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Active Disturbance Rejection Control Based on a Simultaneous Adaptive Observer and a Time Varying Parameter Identifier

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Abstract— **Design of adaptive observers has been an active field in the systems theory. Until know, most of the existing solutions use a class of regular form with a set of unknown parameters. A different scheme used an estimation of external perturbations that can be compensated by the adaptive gain associated to the observer. In this paper, the presence of external perturbations for a model defined by a chain of integrators is considered. The adaptive observer used an identifier to obtain the time varying parameters used by the observer. Simultaneously, an adaptive gain associated to the observer adjusts the observer trajectories to provide the convergence between the states of the uncertain nonlinear system and the ones associated to the estimator suggested in this paper. Once the states of the uncertain system were obtained, a simple feedback controller was able to reject actively the perturbations that affect the nonlinear system. A simple third order uncertain system was evaluated in a numerical simulation for proving the performance of this observer/identifier. The same system was controlled using the estimated trajectories provided by the observer.**

Index Terms— **Adaptive observer, continuous-time least mean squares, time varying identification, active disturbance rejection.**

I. INTRODUCTION

Differential flatness property allows a complete parametrization of all system variables in terms of a limited set of special, differentially independent, output variables, called the flat outputs, and a finite number of their time derivatives [1], [2]. One of the implications of a flat system is the fact that it can be equivalent to a linear controllable system with a number of flat outputs equal to its number of inputs [3]. Differentially flat systems also can be expressed in a Brunovsky Canonical Form [4].

In flat systems, it is suitable to view the trajectory tracking problem as the control of a disturbed chain of integrators, where the disturbance signal is a lumped function containing non-modeled dynamics, perturbation and disturbance functions, to be canceled out by means of a robust algorithm. A first attempt for this control scheme consisted in considering a complete knowledge of the system, but subjected to external disturbance inputs and possible presence of non-modeled dynamics or uncertainties to be compensated through observation techniques [5].

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In [6], it was shown an observer based disturbance estimation, and subsequent cancelation, via the control law considering both internal and external lumped disturbances, which are actively compensated. This scheme has been called Active Disturbance Rejection Control (ADRC) and it has reported numerous laboratory and industrial applications [7].

An alternative linear-based disturbance estimation technique, by means of approximate, yet accurate, linear Generalized Proportional Integral (GPI) observers was proposed in [8]. The high gain GPI observer naturally includes a, self-updating, lumped, time-polynomial model of the nonlinear state-dependent perturbation; Once the disturbance is estimated, it is delivered to the controller for on-line cancelation while simultaneously estimating the phase variables related to the measured output. Some reported applications from different nature are found in literature (see [9], [10] and references therein).

Due to its high gain nature, the GPI observer accuracy relies on the linear dominant injection dynamics, which is increased while the poles are further away from the imaginary axis of the complex plane, into the left half. Thus, there is a necessity of a good gain tuning process, which needs to guarantee accurate estimations while avoiding high gain undesired effects such as peaking, instabilities or noisy estimations [11], [12].

To overcome the possible arising of this set of problems, it can be proposed an adaptive scheme to tune the gains of the observer. It has been shown that the incorporation of gain adaptation laws improved the high gain observers performance (see [13], [14], [15]).

The objective of this paper is to solve the problem of active disturbance rejection control for differentially flat systems by means of adaptive observers, in trajectory tracking tasks, based on the on-line reconstruction of the disturbance input, say $\xi(x_t)$, as well as the estimated states. The adaptive scheme is believed to enhance the control response in presence of noisy measurements, as well as the avoidance of peaking effects in the observation technique. We are proposing an adaptive observer with time varying gains that should provide an approximation of $\xi(x_t)$ and the estimates of all unmeasurable states. The form in which the approximation of the lumped disturbance signal is performed consisted in the combination of the state variables, having time varying coefficients. The convergence of the observer based control is proven in terms of the second method of Lyapunov.

The outline of the article is given as follows: The class of systems to be analyzed and the active disturbance rejection problem formulation are given in section II, some remarks about classic control schemes and the GPI observer based control are given in section III. Section IV deals with the design of the adaptive disturbance observer. The strategy is tested by means of a control system subject to nonlinearities, nonmodeled dynamics and noisy measurements, which is shown in section V. Finally, some concluding remarks are given.

A. Notation

In this paper, the following notation is used: $\Re^n$ represents the n dimensional real vector space. $\intercal$ is used to define the transpose operation. $\|x\|_H := x^\intercal Hx$ is the weighted norm of the real-valued vector $x \in \Re^n$ with weight matrix $H > 0$, $H = H^\intercal$, $H \in \Re^{n\times n}$. The limit superior $\overline{\lim_{t\to\infty}} c_t$ is used here to represent $\overline{\lim_{t\to\infty}} c_t := \inf_{\varepsilon\to 0}\{\sup\{g(x) : x \in E \cap U\} : U \text{ open}, E\cap U \neq \emptyset\}$. If two matrices $N \in \Re^{n\times n}$ and $M \in \Re^{n\times n}$, fulfill $M > N$ $(\geq)$, that means that $M - N$ is a positive definite matrix. Trace operation is defined as $\mathrm{tr}\{H\}$.

II. Class of systems and problem statement

Let us consider the following dynamic model defined by:

$$\begin{aligned} z_t^{(n)} &= g(z, \dot{z}, \ldots, z^{(n-1)})u_t + \xi_t(z, \dot{z}, \ldots, z^{(n-1)}) \\ y_t &= z_t \end{aligned} \tag{1}$$

where $u_t \in \Re$ is the input, $y_t \in \Re$ is the output signal, $g : \Re^n \to \Re$ is the known control gain, and $\xi : \Re^{n+1} \to \Re$ represents a function that includes external disturbances and unmodeled dynamics. The system is differentially flat with a flat output given by z. Let assume that for any bounded solution of z_t, the disturbance $\xi(\cdot)$ is bounded with bounded finite time derivatives. The output function g is fulfilling the fact that $0 < g^- \leq |g(z)| \leq g^+$, $g^-, g^+ \in \Re^+$, $\forall z \in \Re^n$.

The system (1) can be rewritten as follows

$$\dot{x}_t = Ax_t + G(x_t)u_t + f(x_1)$$
$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ G(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(x_t) \end{bmatrix},$$
$$f^\intercal(x_1) := \begin{bmatrix} 0 & 0 & \cdots & 0 & \xi(x_1) \end{bmatrix} \tag{2}$$

where $x_t = [x_{1,t}, x_{2,t}, \ldots, x_{n,t}]^\intercal$, $x_{k,t} = d^{k-1}y_t/dt^{k-1}$.

The uncertain nonlinear section $f(x_t)$ fulfills by assumption the following inequality

$$\|f(x_1)\|^2 \leq f_0 + \beta\|x_1\|^2 \tag{3}$$

with f_0 and β, positive scalars.

A. Problem formulation

Given an output reference trajectory, x^* for the system (2), devise an output feedback controller, which regardless of the unknown nonmodeled dynamics, or external disturbances, both lumped in an additive signal $\xi(x_t)$, forces the flat output $x_{1.t}$ to track the desired reference signal, with a tracking error restricted to a small neighborhood of the origin in the error phase space.

III. Classic Approach and Active Disturbance Rejection Approach

A. Classic output feedback linearization

A classical controller for this class of systems is given by:

$$u_t = \frac{1}{g(x_t)}\Big(-\xi(x_t) - h(x_t^*) - \sum_{j=1}^{n} k_j(x_{j,t} - x_{j,t}^*)\Big)$$

where $x_{j,t}^*$ is the $j-th$ component of a predefined reference trajectory, characterized by the system $\dot{x}_t^* = Ax_t^* + h(x_t^*)$.

The function $h(x_t^*)$ is assumed to be uniformly continuous on t, that is $\|h(\eta_1) - h(\eta_2)\| \leq h^+\|\eta_1 - \eta_2\|$, $\forall \eta_1, \eta_2 \in \Re^n$.

If the characteristic polynomial of the linearized system, say $s^n + k_n s^{n-1} + \ldots + k_2 s + k_1$ has its roots in the left half side of the complex plane, then x_t converges asymptotically to x_t^*.

However, the aforementioned design has two important inconveniences: 1) the control requires the perfect knowledge of $\xi(x_t)$ and all the states $[x_{1,t}, \ldots, x_{n,t}]$ must be available on-line. The active disturbance rejection control can solve the problem by means of different techniques.

B. The GPI observer using Taylor series approximation

In this approach, the lumped disturbance signal ξ_t is assumed to be unknown, and the controller needs to incorporate an estimator. One alternative is using a class of extended Luenberger class observers, named GPI observers [8]. The system[1], which is given by

$$\begin{bmatrix} \dot{\hat{x}}_t \\ \dot{\hat{\rho}}_t \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} \hat{x}_t \\ \hat{\rho}_t \end{bmatrix} + \begin{bmatrix} G(\hat{x}_t)u_t \\ 0 \end{bmatrix} + \Lambda e_t \tag{4}$$
$$e_t = x_{1,t} - \hat{x}_{1,t}$$

constitutes a GPI observer for the system (2), where $\hat{x}_t = [\hat{x}_1, \ldots, \hat{x}_n]^\intercal$ is the state estimates vector, $\rho_t = [\rho_{1,t}, \ldots, \rho_{m,t}]^\intercal$ is the set of extended states to estimate the disturbance input, by means of the Taylor series decomposition. The vector $\Lambda = [\lambda_1, \ldots, \lambda_{m+n}]^\intercal$ and the matrices A_{11}, A_{12} and A_{22} are given by: $A_{11} = A$,

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

[1]In this case, the observer is given in terms of a state space realization

This observer was successfully implemented to solve the ultimate boundedness problem for the state estimation problem. This observer proposal is considering the self updating approximation of $\xi(x_t)$ by means of a family of time polynomials of degree $m-1$ given by $\xi(x_t) = \sum_{i=1}^{m-1} c_i t^i + r_t$, $c_i \in \Re$, and r_t denotes the set of high order terms to complete the series.

Regardless of the benefits achieved by the method, the convergence region of the observer depends on possible high values of the gain vector Λ, which may have problems in presence of noisy signals, or some peaking effects in a wrong tuning.

An alternative disturbance estimation considers to change the strategy to obtain the approximation of $\xi(x_t)$ from the Taylor series to the structure given by $\xi(x_t) = \Lambda_t^0 x_t + \tilde{\xi}(x_t)$. In this option, Λ_t^0 is a vector formed by time varying constants that shall be adjusted by a special law and fulfilling a companion form for all time, that is $\Lambda_t^0 = \begin{bmatrix} -\alpha_{1,t}^0 & -\alpha_{2,t}^0 & -\alpha_{3,t}^0 & \cdots & -\alpha_{n,t}^0 \end{bmatrix}$.

In [16], this problem has been tackled by the high order differentiator schemes, the homogeneity approach and some others. However, all of them are forcing the convergence between the states of observer and the uncertain system despite the nature of $\xi(x_t)$. Once the estimation has been completed, the parameter identification method is turned on. Here, a mixed form having both options is considered, that is, we are proposing an adaptive observer with time varying gains that should provide an approximation of $\xi(x_t)$ and the estimates of all unmeasurable states.

IV. THE ADAPTIVE OBSERVER

The nonlinear adaptive observer for the system (2) is proposed as (with the corresponding feedback controller using the observer states $u_t := u(\hat{x}_t)$):

$$
\begin{aligned}
&\dot{\hat{x}}_t = \hat{A}_t \hat{x}_t + G_t u_t + K_t e_t \qquad (5)\\
&e_t := y_t - C\hat{x}_t\\
&u_t = -H\hat{x}_t\\
&\hat{A}_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{1,t} & -\alpha_{2,t} & -\alpha_{3,t} & \cdots & -\alpha_{n,t} \end{bmatrix}\\
&C := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}
\end{aligned}
$$

here, $\alpha_{i,t}$, $i \in \{1, \ldots, n\}$ are time varying parameters that must be adjusted by an adaptive law based on the least mean square method. The matrix $H \in \Re^{1\times n}$ is the gain matrix. Besides, the time variable gain K_t obeys the following dynamics

$$\dot{K}_t = -k_1 P_t \hat{x}_t e_t \qquad (6)$$

where $k_1 \in \Re$ and P_t satisfies the time-varying Riccati Equation

$$\dot{P}_t + P_t \bar{A}_t + \bar{A}_t^{\intercal} P_t + P_t R_t P_t + Q_t = 0 \qquad (7)$$

where $\bar{A}_t = A - K_t C$, $R_t := \Lambda_{1,t} + \Lambda_{2,t} > 0$, $\Lambda_{1,t}, \Lambda_{2,t} \in \Re^{n\times n}$, $Q_t := Q_1 + C^{\intercal} K_t \Lambda_{3,t} K_t C + \beta_1 \Lambda_{4,t}$, and $Q_1 \in \Re^{n\times n}$. The matrices $\Lambda_{1,t}$, $\Lambda_{2,t}$, $\Lambda_{3,t}$, $\Lambda_{4,t}$ are piecewise continuous integrable, positive definite for all t and all they are square $n \times n$ matrices, $Q_1 = Q_1^{\intercal}, Q_1 > 0$. These matrices always can be selected in that form way because they are designer choices.

The matrix $\hat{A}_t$ is having time varying parameters in its last row. Let us define the vector $\bar{a}_t = \begin{bmatrix} -\alpha_{1,t} & -\alpha_{2,t} & -\alpha_{3,t} & \cdots & -\alpha_{n,t} \end{bmatrix}$, which represents the last row in the matrix $\hat{A}_t$. The dynamics of $\bar{a}_t$ is governed by

$$\bar{a}_t = \left[\int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^{\intercal} d\tau \right]^{-1} \left[\int_{t-h}^{t} w_\tau \hat{x}_\tau^{\intercal} d\tau \right]^{\intercal} \qquad (8)$$

with h a positive scalar. Besides:

$$
\begin{aligned}
&w_t = M\left(\hat{x}_t - \hat{x}_{t-h} - \int_{t-h}^{t} \left(G(\hat{x}_\tau) u_\tau + K_\tau e_\tau \right) d\tau \right)\\
&M^{\intercal} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad (9)
\end{aligned}
$$

The convergence result for the adaptive observer is given in the following theorem.

Theorem. 1: Let us consider the class of nonlinear uncertain systems (1) fulfilling the condition (3) with incomplete information. Now, define the state observer given in (5) with the time varying gain K_t adjusted by (6) and the time varying parameters $\overline{a}_t$ tuned by the least mean square for continuous time described in (8). If there exist positive definite matrices Q_t and $\bar{Q}_t$ time varying matrix parameters such that (5) and the time varying Riccati equations

$$\dot{S}_t + S_t\left(\hat{A}_t - G_t H\right) + \left(\hat{A}_t - G_t H\right)^{\intercal} S_t + S_t \bar{R}_t S_t + \bar{Q}_t = 0 \qquad (10)$$

with $\bar{R}_t = \Lambda_{3,t}^{-1}$, $\bar{Q}_t = Q_2 + A^{\intercal} \Lambda_{2,t}^{-1} A + \beta_2 \Lambda_{4,t} > 0$, $\bar{Q}_t \in \Re^{n\times n}$, $Q_2 = Q_2^{\intercal} > 0$ have positive definite solutions P_t and S_t for all t, then the tracking and estimation errors, say $\delta_t = \hat{x}_t - x_t^*$ and $\Delta_t = x_t - \hat{x}_t$ converge asymptotically to a bounded neighborhood of the origin.

Proof: The best reachable parameter for $\hat{A}^*$ is found as the solution of the following problem

$$
\begin{aligned}
&\bar{a}_t := \arg\min \left[\int_{\tau=0}^{t} \left(M w_\tau - \bar{a}_\tau \hat{x}_\tau \right) d\tau \right]\\
&w_t = \hat{x}_t - \hat{x}_{t-h} - \int_{t-h}^{t} \left(G(\hat{x}_\tau) u_\tau + K_\tau e_\tau \right) d\tau
\end{aligned}
$$

The estimation error obeys the following structure

$$\dot{\Delta}_t = A\Delta_t + \tilde{A}_t \hat{x}_t + K_t e_t + f(x_t)$$

The variable $\tilde{A}_t$ stands for

$$\tilde{A}_t = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\tilde{\alpha}_{1,t} & -\tilde{\alpha}_{2,t} & \cdots & -\tilde{\alpha}_{n,t} \end{bmatrix}$$

where $\tilde{\alpha}_{i,t} := \alpha_{i,t} - \alpha_{i,t}^0$. ■

To prove the convergence of the estimation error, let us propose the following quadratic form as a Lyapunov Candidate Function.

$$V(\Delta_t, e_t, \hat{x}_t, K_t) = \Delta_t^\intercal P_t \Delta_t + \hat{x}_t^\intercal S_t \hat{x}_t + k_1^{-1} tr\left\{K_t^\intercal K_t\right\} + \eta_t^\intercal \eta_t$$
$$\eta_t = \int_0^t \Big(\int_{s-h}^{s} w_\tau \hat{x}_\tau^\intercal d\tau - \bar{a}_s^\intercal \int_{s-h}^{s} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau \Big) ds \qquad (11)$$

The full time derivative of the proposed energetic function was calculated as

$$\dot{V}(\Delta_t, e_t, \hat{x}_t, K_t) = 2\Delta_t^\intercal P_t \dot{\Delta}_t + \Delta_t \dot{P}_t \Delta_t + 2\hat{x}_t^\intercal S_t \dot{\hat{x}}_t + 2\hat{x}_t \dot{S}_t \hat{x}_t + 2k_1 tr\left\{K_t^\intercal \dot{K}_t\right\} + 2\eta_t^\intercal \dot{\eta}_t$$
$$\dot{\eta}_t = \int_{t-h}^{t} w_\tau \hat{x}_\tau^\intercal d\tau - \bar{a}_t^\intercal \int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau$$

That is:

$$\dot{V}(\Delta_t, e_t, \hat{x}_t, K_t) = 2\Delta_t^\intercal P_t \left[A_t \Delta_t + \tilde{A}_t \hat{x}_t + K_t e_t + f(t,x)\right] + \Delta_t^\intercal \dot{P}_t \Delta_t + 2\hat{x}_t S_t \dot{\hat{x}}_t + 2\hat{x}_t \dot{S}_t \hat{x}_t + 2k_1 tr\left\{K_t^\intercal \dot{K}_t\right\} + 2\eta_t^\intercal \dot{\eta}_t$$

Using the relationship between the state estimation error and the scalar output error $e_t = C\Delta_t$, and using its corresponding time derivative

$$\dot{e}_t = C\dot{\Delta}_t = M\left(A_t \Delta_t + \tilde{A}_t \hat{x}_t + K_t e_t + f(x_t)\right)$$

The direct substitution of the previous equation in the full time derivative of the energetic function (11) yields

$$\dot{V}(\Delta_t, e_t, \hat{x}_t, K_t, \eta_t) = 2\Delta_t^\intercal P_t [A_t \Delta_t + \tilde{A}_t \hat{x}_t + K_t e_t + f(t,x)] + \Delta_t^\intercal \dot{P}_t \Delta_t + 2\hat{x}_t^\intercal P_t[\hat{A}_t \hat{x}_t + G(x_t)u_t + K_t e_t] + 2\hat{x}_t \dot{S}_t \hat{x}_t + 2k_1 tr\{K_t^\intercal \dot{K}_t\} + 2\eta_t^\intercal \Big[\int_{t-h}^{t} w_\tau \hat{x}_\tau^\intercal d\tau - \bar{a}_t^\intercal \int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau\Big]$$

The direct application of the matrix inequality: $X^\intercal Y + Y^\intercal X \le X^\intercal M X + Y^\intercal M^{-1} Y$, valid for any $X, Y \in R^{r\times s}$ and any $0 < M = M^\intercal \in R^{s\times s}$ leads to

$$\dot{V}(\Delta_t, e_t, \hat{x}_t, K_t, \eta_t) \le \Delta_t^\intercal (\dot{P}_t + P_t A_t + A_t^\intercal P_t + P_t R_t P_t + Q_t)\Delta_t + \hat{x}_t^\intercal (\dot{S}_t + S_t \hat{A}_t + \hat{A}_t^\intercal S_t + S_t \bar{R}_t S_t + \bar{Q}_t)\hat{x}_t + f_0 + 2k_1 tr\{K_t^\intercal \dot{K}_t\} - \Delta_t^\intercal \left(Q_t + \bar{Q}_t\right)\Delta_t - \hat{x}_t^\intercal S_t \hat{x}_t + 2\eta^\intercal(t)\Big[\int_{t-h}^{t} w(\tau)\hat{x}^\intercal(\tau) d\tau - \bar{a}_t^\intercal \int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau\Big]$$

In the previous equation, the following inequality was used

$$\|f(x_t)\|^2 \le f_0 + \beta_1 \|\Delta\|^2 + \beta_2 \|\hat{x}_t\|^2$$

that is obtained directly from (3). In the previous inequality $\beta_1 > 2\beta$ and $\beta_2 > 2\beta$, Now let us take the term inside the brackets in last equation equal to zero

$$\Big[\int_{t-h}^{t} w_\tau \hat{x}_\tau^\intercal d\tau - \bar{a}_t^\intercal \int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau\Big] = 0$$

This is possible if

$$\bar{a}_t = \Big[\int_{t-h}^{t} \hat{x}_\tau \hat{x}_\tau^\intercal d\tau\Big]^{-1} \Big[\int_{t-h}^{t} w_\tau \hat{x}_\tau^\intercal d\tau\Big]^\intercal$$

By the adjustment law for the matrix K_t, the selected method to estimate $\bar{a}_t$ in its integral form and the assumption on the positive definiteness of the solution for the set of two Riccati equations (7), (10) one has:

$$\dot{V}(\Delta_t, e_t, \hat{x}_t, K_t, \eta_t) \le -\Delta_t^\intercal Q_1 \Delta_t - \hat{x}_t^\intercal Q_2 \hat{x}_t + f_0$$

Therefore, the trajectories of Δ_t and $\hat{x}_t$ converge asymptotically to a bounded zone characterized by

$$\overline{\lim_{t\to\infty}} \left\| \begin{bmatrix} \Delta_t \\ \hat{x}_t \end{bmatrix} \right\|^2 \le \max(\lambda_{\min}^{-1}(Q_{12}),\ \lambda_{\min}^{-1}(Q_3)) f_0$$

and using the Barbalat theorem [17], the statement of the theorem is proven.

Remark 1: The time varying Riccati equations have positive definite solutions if both matrices R_t and Q_t are piecewise continuous integrable [18]. But, they were selected in such a manner from the beginning. Therefore, both Riccati equations can have positive definite solutions.

V. Example: A Disturbed Third Order System

A third order uncertain system was selected to test the output based controller suggested in this paper. The aforementioned system is

$$\theta_t^{(3)} = f(\theta_t) + g(\theta_t)u_t + \omega_t$$
$$y_t = \theta_t \qquad (12)$$

where $f(\theta_t) = (a_0 + a_t)^\intercal \vec{\theta}_t$, $g(\theta)_t = 2$, $\omega_t := \cos(60\pi t) + WN(0,1)$, $\vec{\theta}_t^\intercal = \begin{bmatrix}\theta_t & \dot{\theta}_t & \ddot{\theta}_t\end{bmatrix}$, $a_0^\intercal = \begin{bmatrix}-2.3 & -1.5 & -4.2\end{bmatrix}$, $a_t^\intercal = \begin{bmatrix}0.29\sin(8\pi t) & 0.16\sin(\pi t) & 0.23\sin(2.4\pi t)\end{bmatrix}$.

The term $WN(0,1)$ refers to a pseudo white noise implemented numerically with zero mean and unitary variance. The uncertainty in the previous system is because $f(\theta_t)$ is assumed to be unknown.

Using the state variable representation, the system (12) can be transformed to

$$
\begin{aligned}
\dot{z}_t &= \Phi_t z_t + \Gamma_t u_t + \bar{\omega}_t \\
\Phi_t &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (a_0 + a_t)^{\intercal} \\
\Gamma_t &:= \begin{bmatrix} 0 \\ 0 \\ g(z_1) \end{bmatrix}, \quad \bar{\omega} := \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{bmatrix} = \begin{bmatrix} \theta_t \\ \dot{\theta}_t \\ \ddot{\theta}_t \end{bmatrix}
\end{aligned}
$$

A. Observer design

The observer was designed following the ideas presented in last section. The observer is governed by the following Ordinary Differential Equation:

$$
\begin{aligned}
\dot{\hat{z}}_t &= \hat{\Phi}_t \ \hat{z}_t + \Gamma_t u_t + K_t(z_{1,t} - C\hat{z}_t) \\
C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \hat{\Phi}_t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \hat{a}_t^{\intercal}
\end{aligned}
$$

where $\hat{a}_t$ and K_t are governed by (8) and (6) respectively.

B. Controller design

Based on the states provided by the observer, the control action was designed as $u_t = H^{\intercal}(\hat{z}_t - z_t^*)$.

The reference trajectory z_t^* was selected as follows:

$$
\begin{aligned}
\dot{z}_t^* &= \Phi^* z_t^* + \Gamma v_t \\
\Phi^* &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.10 & -0.99 & -0.22 \end{bmatrix}; v_t = 2\sin(4\pi t)
\end{aligned}
$$

The gain matrix H was selected using the Ackermann formula for the dynamical system representing the tracking error $\delta_t = \hat{z}_t - z_t^*$.

C. Simulation results

1) The observer performance: A set of numerical simulations was developed in Matlab. The observer trajectories converged to a bounded zone around the real trajectories of (12). The observer adaptive gain was having a transient behavior with no oscillations. uncertain system in less than 15 seconds. Figure 1 shows the reconstruction of all the three states of the uncertain system in solid line. In the same figure, the observer states are shown in dashed line. Notice that the first state is completely reconstructed within the first 5 seconds and actually, no visible difference is recognized between the system and the observer trajectory. The remainder states are reproduced by the corresponding observer trajectories some seconds later. Besides, Figure 2 shows simultaneously the initial behavior of the trajectories produced by the uncertain system (solid line) and the observer (dashed line). This close view to the trajectories evolution demonstrates the observer performance when the states of the uncertain system are estimated.

The adaptation law for the uncertain parameters forced a transient behavior on the time varying estimation. These oscillations follow the states variation through the time as seen in the figure 3. These oscillations do not coincide in frequency with those used in the simulations. However, this was an expected behavior considering that poor information about the uncertain system provided to the observer.

The adaptive gains are adjusted before some miliseconds and then they remain constant from that moment and so on. Figure 4 (top) shows the beginning of the K_t evolution.

2) The controller performance: The output based controller forced the trajectories of the uncertain system to a reference trajectory described above. The reference trajectories were achieved by the uncertain system within the first 10 seconds. Figure 1 shows all the three states of the uncertain system in solid line but also contains the reference trajectories depicted by a dotted line. As one can expect, the controller forces the converge between the reference and the uncertain system states but always after the observer trajectories have converged to the states of (12). Figure 2 gives a detail about the first steps on the controller performance.

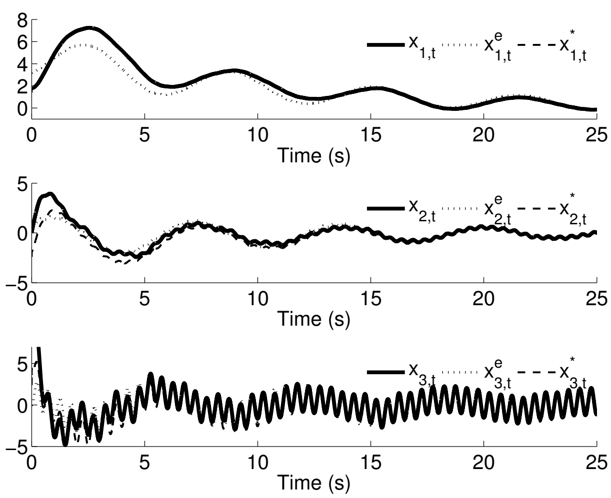

Fig. 1. System states and estimations.

The control action is depicted in figure 4 (middle). Although the controller suggested is a high gain type, the overshooting effects are eased by the observer disturbance estimation as well as it forced the convergence between the uncertain system trajectories and the reference states.

Finally, the mean square error between the norm of the system trajectories and the norm of the reference states is depicted in figure 4 (bottom). Notice that this performance index is going to zero from the beginning. The same figure is also showing the mean square of Δ_t. These two graphics confirmed how the observer with time varying gain can provide a faster convergence for Δ_t than the controller that is using a constant gain.

VI. Conclusions

This paper describes a mixed structure based on adaptive observation and parameter estimation. This mixed adaptive

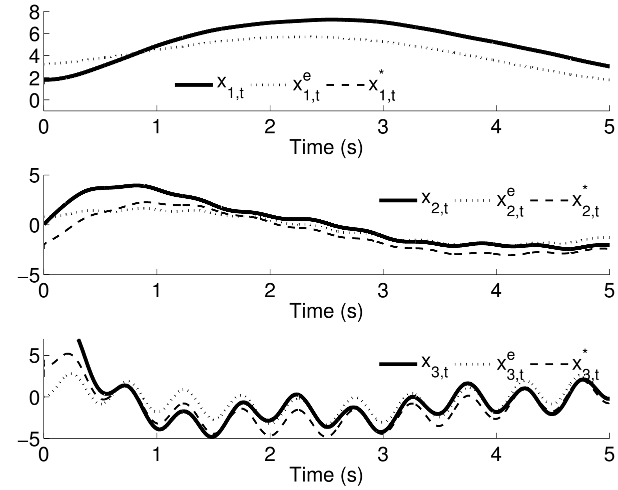

Fig. 2. Initial behavior of the observer trajectories.

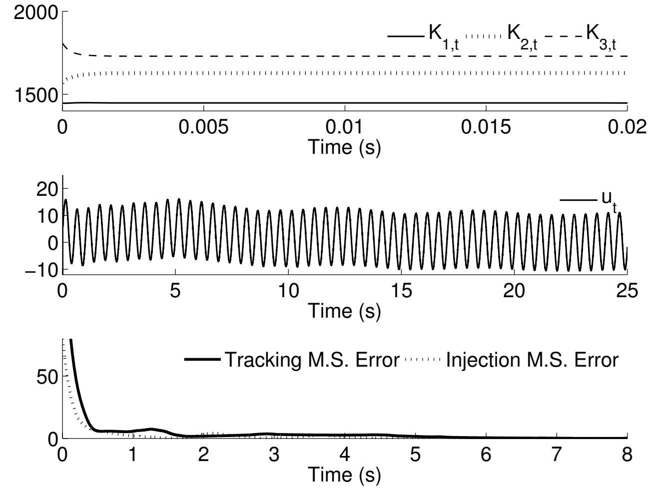

Fig. 4. Adaptive tuning of the control input and performance.

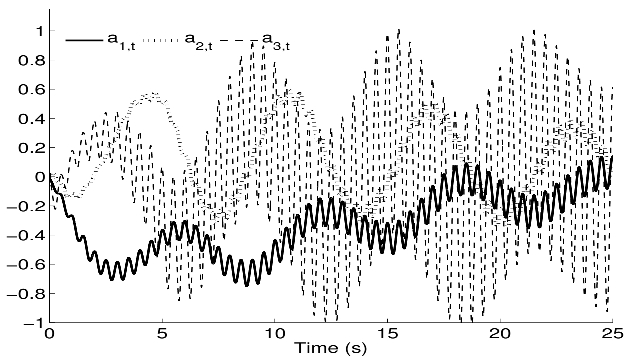

Fig. 3. Parameter adaptation.

observer can be used to control a class of uncertain nonlinear systems presented in the regular controllable Brunovsky form. The convergence of this adaptive observer was provided by means of a time varying Lyapunov function. The analysis presented here has provided the adjustment law for the gain matrix associated to the observer as well as the parameter adjustment method based on the continuous least mean square scheme. A simple linear feedback controller was designed to show how the observer states can be used to substitute the real variables of the uncertain system. A basic third order uncertain nonlinear system was used to generate a numerical simulation that validated the theoretical result achieved in this study.

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