2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Optimal Wells Scheduling of a Petroleum Reservoir

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Abstract— **This paper discusses an optimal control approach to the life cycle optimization of water-flooding problem, which can be viewed as a scheduling problem, by calculating the optimal switching times given the optimal wells settings to be bang-bang controls. The optimal switching times are obtained from the adjoint method combined with the halving interval method. A numerical example using realistic reservoir settings is included to illustrate the application of the proposed method.**

I. INTRODUCTION

Subsurface petroleum production consists of two main systems; reservoirs and wells. The reservoirs contain a large amount of hydrocarbon deposits, located hundreds or thousands of meters beneath the ground, trapped by overlaying rock formations with zero permeability. In order to produce the hydrocarbon, particularly oil, a number of production wells are drilled through the reservoir formations. Traditionally, the difference between the reservoir pressure and the surface pressure is the only driving force to lift the oil from the reservoir toward the surface facilities. However, this method is most effective to produce oil near the wells. As a result a large amount of oil far from the production wells may still remain in the reservoir. Further, as time goes by the pressure will decline and the wells might be shut down. To sweep the remaining oil, injection wells are drilled and fluids (water, gas, polymer, or surfactant) are injected. When water is used the process is called water-flooding. Once the water is injected, the water front will move according to the permeability and porosity distribution of the fields (**Fig. 1**).

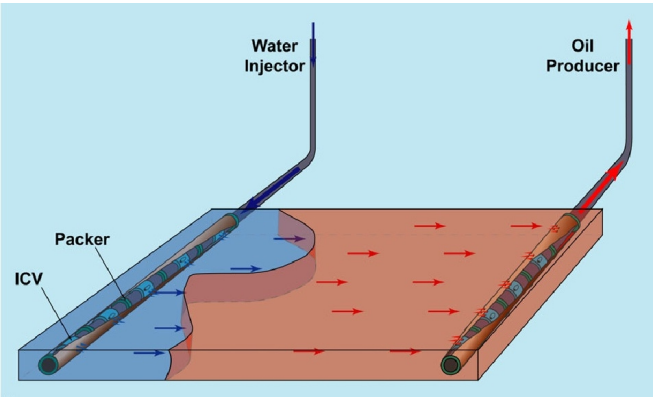

Fig. 1. Illustration of water-flooding process in a thin reservoir [13].

This paper was supported by the Center for Integrated Operations In The Petroleum Industry, Norwegian University of Science and Technology (NTNU).

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To track the water movement, a state of art reservoir simulator based on mass conservation and Darcy's law is usually employed where the inputs may be collected from geological data, e. g., seismic, core analysis, and well testing ([1], [2], [3]).

A. Production Settings

The normal way to operate the production wells are to produce at the maximum rates in the beginning of the production. Later, wells will be completely closed if the water production is too high since larger water production is costly. The presence of high permeability channels between injection and production wells will cause early water breakthrough. Thus, water production will be dominant in the production wells and the wells may be closed early. On the other hand, low permeability regions will not be swept even though they contain large amounts of oil. Hence, large amounts of recoverable oil will remains in the reservoir.

The production optimization problem may be viewed as a scheduling problem. Intuitively, to sweep efficiently, the production wells in a high permeability region should be closed early on while the production wells in a low permeability region should be opened initially. It should be mentioned that the general switching time problem using optimal control theory has received much attention, see e.g., [4], [5], [6], [7], [8].

B. Optimal Control Theory

A standard continuous time optimal control problem can be stated as follows

$$\min_{\mathbf{u}} J(\mathbf{u}) = \psi(\mathbf{x}(T)) + \int_0^T \mathscr{L}(\mathbf{x}(t),\mathbf{u}(t))\,\mathrm{d}t \quad (1)$$

$$\text{subject to } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \quad (2)$$

$$\mathbf{x}(0) = \bar{\mathbf{x}}_0 \quad (3)$$

$$\mathbf{h}(\mathbf{x}(t),\mathbf{u}(t)) \leq 0 \quad (4)$$

$$\mathbf{u}(t) \in \mathscr{U},\ \forall t \in [0,T] \quad (5)$$

$$\mathscr{U} = \{\mathbf{q} \in \mathbb{R}^m : \mathbf{u}_{\min} \leq \mathbf{q} \leq \mathbf{u}_{\max}\} \quad (6)$$

Here, we want to minimize (or maximize) the objective functional J subject to a dynamical system (2)-(3), a path constraint (4), and control constraints (5)-(6) in a fixed time interval $[0,T]$. There are three families of approaches to solve the above problem: (i) Hamilton-Jacobi-Belmann equation (dynamics programming), (ii) indirect method (Pontryagin maximum principle), and (iii) direct method.

In this paper, we proposed an indirect method to calculate the optimal switching time. The idea is to calculate the gradient of the objective functional J with respect to the

control variables $\mathbf{u}$. The objective functional may be the net present value (NPV) or recovery factor, while the control variables are the switching times. Employing calculus of variation, the indirect method requires two simulations; a forward simulation to calculate the reservoir state and a backward simulation to calculate the Lagrange multiplier by solving a costate equation ([9], [10], [11]). In some literatures, the costate equation and the Lagrange multiplier are also referred to the adjoint equation and the adjoint variable, respectively. Hence the indirect method is sometime called the adjoint method.

For simplicity, we regard a problem without inequality constraints. Define the so-called Hamiltonian as

$$H(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)) = \mathscr{L}(\mathbf{x}(t),\mathbf{u}(t)) + \boldsymbol{\lambda}^{\intercal}(t)\left(\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))\right) \quad (7)$$

where $\boldsymbol{\lambda}$ denote the adjoint variable and $\intercal$ is the transpose. The adjoint equation can be written as follows

$$\dot{\boldsymbol{\lambda}}^{\intercal}(t) = -\frac{\partial H(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t))}{\partial \mathbf{x}} \quad (8)$$

$$\boldsymbol{\lambda}^{\intercal}(T) = \frac{\partial \psi(\mathbf{x}(T))}{\partial \mathbf{x}} \quad (9)$$

If we know the state $\mathbf{x}$ and the adjoint variable $\boldsymbol{\lambda}$ at a point on the optimal trajectory, than we would obtain the optimal control inputs from the Pontryagin Maximum Principle: $\mathbf{u}_{\text{opt}}(t) = \arg\min_{\mathbf{u}} H(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t))$. To calculate the optimal switching times, first the dynamical system is simulated on the horizon $[0,T]$ to obtain the state vector $\mathbf{x}$. Afterward, the adjoint equation (8)-(9) is solved backward to obtain the adjoint variable $\boldsymbol{\lambda}$. The state and the multiplier are used to calculate the gradient. The gradient can be combined with a line search algorithm to find the optimal switching times.

An interesting solution to the optimal control of water-flooding problem is the existence of bang-bang type control [12]. This observation has been investigated also by [13]. According to [14], if the objective functional is linear in the control and the only constraints are upper and lower bounds on the control, due to the particular structure, the problem will sometimes have bang-bang optimal solutions. The bang-bang control has a major practical advantage since it can be implemented with simple on-off valves.

C. Contributions of This Paper

A novel optimal control problem is formulated in order to maximize NPV. The embedded gradient calculation for the bang-bang control comes with a proof. An example of a realistic reservoir setting is presented to show the potential of the proposed method.

II. RESERVOIR MODEL

Ideally, to model fluid flow in reservoirs, each component should be considered individually. However, this is a non-trivial task since the number of components can be quite high which leads to complex fluid flow models. As a consequence the computation time may be extensive. To overcome this issue, a simpler model with only considers oil, water, and gas, which is commonly used in the petroleum industry, is presented. Such a model is known as the black-oil model.

A. Black-Oil Model

A standard oil-water system in black oil formulation may be written as follow [1]

$$\frac{\partial}{\partial t}\left(\phi\rho_o[1-s]\right) = \nabla\cdot\left(k\frac{k_{ro}}{\mu_o}\rho_o\nabla p\right) + q_o \quad (10)$$

$$\frac{\partial}{\partial t}\left(\phi\rho_w s\right) = \nabla\cdot\left(k\frac{k_{rw}}{\mu_w}\rho_w\nabla p\right) + q_w \quad (11)$$

where ϕ denotes the reservoir porosity, ρ is the fluid density, μ is the fluid viscosity, k is the reservoir permeability, and k_r is the relative permeability. Here, the unknown variables to be determined are oil pressure p and water saturation s. The subscripts o and w will refer to oil and water, respectively. The divergence operator is denoted by ∇ while the source or the sink terms, i.e., the well flow rates, are denoted by q. The fluid compressibility are defined as

$$c_i(p) = \frac{1}{\rho_i(p)}\frac{\mathrm{d}p_i(p)}{\mathrm{d}p}, \quad i\in\{o,w\} \quad (12)$$

Since petroleum reservoirs are generally heterogeneous with respect to permeability and porosity, solving (10)-(11) analytically is nearly impossible. The common procedure is to discretize the equations spatially into a finite number of grid blocks assuming homogeneous geological properties in each blocks.

Let us assume the reservoir is modeled by $N\times 1\times 1$ grid blocks (1D) of fixed volume and there are only two wells (an injector in grid block 1 and a producer in grid block N). Define the state vector of pressure and saturation in each grid block as $\mathbf{x} = [\mathbf{p}^{\intercal}\ \mathbf{s}^{\intercal}]^{\intercal}\in\mathbb{R}^{2N}$ where $\mathbf{p} = [p^1\cdots p^N]^{\intercal}\in\mathbb{R}^N$ and $\mathbf{s} = [s^1\cdots s^N]^{\intercal}\in\mathbb{R}^N$. After discretization of (10)-(11) using finite difference, the state space model can be written as [15]

$$\mathbf{E}(\mathbf{x}(t))\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}(\mathbf{x}(t))\mathbf{x}(t) + \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{u}(t) \quad (13)$$

$$\mathbf{x}(0) = \bar{\mathbf{x}}_0 \quad (14)$$

where

$$\mathbf{E}(\mathbf{x}(t)) = \left[\begin{array}{c|c} \mathscr{D}(\phi^j(1-s^j)\rho_o c_o(p^j)) & \mathscr{D}(-\phi^j\rho_o) \\ \hline \mathscr{D}(\phi^j s^j\rho_w c_w(p^j)) & \mathscr{D}(\phi^j\rho_w)\end{array}\right] \quad (15)$$

with $\mathscr{D}$ denotes diagonal entry and j denotes the grid block number. The input matrix

$$\tilde{\mathbf{B}}(\mathbf{x}(t)) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{\rho_o(p^N)}{v}\left(1-f_w(s^N)\right) \\ \hline \frac{\rho_w(p^1)}{v} & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{\rho_w(p^N)}{v}f_w(s^N)\end{array}\right] \quad (16)$$

while the state matrix

$$\tilde{\mathbf{A}}(\mathbf{x}(t)) = \left[\begin{array}{c|c} \mathbf{A}_o(\mathbf{x}(t)) & 0 \\ \hline \mathbf{A}_w(\mathbf{x}(t)) & 0 \end{array}\right] \quad (17)$$

where

$$\mathbf{A}_i = \begin{bmatrix} -a_i^{1,2} & a_i^{1,2} & & \mathbf{0} & \mathbf{0} \\ a_i^{2,1} & -(a_i^{2,1}+a_i^{2,3}) & a_i^{2,3} & & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \\ \mathbf{0} & \mathbf{0} & & a_i^{N-1,N} & -a_i^{N-1,N} \end{bmatrix} \quad (18)$$

and

$$a_i^{j,j\pm1} = \frac{\rho_i(p^j)}{\mu_i}\frac{k^{j,j\pm1}}{\Delta x^2}k_{ri}^{j,j\pm1}(p^j,p^{j\pm1},s^j,s^{j\pm1}) \quad (19)$$

$$k^{j,j\pm1} = \frac{2}{\frac{1}{k^j}+\frac{1}{k^{j\pm1}}} \quad (20)$$

$$k_{ri}^{j,j\pm1} = \begin{cases} k_{ri}(s^j), & \text{if } p^j \geq p^{j\pm1} \\ k_{ri}(s^{j\pm1}), & \text{if } p^j < p^{j\pm1} \end{cases}, \quad i \in \{o,w\} \quad (21)$$

Remark that if the reservoir is compressible, i.e., c_w and c_o are not zero, matrix $\mathbf{E}$ is invertible. Let N_w denote the total number of wells. The matrices $\mathbf{E} \in \mathbb{R}^{2N\times 2N}$, $\tilde{\mathbf{A}} \in \mathbb{R}^{2N\times 2N}$, and $\tilde{\mathbf{B}} \in \mathbb{R}^{2N\times N_w}$ are sparse matrices. $\mathbf{E}$ and $\tilde{\mathbf{A}}$ store information regarding rock and fluid properties while $\tilde{\mathbf{B}}$ store information regarding fractional flow of the wells. The control vector consists of the well models and can be written as $\mathbf{u}(t) = [q^1(t)\ q^N(t)]^\intercal$. By defining $\mathbf{A}(\mathbf{x}(t)) = \mathbf{E}^{-1}(\mathbf{x}(t))\tilde{\mathbf{A}}(\mathbf{x}(t))$ and $\mathbf{B}(\mathbf{x}(t)) = \mathbf{E}^{-1}(\mathbf{x}(t))\tilde{\mathbf{B}}(\mathbf{x}(t))$ the system may be written in a standard form of state space equation as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) \quad (22)$$

$$\mathbf{x}(0) = \bar{\mathbf{x}}_0 \quad (23)$$

From (2) and (22), $\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t)$.

B. Well Model

The wells are modeled using the standard Peaceman well model as follows

$$q^j(t) = \alpha^j(t)w^j(s^j)(p_R - p_{bh}^j(t)), \quad j \in \{\mathscr{N}_{inj}, \mathscr{N}_{prod}\} \quad (24)$$

where α^j denote the valves opening, p_R is the average reservoir pressure, and w^j is the productivity index.

If the valve setting $\alpha^j(t)$ is considered as the control variable, then the term $q^j(t)$ in $\mathbf{u}$ is replaced by $\alpha^j(t)$ and the corresponding entries in $\tilde{\mathbf{B}}$ are modified to include the term $w^j(s^j)(p_R^j - p_{bh}^j(t))$. On the other hand, if p_{bh}^j is considered as the control variable, then the term $q^j(t)$ in $\mathbf{u}$ is replaced by p_{bh}^j and the corresponding entries in $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{A}}$ are modified to include the term $w^j(s^j)$.

III. GRADIENT FORMULA

In this section, the gradient of the switching times control is calculated using the steps presented in [16]. Suppose the valves settings are considered as the control variables. We start with parameterizations of the control input $\mathbf{u}$ over the time horizon $[0,T]$.

The j^{th} component α^j for $j = 1,\cdots,N_w$ of the input control vector $\mathbf{u} \in \mathbb{R}^{N_w}$ is considered as a piecewise constant functions over the interval $[0,T]$ with switches at $t_1^j,\cdots,t_{M_j}^j$. Let $t_0^j = 0$, $t_{M_{j+1}}^j = T$, and the switching times t_k^j, $k = 1,\cdots,M_j$ may be distinguished such that $0 = t_0^j < t_1^j < \cdots < t_{M_j}^j < t_{M_{j+1}}^j = T$. Mathematically, $\alpha^j(t)$ may be written in the following form

$$\alpha^j(t) = \sum_{k=0}^{M_j} \gamma_k^j \chi_{[t_k^j, t_{k+1}^j)}(t) \quad (25)$$

where the basis of the piecewise constant function is

$$\chi_I(t) = \begin{cases} 1, & t \in I \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

and the multiplier coefficient γ_k^j describing bang-bang control is given by

$$\gamma_k^j = \begin{cases} 1, & k \geq 0 \text{ is an even number} \\ 0, & k \geq 0 \text{ is an odd number} \end{cases} \quad (27)$$

Let's define vectors of the switching times as $\mathbf{v}^j = [t_1^j,\cdots,t_{M_j}^j]^\intercal$, the augmented vector of the switching times as $\mathbf{v} = [(\mathbf{v}^1)^\intercal,\cdots,(\mathbf{v}^{N_w})^\intercal]^\intercal$ and $\mathbf{u}(\cdot|\mathbf{v})$ will be use to described a control input determined by the switching vector $\mathbf{v}$.

Theorem 1: For each $j = 1,\cdots,N_w$, the gradient of the functional J with respect to $\mathbf{v}^j$ is given by

$$\frac{\partial J(\mathbf{v})}{\partial \mathbf{v}^j} = \begin{bmatrix} H(\mathbf{x}(t_1^-|\mathbf{v}),\mathbf{u}(t_1^-|\mathbf{v}),\boldsymbol{\lambda}(t_1^-|\mathbf{v})) \\ -H(\mathbf{x}(t_1^+|\mathbf{v}),\mathbf{u}(t_1^+|\mathbf{v}),\boldsymbol{\lambda}(t_1^+|\mathbf{v})) \\ \vdots \\ H(\mathbf{x}(t_{M_j}^-|\mathbf{v}),\mathbf{u}(t_{M_j}^-|\mathbf{v}),\boldsymbol{\lambda}(t_{M_j}^-|\mathbf{v})) \\ -H(\mathbf{x}(t_{M_j}^+|\mathbf{v}),\mathbf{u}(t_{M_j}^+|\mathbf{v}),\boldsymbol{\lambda}(t_{M_j}^+|\mathbf{v})) \end{bmatrix} \quad (28)$$

where the right and left limit are defined as $H(t^+|\mathbf{v}) = \lim_{\tau\to t^+} H(\tau|\mathbf{v})$ and $H(t^-|\mathbf{v}) = \lim_{\tau\to t^-} H(\tau|\mathbf{v})$.

Proof: Let $s_0^j = s_{M_{j+1}}^j = 0$ and $\mathbf{s}^j = [s_1^j,\cdots,s_{M_j}^j]^\intercal$ be a arbitrary but fixed dummy vector of time. Furthermore, let ε be a sufficiently small positive constant such that $0 \leq t_1^j + \varepsilon s_1^j \leq \cdots \leq t_{M_j}^j + \varepsilon s_{M_j}^j \leq T$. Define a perturbation vector of well j as $\boldsymbol{\vartheta}^j(\varepsilon) = [\cdots,(\mathbf{v}^{j-1})^\intercal,(\mathbf{v}^j + \varepsilon s^j)^\intercal,(\mathbf{v}^{j+1})^\intercal,\cdots]$. Let $t \in [t_l^j + \varepsilon s_l^j,\ t_{l+1}^j + \varepsilon s_{l+1}^j]$ for some $l \in \{0,\cdots,M_j\}$, then the explicit solution for (22) may be written as follows

$$\begin{aligned} &\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon)) \quad (29)\\ =\ &\mathbf{x}(0) + \sum_{k=0}^{l-1}\int_{t_k^j+\varepsilon s_k^j}^{t_{k+1}^j+\varepsilon s_{k+1}^j} \\ &\left(\mathbf{A}(\mathbf{x}(\tau|\boldsymbol{\vartheta}^j(\varepsilon)))\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon)) + \mathbf{B}(\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon)))\gamma_k^j\right)\mathrm{d}\tau \\ &+ \int_{t_l^j+\varepsilon s_l^j}^{t} \\ &\left(\mathbf{A}(\mathbf{x}(\tau|\boldsymbol{\vartheta}^j(\varepsilon)))\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon)) + \mathbf{B}(\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon)))\gamma_k^j\right)\mathrm{d}\tau \end{aligned}$$

The first variation of the state vector $\mathbf{x}$ may be computed as

follows

$$\Delta\mathbf{x}(t) \triangleq \frac{\mathrm{d}\mathbf{x}(t|\boldsymbol{\vartheta}^j(\varepsilon))}{\mathrm{d}\varepsilon}|_{\varepsilon=0} = \tag{30}$$
$$\int_0^t \frac{\partial\left(\mathbf{A}(\mathbf{x}(\tau|\mathbf{v}))\mathbf{x}(\tau|\mathbf{v})+\mathbf{B}(\mathbf{x}(\tau|\mathbf{v}))\mathbf{u}(\tau|\mathbf{v})\right)}{\partial\mathbf{x}}\Delta\mathbf{x}(\tau)\,\mathrm{d}\tau$$
$$+\int_0^t \sum_{k=0}^{M_j}\left(\mathbf{A}(\mathbf{x}(\tau^-|\mathbf{v}))\mathbf{x}(\tau^-|\mathbf{v})+\mathbf{B}(\mathbf{x}(\tau^-|\mathbf{v}))\mathbf{u}\right)s_k^j\delta\,\mathrm{d}\tau$$
$$-\int_0^t \sum_{k=0}^{M_j}\left(\mathbf{A}(\mathbf{x}(\tau^+|\mathbf{v}))\mathbf{x}(\tau^+|\mathbf{v})+\mathbf{B}(\mathbf{x}(\tau^+|\mathbf{v}))\mathbf{u}\right)s_k^j\delta\,\mathrm{d}\tau$$

where $\delta=\delta(\tau-t_k^j)$ is the Dirac delta function while the right and left limit are defined as $\mathbf{x}(t^+|\mathbf{v})=\lim_{\tau\to t^+}\mathbf{x}(\tau|\mathbf{v})$ and $\mathbf{x}(t^-|\mathbf{v})=\lim_{\tau\to t^-}\mathbf{x}(\tau|\mathbf{v})$. The derivative of the first variation of the state vector $\mathbf{x}$ may be computed as follows

$$\Delta\dot{\mathbf{x}}(t) = \frac{\partial\left(\mathbf{A}(\mathbf{x}(t|\mathbf{v}))\mathbf{x}(t|\mathbf{v})+\mathbf{B}(\mathbf{x}(t|\mathbf{v}))\mathbf{u}(t|\mathbf{v})\right)}{\partial\mathbf{x}}\Delta\mathbf{x}(t)$$
$$+\sum_{k=0}^{M_j}\left(\mathbf{A}(\mathbf{x}(t^-|\mathbf{v}))\mathbf{x}(t^-|\mathbf{v})+\mathbf{B}(\mathbf{x}(t^-|\mathbf{v}))\mathbf{u}(t^-|\mathbf{v})\right)s_k^j\delta$$
$$-\sum_{k=0}^{M_j}\left(\mathbf{A}(\mathbf{x}(t^+|\mathbf{v}))\mathbf{x}(t^+|\mathbf{v})+\mathbf{B}(\mathbf{x}(t^+|\mathbf{v}))\mathbf{u}(t^+|\mathbf{v})\right)s_k^j\delta \tag{31}$$

The first variation of the objective functional J may be computed as follow

$$\frac{\mathrm{d}J(\boldsymbol{\vartheta}^j(\varepsilon))}{\mathrm{d}\varepsilon}|_{\varepsilon=0} =$$
$$\frac{\partial\psi(\mathbf{x}(t|\mathbf{v}))}{\partial\mathbf{x}}\Delta\mathbf{x}(T)+\int_0^T\frac{\partial\mathscr{L}(\mathbf{x}(t|\mathbf{v}),\mathbf{u}(t|\mathbf{v}))}{\partial\mathbf{x}}\Delta\mathbf{x}(t)\,\mathrm{d}t$$
$$+\int_0^T\sum_{k=0}^{M_j}\left(\mathscr{L}(\mathbf{x}(t^-|\mathbf{v}),\mathbf{u}(t^-|\mathbf{v}))\right)s_k^j\delta\,\mathrm{d}t$$
$$-\int_0^T\sum_{k=0}^{M_j}\left(\mathscr{L}(\mathbf{x}(t^+|\mathbf{v}),\mathbf{u}(t^+|\mathbf{v}))\right)s_k^j\delta\,\mathrm{d}t \tag{32}$$

Using (31) and the following integration by parts

$$\int_0^T\boldsymbol{\lambda}^T\Delta\dot{\mathbf{x}}(t)\,\mathrm{d}t = \frac{\partial\psi(\mathbf{x}(T|\mathbf{v}))}{\partial\mathbf{x}}\Delta\mathbf{x}(T) \tag{33}$$
$$+\int_0^T\frac{\partial H(\mathbf{x}(t|\mathbf{v}),\mathbf{u}(t|\mathbf{v}),\boldsymbol{\lambda})}{\partial\mathbf{x}}\Delta\mathbf{x}(t)\,\mathrm{d}t$$

the second term of the right hand side of (32) may be computed as follows

$$\int_0^T\frac{\partial\mathscr{L}(\mathbf{x}(t|\mathbf{v}),\mathbf{u}(t|\mathbf{v}))}{\partial\mathbf{x}}\Delta\mathbf{x}(t)\mathrm{d}t \tag{34}$$
$$= -\frac{\partial\psi(\mathbf{x}(T|\mathbf{v}))}{\partial\mathbf{x}}\Delta\mathbf{x}(T)$$
$$+ \int_0^T\boldsymbol{\lambda}^\intercal\sum_{k=1}^{M_j}\left(\mathbf{A}(\mathbf{x}(t^-|\mathbf{v}))\mathbf{x}(t^-|\mathbf{v})+\mathbf{B}(\mathbf{x}(t^-|\mathbf{v}))\mathbf{u}\right)s_k^j\delta\,\mathrm{d}t$$
$$- \int_0^T\boldsymbol{\lambda}^\intercal\sum_{k=1}^{M_j}\left(\mathbf{A}(\mathbf{x}(t^+|\mathbf{v}))\mathbf{x}(t^+|\mathbf{v})+\mathbf{B}(\mathbf{x}(t^+|\mathbf{v}))\mathbf{u}\right)s_k^j\delta\,\mathrm{d}t$$

Since $\frac{\mathrm{d}J(\boldsymbol{\vartheta}^j(\varepsilon))}{\mathrm{d}\varepsilon}|_{\varepsilon=0}=\frac{\mathrm{d}J(\mathbf{v})}{\mathrm{d}\mathbf{v}^j}\mathbf{s}^j$, substituting (34) into (32), yields

$$\frac{\partial J(\mathbf{v})}{\partial\mathbf{v}^j}\mathbf{s}^j = \begin{bmatrix} H(\mathbf{x}(t_1^-|\mathbf{v}),\mathbf{u}(t_1^-|\mathbf{v}),\boldsymbol{\lambda}(t_1^-|\mathbf{v})) \\ -H(\mathbf{x}(t_1^+|\mathbf{v}),\mathbf{u}(t_1^+|\mathbf{v}),\boldsymbol{\lambda}(t_1^+|\mathbf{v})) \\ \vdots \\ H(\mathbf{x}(t_{M_j}^-|\mathbf{v}),\mathbf{u}(t_{M_j}^-|\mathbf{v}),\boldsymbol{\lambda}(t_{M_j}^-|\mathbf{v})) \\ -H(\mathbf{x}(t_{M_j}^+|\mathbf{v}),\mathbf{u}(t_{M_j}^+|\mathbf{v}),\boldsymbol{\lambda}(t_{M_j}^+|\mathbf{v})) \end{bmatrix}^\intercal \mathbf{s}^j \tag{35}$$

This completes the proof. ■

IV. THE HALVING INTERVAL METHOD

The switching times gradient obtained from (28) is used to determine the optimal switching times. The idea is to shrink the interval of the time horizon by one-half several times. The algorithm is presented bellow.

Input: t_0, t_f, ε
Output: t_s
while $|t_f - t_0| \geq \varepsilon$ **do**
  $t_s = \frac{t_f - t_0}{2}$;
  Compute $\frac{\partial J(t_s)}{\partial t}$;
  if $\frac{\partial J(t_s)}{\partial t} < 0$ **then**
    $t_0 = t_s$;
  else
    $t_f = t_s$;
  end
end
Algorithm 1: The halving interval method algorithm.

Since the method is gradient based, the solution is locally optimal when it converges. An illustration of the halving interval method can be seen in **Fig. 2** where the solution converges in 8 iterations.

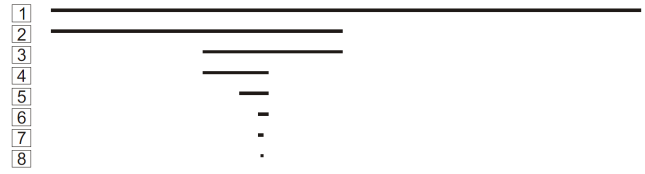

Fig. 2. Illustration of the halving interval method.

V. ISSUES ON PRACTICAL IMPLEMENTATIONS

It has been shown that for regular linear and nonlinear optimal control problems involving n^{th} order systems, the optimal solution has at most n switching times [17]. Because in practice the number of grid blocks might be in the millions and the number of wells might be in the dozens, the method does not appear to have much practical usefulness. Therefore, a pragmatic approach appears to be possible by setting the number of switching times to a reasonable number in comparison to the computational time.

Another limitation occurs when computing the left and right limit of the Hamiltonian sufficiently small time steps

are required. On the other hand, one step of the reservoir simulation will require a significant amount of computational time, hence, in practice larger time steps are usually used. Therefore to ensure the algorithm leads to the optimal solution, the quality of the gradient could be checked by the finite difference.

To implement the switching time method, a state of art reservoir simulator called Matlab Reservoir Simulator Toolbox (MRST) is employed. The simulator is completed with the adjoint code [18] which can be used to calculate the gradient of the switching time (28).

VI. NUMERICAL EXAMPLE

Consider a realistic three dimensional oil reservoir given in **Fig. 3**. The reservoir is taken from [19]. The number of active grid blocks is 13,420. The reservoir is completed with two injection wells and one multi-lateral production well. The multi-lateral production well is connected to two horizontal wells. Each horizontal well is installed with inflow control devises which can be used to control flow rates or BHPs (i.e. both horizontal wells are controllable) via valve opening. Permeability distribution is given in **Fig. 4**.

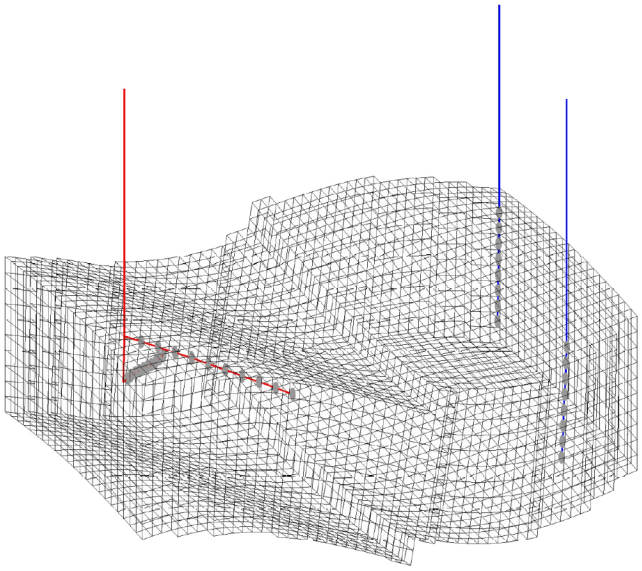

Fig. 3. Grid and wells positions. Blue are injectors and red producers.

The injection rates are set constant for both injection wells at 500 $\mathrm{m}^3/\mathrm{day}$. Oil and water viscosity are set at 5×10^{-3} Pa.s and 1×10^{-3} Pa.s, while their density is 859 kg/m^3 and 1014 kg/m^3, respectively. The average reservoir pressure p_R is 400 bara.

The oil and water production price are $r_o^p = 630\ \mathrm{USD}/\mathrm{m}^3$ and $r_w^p = 63\ \mathrm{USD}/\mathrm{m}^3$, while the water injection price is $r_w^i = -63\ \mathrm{USD}/\mathrm{m}^3$. The discount factor is $I = 10\%$. Two methods are compared. First, reactive control is used where the well is operated at the maximum production rates and will be shut after the well is not profitable. In this case the water cut threshold is 0.82. Reactive control is commonly used in petroleum industry. The second method is to use the optimal switching times algorithm. The control variables are the switching times of the valve opening in both horizontal production wells, namely t_1^1, t_2^1, t_1^2, and t_2^2, i.e., $M_1 = M_2 = 2$. The optimization is run for 512 days. The objective is to maximize the following NPV

$$\max_{\mathbf{v}} J_{NPV} = \int_0^T \left(\sum_{j\in\mathcal{N}_{prod}} r_o^p(t) q_o^j(t) - \sum_{j\in\mathcal{N}_{prod}} r_w^p q_w^j(t) + \sum_{j\in\mathcal{N}_{inj}} r_w^i q_w^j(t) \right) \times \frac{1}{\left(1+\frac{I}{100}\right)^{ct}}\, \mathrm{d}t \qquad (36)$$

subject to the dynamical system described in (22)-(23).

In practice, the wells will be produced at the maximum production rates in the first weeks in order to clean up the well from the brine set during completion and to get a good clean up of the near well reservoir. Therefore, in the switching times method the wells will initially be fully open. In this case, the initial switching times are: $t_1^1 = 128$ days, $t_2^1 = 384$ days, $t_1^2 = 256$ days, and $t_2^2 = 384$ days, i.e., the well is closed between 128 days and 384 days for horizontal well no. 1 and between 256 days and 384 days for horizontal well no. 2.

Fig. 5 shows the optimal switching times control for both horizontal wells. Using the proposed method, we obtain $t_1^1 = 222$ days, $t_2^1 = 342$ days, $t_1^2 = 342$ days, and $t_2^2 = 510$ days, i.e., the well is shut between 222 days and 342 days for horizontal well no. 1 and between 342 days and 510 days for horizontal well no. 2. The reactive control method is shown by the dashed line. **Fig. 6** shows the comparison of the objective functional of the switching times method and the reactive control. It can be seen that the switching times method gives a higher NPV. Typical run-time for a single simulation in MRST was 4 minutes on an Intel core duo processor 1.66 GHz computer.

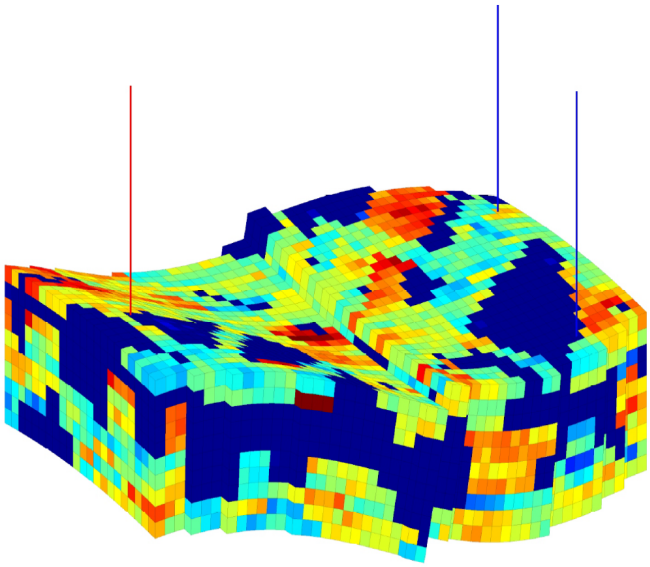

Fig. 4. Permeability distribution.

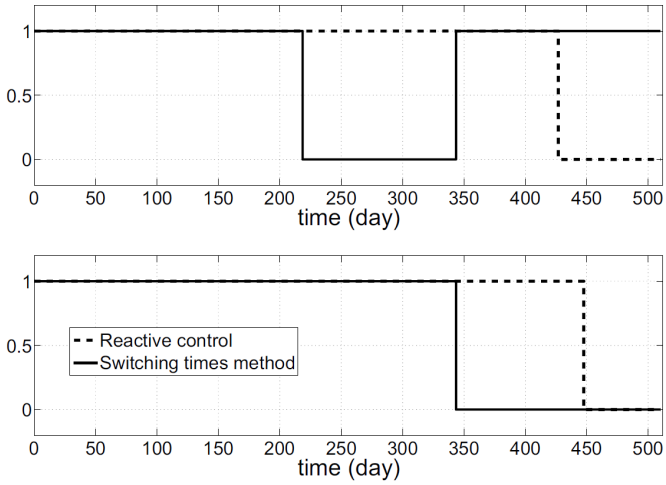

Fig. 5. Valves opening of the horizontal production well no. 1 (up) and no. 2 (down).

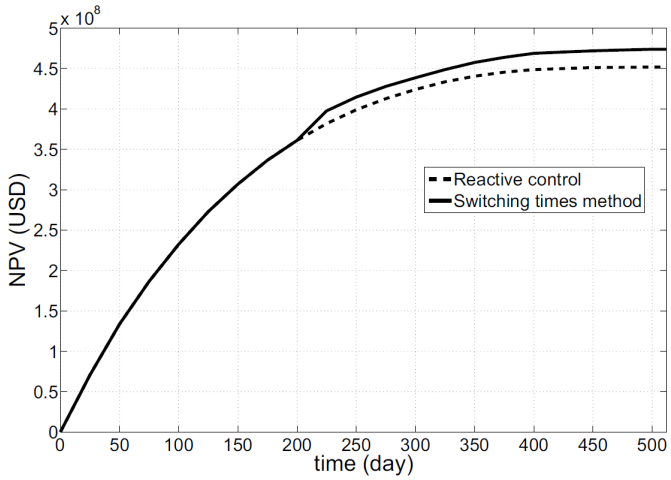

Fig. 6. Comparison of the objective functional.

VII. CONCLUSIONS

In this paper, an optimal switching times method for an oil reservoir, given the optimal settings of the wells to be bang-bang controls, has been presented. The bang-bang control has a major practical advantage since it can be implemented with simple on-off valves. The reservoir model is written in a state space representation and was assumed to be free from uncertainty. The optimal control settings are obtained based on gradient calculation using the adjoint method combined with the halving interval method. A state of art reservoir simulator called MRST is used for this purpose. An example of a realistic reservoir show that the proposed method can be used to increase oil revenue significantly.

ACKNOWLEDGMENTS

This paper was supported by the Center for Integrated Operations in the Petroleum Industry, NTNU.

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