali, H. Rafaralahy, and M. Zasadzinski, "$\mathcal{H}_\infty$ functional filtering for stochastic bilinear systems with multiplicative noises," *Automatica*, vol. 45, pp. 1038–1045, 2009.

[9] S. Xu and T. Chen, "Robust $\mathcal{H}_\infty$ filtering for uncertain impulsive stochastic systems under sampled measurements," *Automatica*, vol. 39, pp. 509–516, 2003.

[10] E. Gershon, D. Limebeer, U. Shaked, and I. Yaesh, "Robust $\mathcal{H}_\infty$ filtering of stationary continuous-time linear systems with stochastic uncertainties," *IEEE Trans. Aut. Control*, vol. 46, pp. 1788–1793, 2001.

[11] L. Dai, *Singular Control Systems*, ser. Lecture Notes in Control and Information Sciences. New York : Springer-Verlag, 1989, vol. 118.

[12] K. Takaba, N. Morihira, and T. Katayama, "$\mathcal{H}_\infty$ control for descriptor systems – A $J$-spectral factorization approach," in *Proc. IEEE Conf. Decision & Control*, Lake Buena Vista, USA, 1994.

[13] I. Masubuchi, Y. Kamitane, A. Ohara, and N. Suda, "$\mathcal{H}_\infty$ control for descriptor systems : a matrix inequality approach," *Automatica*, vol. 33, pp. 669–672, 1997.

[14] E. Uezato and M. Ikeda, "Strict LMI conditions for stability, robust stabilization, and $\mathcal{H}_\infty$ control of descriptor systems," in *Proc. IEEE Conf. Decision & Control*, Phoenix, USA, 1999.

[15] I. Masubuchi, "Dissipativity inequalities for continuous-time descriptor systems with applications to synthesis of control gains," *Syst. & Contr. Letters*, vol. 55, pp. 158–164, 2006.

[16] ——, "Output feedback controller synthesis for descriptor systems satisfying closed-loop dissipativity," *Automatica*, vol. 43, pp. 339–345, 2007.

[17] R. Winkler, "Stochastic DAEs in transient noise simulation," in *Proc. Scientific Computing in Electrical Engineering*, Eindhoven, The Netherlands, 2002.

[18] ——, "Stochastic differential algebraic equations of index 1 and applications in circuit simulation," *J. Computational & Applied Mathematics*, vol. 157, pp. 477–505, 2003.

[19] C. Rao and S. Mitra, *Generalized Inverse of Matrices and its Applications*. New York : Wiley, 1971.

[20] L. Hu and X. Mao, "Almost sure exponential stabilisation of stochastic systems by state-feedback control," *Automatica*, vol. 44, pp. 465–471, 2008.

[21] R. März and R. Winkler, "Analysis und Numerik für Algebro-Differentialgleichungen," Humboldt-Universität zu Berlin, Berlin, Germany, Tech. Rep., 2001.

[22] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Englewood Cliffs, New Jersey : Prentice Hall, 1993.

[23] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, New Jersey : Prentice Hall, 1996.

[24] P. Gahinet and P. Apkarian, "A Linear Matrix Inequality approach to $\mathcal{H}_\infty$ control," *Int. J. Robust & Nonlinear Contr.*, vol. 4, pp. 421–448, 1994.

[25] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia : SIAM, 1994.

[26] I. Masubuchi and A. Suzuki, "Gain-scheduled controller synthesis based on new LMIs for dissipativity of descriptor LPV systems," in *Proc. Triennal IFAC World Congress*, Seoul, Korea, 2008.