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# A Stopping Rule for Simultaneous Perturbation Stochastic Approximation

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*Abstract*— **A stopping rule is developed for simultaneous perturbation stochastic approximation (SPSA) which is an iterative method for minimizing an unknown objective function via its noise corrupted measurements. It is shown that, when the number of iterations reaches a constant determined by the stopping rule, SPSA for the quadratic convex problem provides us with a suboptimal solution which is close to the optimal solution with a specified probabilistic guarantee. Furthermore, the number is determined by the specified guarantee and polynomial in the problem size.**

## I. INTRODUCTION

Simultaneous perturbation stochastic approximation (SPSA) [1], [2] gives us the optimal solution of an unconstrained optimization which is to minimize the unknown objective function. This method has various areas of application, for example, adaptive control [3], [4], on-line parameter tuning of automotive engine [5], inventory management in supply chains [6], an application to the servo system [7], and so on.

SPSA estimates the gradient of the objective function via *simultaneous perturbation* of all elements in decision variable [1], [8]. In fact, SPSA employs difference quotient of a directional derivative of the objective function along a given perturbation vector instead of the gradient. Its random selection of perturbation vector leads to an adequate gradient estimation of the objective function in a probabilistic sense. The significant advantage of SPSA is that the number of measurements of the objective function is less than that of the finite-difference stochastic approximation (FDSA) when we search for a solution with the same accuracy. There are many researches on SPSA from theoretical aspects, for example, a sufficient condition on convergence [1], the asymptotic rate of convergence [9], and so on [10], [11]. Since sufficient conditions for convergence to the optimal solution is well established [2], we have already understood that the candidate of the solution is close to the optimal solution when the number of iterations becomes large enough.

In this paper, we focus on stopping rules for SPSA. Our question is *when we stop SPSA procedure, that is, what is a principal criterion for termination of its procedure?* We can execute the procedure with only finite iterations in practical situations. We therefore need a stopping rule which guarantees quality on solution candidates. We are especially interested in stopping rules with theoretical guarantee. From the theoretical aspects, the stopping rules clarifies detailed analysis of the rate of convergence of SPSA, while the rate of convergence, when the number of iterations becomes large enough, is given by [9]. On the other hand, we have already developed stopping rules for stochastic approximation of unknown linear equation [12] and that for FDSA [13]. However, it should be noted that construction of stopping rule for SPSA is more difficult than these cases due to its random perturbation vector for gradient estimation. In this paper, we consider an unconstrained optimization problem with a convex quadratic objective function. Then, we show a stopping rule which guarantees that effect of initial error and that of measurement noise are small enough within specified value in theoretical sense. In addition to this, we clarify that the number of iterations which is derived from the stopping rule is a polynomial of problem size.

We hasten to note that the conventional stopping rules on stochastic approximation for an unknown equation, for example, [14]–[18], can not apply to SPSA. This is because these stopping rules require that variance of gradient estimation is bounded. Note also that stopping rules via surrogate process [19]–[21] for stochastic optimization are useful in practical situations, though they do not give a rigorous theoretical guarantee. On the other hand, the results in this paper give a theoretical guarantee to a stopping rule.

This paper is organized as follows. In Section II, we state the problem formulation and a stopping rule of SPSA which is a main result in this paper. In Section III, we provide proof of the theorem, while in Section IV we supply verification on the stopping rule. Numerical examples are shown in Section V. Finally, we state some concluding remarks in Section VI.

## II. PROBLEM FORMULATION AND MAIN RESULT

Let us consider an unconstrained optimization

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the decision variable. Here, we assume that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is unknown, while its noisy measurement $y \in \mathbb{R}$ is available. That is, for a given $x$, we can obtain a measurement

$$y = f(x) + \varepsilon$$

where $\varepsilon \in \mathbb{R}$ is measurement noise. An update rule of simultaneous perturbation stochastic approximation is

$$x_{k+1} = x_k - a_k \hat{g}_k, \tag{2}$$

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where $x_k \in \mathbb{R}^n$ denotes the $k$-th candidate of the solution, that is, the subscript $k$ means the number of iterations, the gain $a_k \in \mathbb{R}$ is monotone decreasing sequence, and $\hat{g}_k$ is the $k$-th gradient estimation which is given by

$$\begin{aligned}
\hat{g}_k &= \begin{bmatrix} \hat{g}_{k,1} & \hat{g}_{k,2} & \cdots & \hat{g}_{k,n} \end{bmatrix}^{\mathrm{T}}, \\
\hat{g}_{k,i} &= \frac{y_k^+ - y_k^-}{2c_k\delta_{k,i}}, \qquad\qquad i = 1,2,\ldots,n, \\
y_k^+ &= f(x_k + c_k\delta_k) + \varepsilon_k^+, \\
y_k^- &= f(x_k - c_k\delta_k) + \varepsilon_k^-.
\end{aligned}$$

The vector $\delta_k \in \mathbb{R}^n$ is a random perturbation vector, its $i$-th element is $\delta_{k,i}$, and $c_k \in \mathbb{R}$ is a monotone decreasing step size.

In this paper, we assume that the objective function is convex quadratic

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Hx + h^{\mathrm{T}}x + d, \qquad \mu I \preceq H \preceq \eta I$$

for construction of a stopping rule. Note that we do not know the exact coefficients $H$, $h$, and $d$. Furthermore, we set $\mu = 1/2$ and $\eta = 1$ for simplicity, though $\eta > \mu > 0$ is sufficient to construct our stopping rule. Measurement noise $\varepsilon_k^{\pm} \in \mathbb{R}$ is according to independent and identically distribution with mean 0 and variance $\sigma^2$. The problem (1) has a unique solution $x^* = -H^{-1}h$ due to

$$f(x) = \frac{1}{2}\left(x + H^{-1}h\right)^{\mathrm{T}} H \left(x + H^{-1}h\right) - \frac{1}{2}h^{\mathrm{T}}H^{-1}h + d.$$

We set $a_k = (n+k)^{-1}$, $c_k = (n+k)^{-1/6}$ whose setting has the best performance in a theoretical sense [2]. We select any $i$-th element $\delta_{k,i}$ as a random variable which takes 1 with probability $1/2$ and $-1$ with probability $1/2$, that is, its probability distribution is the symmetric Bernoulli distribution. This distribution is the most preferable one for SPSA [22].

Now, we give a stopping rule for SPSA which is the main result in this paper.

*Theorem 1:* For given constants $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, and $\gamma \in (0,1)$, select $\bar{k} \in \mathbb{N}$ satisfying

$$\bar{k} \geq \max\{\tau_1, \tau_2\}, \tag{3}$$

$$\tau_1 \doteq \frac{n+1}{\alpha^2}\left(1 + \sqrt{\frac{n\exp(1)}{\gamma}}\right) - n, \tag{4}$$

$$\tau_2 \doteq 2\left(\frac{6n\sigma^2\exp(1)}{\beta^2\gamma}\right)^{3/2} - n. \tag{5}$$

Then, the $\bar{k}$-th candidate $x_{\bar{k}}$ of the solution satisfies

$$\mathsf{P}^{\infty}\left(\|x_{\bar{k}} - x^*\|_2 \leq \alpha\|x_1 - x^*\|_2 + \beta\right) \geq 1 - \gamma \tag{6}$$

for any initial candidate $x_1 \in \mathbb{R}^n$ of the solution, where $\|\cdot\|_2$ denotes the Euclidean norm and $\mathsf{P}^{\infty}$ is the probability measure according to the measurement noise sequence and the random perturbation vector sequence.

This theorem says that how close to the optimal solution $x^*$ the candidate $x_{\bar{k}}$ is. The smaller $\alpha$ and $\beta$ are selected, the smaller distance between $x_{\bar{k}}$ and $x^*$ is guaranteed with probability $1-\gamma$. Notice that SPSA is a randomized algorithm, that is, we could not have the same result each execution even if we set the same parameters. This is why Theorem 1 is a statement in a probabilistic sense. The theorem also states that the number $\bar{k}$ is polynomial of problem size. We hasten to note that we have conventional results as application of Theorem 1 since we can select any small parameters $\alpha$, $\beta$, and $\gamma$

*Corollary 1:* [1] For any initial candidate $x_1 \in \mathbb{R}^n$ of the solution, a sequence $x_1$, $x_2$, ... of candidates of the optimal solution converges in probability towards $x^*$.

This implies that Theorem 1 is a natural extension of conventional results.

*Remark 1:* We can also guarantee the quality on the objective function value when we employ our stopping rule. In fact, the $\bar{k}$-th candidate $x_{\bar{k}}$ of the solution satisfies

$$\mathsf{P}^{\infty}\left(f(x_{\bar{k}}) - f(x^*) \leq \frac{1}{2}\left(\alpha\|x_1 - x^*\|_2 + \beta\right)^2\right) \geq 1 - \gamma,$$

where $\bar{k}$ is defined by (3). We can validate this property via easy computation. We first note that

$$f(x^*) = -\frac{1}{2}h^{\mathrm{T}}H^{-1}h + d$$

due to

$$f(x) = \frac{1}{2}(x + H^{-1}h)^{\mathrm{T}}H(x + H^{-1}h) - \frac{1}{2}h^{\mathrm{T}}H^{-1}h + d.$$

This implies that

$$\begin{aligned}
f(x) - f(x^*) &= \frac{1}{2}(x + H^{-1}h)^{\mathrm{T}}H(x + H^{-1}h) \\
&= \frac{1}{2}(x - x^*)^{\mathrm{T}}H(x - x^*).
\end{aligned}$$

From the assumption $I/2 \preceq H \preceq I$,

$$f(x_{\bar{k}}) - f(x^*) = \frac{1}{2}(x_{\bar{k}} - x^*)^{\mathrm{T}}H(x_{\bar{k}} - x^*) \leq \frac{1}{2}\|x_{\bar{k}} - x^*\|_2^2.$$

That is, when the number of iterations reaches $\bar{k}$, the above inequality implies that

$$f(x_{\bar{k}}) - f(x^*) \leq \frac{1}{2}\|x_{\bar{k}} - x^*\|_2^2 \leq \frac{1}{2}\left(\alpha\|x_1 - x^*\|_2 + \beta\right)^2$$

with probability $1-\gamma$. We can conclude that the distance between a solution candidate and the optimal solution is proportional to the error of the objective function value.

## III. Proof of Theorem 1

The gradient estimation $\hat{g}_{k,i}$ with respect to the $i$-th element of $x$ is rewritten as

$$\begin{aligned}
\hat{g}_{k,i} &= \frac{y_k^+ - y_k^-}{2c_k\delta_{k,i}} \\
&= \frac{f(x_k + c_k\delta_k) - f(x_k - c_k\delta_k) + \varepsilon_k^+ - \varepsilon_k^-}{2c_k\delta_{k,i}} \\
&= \frac{x_k^{\mathrm{T}}H\delta_k + h^{\mathrm{T}}\delta_k}{\delta_{k,i}} + \frac{\varepsilon_k}{2c_k\delta_{k,i}},
\end{aligned}$$

where the symbol

$$\varepsilon_k \doteq \varepsilon_k^+ - \varepsilon_k^-$$

is introduced for simplicity. The first order optimality condition [23]

$$\nabla f(x^*) = Hx^* + h = 0$$

leads to

$$\hat{g}_{k,i} = \frac{1}{\delta_{k,i}} \delta_k^{\mathrm{T}} H (x_k - x^*) + \frac{1}{2c_k \delta_{k,i}} \varepsilon_k.$$

Introducing the error $e_k = x_k - x^*$ between the candidate of the solution and the optimal solution, the $k$-th gradient estimation $\hat{g}_k$ is given by

$$\begin{aligned}\hat{g}_k &= \begin{bmatrix} \frac{1}{\delta_{k,1}} \\ \frac{1}{\delta_{k,2}} \\ \vdots \\ \frac{1}{\delta_{k,n}} \end{bmatrix} \delta_k^{\mathrm{T}} H e_k + \begin{bmatrix} \frac{1}{\delta_{k,1}} \\ \frac{1}{\delta_{k,2}} \\ \vdots \\ \frac{1}{\delta_{k,n}} \end{bmatrix} \frac{1}{2c_k} \varepsilon_k \\ &= \delta_k \delta_k^{\mathrm{T}} H e_k + \frac{\delta_k}{2c_k} \varepsilon_k,\end{aligned}$$

where we notice that

$$\delta_{k,i} = \frac{1}{\delta_{k,i}}$$

due to the $i$-th element $\delta_{k,i} = 1$ or $-1$. Then, we have the difference equation

$$e_{k+1} = \left( I - \frac{1}{n+k} \delta_k \delta_k^{\mathrm{T}} H \right) e_k - \frac{\delta_k}{2c_k(n+k)} \varepsilon_k$$

with respect to the error $e_k$ by substituting the above gradient estimation to the update rule (2) and subtracting the optimal solution $x^*$ from the both sides. We can interpret this difference equation as a linear time-varying system [24] with the state variable $e_k$ and the disturbance $\varepsilon_k$, thus, its state equation solution is

$$e_k = \Phi(k,1) e_1 - \sum_{j=1}^{k-1} \Phi(k, j+1) \frac{\delta_j}{2c_j(n+j)} \varepsilon_j, \qquad (7)$$

where $e_1 \in \mathbb{R}^n$ is the initial state and $\Phi(k,j)$, $k, j \in \mathbb{N}$ is a discrete-time transition matrix which is defined by

$$\Phi(k,j) = \begin{cases} \left( I - \frac{1}{n+k-1} \delta_{k-1} \delta_{k-1}^{\mathrm{T}} H \right) & \\ \quad \cdot \left( I - \frac{1}{n+k-2} \delta_{k-2} \delta_{k-2}^{\mathrm{T}} H \right) & \\ \quad \cdots \left( I - \frac{1}{n+j} \delta_j \delta_j^{\mathrm{T}} H \right) & \text{if } k > j, \\ I & \text{if } k \le j. \end{cases}$$

For completion of the proof, we have to derive a condition with respect to $k$ which satisfies

$$\|\mathsf{E}[e_k]\|_2 \le \|\mathsf{E}[\Phi(k,1)]\|_2 \|e_1\|_2 \le \alpha \|e_1\|_2, \qquad (8)$$

$$\mathsf{P}^{\infty}(\|e_k - \mathsf{E}[e_k]\|_2 \le \alpha \|e_1\|_2 + \beta) \ge 1 - \gamma \qquad (9)$$

for given $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, and $\gamma \in (0,1)$, where $\|\cdot\|_2$ for a matrix denotes the matrix norm induced by the Euclidean norm and $\mathsf{E}$ denotes expectation. This is because

$$\begin{aligned}\|e_k\|_2 &= \|e_k - \mathsf{E}[e_k] + \mathsf{E}[e_k]\|_2 \\ &\le \|e_k - \mathsf{E}[e_k]\|_2 + \|\mathsf{E}[e_k]\|_2 \\ &\le \alpha \|e_1\|_2 + \beta\end{aligned}$$

if the inequalities (8) and (9) hold. We first consider a condition satisfying (8).

*Lemma 1:* The inequality

$$\|\mathsf{E}[\Phi(k,1)]\|_2 \le \sqrt{\frac{n+1}{n+k}}$$

holds when the number of iterations reaches $k$.

*Proof:* Since the perturbation vectors $\delta_\ell$, $\ell = 1,2,\ldots$ are mutually independent random variables, the expectation of transition matrix is product of the expectations of coefficient matrices $I - \delta_\ell \delta_\ell^{\mathrm{T}} H/(n+\ell)$ at each $\ell = 1,2,\ldots,k-1$:

$$\begin{aligned}&\mathsf{E}[\Phi(k,1)] \\ &= \mathsf{E}\left[ I - \frac{1}{n+k-1} \delta_{k-1} \delta_{k-1}^{\mathrm{T}} H \right] \cdot \mathsf{E}\left[ I - \frac{1}{n+k-2} \delta_{k-2} \delta_{k-2}^{\mathrm{T}} H \right] \\ &\quad \cdots \mathsf{E}\left[ I - \frac{1}{n+1} \delta_1 \delta_1^{\mathrm{T}} H \right].\end{aligned}$$

Furthermore, noticing that all elements of the random perturbation vector $\delta_\ell$ are also mutually independent, we have

$$\mathsf{E}[\delta_\ell \delta_\ell^{\mathrm{T}}] = I, \qquad \ell = 1,2,\ldots.$$

We therefore see that the norm of the expectation of the transition matrix $\Phi(k,1)$ is bounded by

$$\begin{aligned}\|\mathsf{E}[\Phi(k,1)]\|_2 &\le \prod_{\ell=1}^{k-1} \left\| I - \frac{1}{n+\ell} H \right\|_2 \\ &\le \prod_{\ell=1}^{k-1} \left( 1 - \frac{1}{2(n+\ell)} \right) \\ &\le \prod_{\ell=1}^{k-1} \exp\left( -\frac{1}{2(n+\ell)} \right) \\ &\le \exp\left( -\int_1^k \frac{1}{2(n+x)} \mathrm{d}x \right) = \sqrt{\frac{n+1}{n+k}},\end{aligned}$$

employing

$$\begin{aligned} I - \frac{1}{n+\ell} H &\preceq I - \frac{I}{2(n+\ell)}, \\ 1 - x &\le \exp(-x), \\ \sum_{\ell=1}^{k-1} \frac{1}{2(n+\ell)} &\le \int_1^k \frac{1}{2(n+x)} \mathrm{d}x, \end{aligned}$$

where the third inequality is derived from Riemann sum. ■

Let us investigate a condition satisfying (9).

*Lemma 2:* For given any constant $\gamma \in (0,1)$,

$$\mathsf{P}^{\infty}\left( \|e_k - \mathsf{E}[e_k]\|_2 \le \sqrt{\frac{n(n+1)\exp(1)}{\gamma(n+k)}}\|e_1\|_2 + \sqrt{\left(\frac{2}{n+k}\right)^{2/3}\frac{6n\sigma^2\exp(1)}{\gamma}}\right) \ge 1-\gamma$$

holds when the number of iterations reaches $k$.

*Proof:* The upper bound of the covariance matrix $V_k$ of the error $e_k$ is

$$\begin{aligned} V_k &= \mathsf{E}\left[(e_k - \mathsf{E}[e_k])(e_k - \mathsf{E}[e_k])^{\mathrm{T}}\right] \\ &= \mathsf{E}\left[e_k e_k^{\mathrm{T}}\right] - \mathsf{E}[e_k]\mathsf{E}[e_k^{\mathrm{T}}] \\ &\preceq \mathsf{E}\left[e_k e_k^{\mathrm{T}}\right], \end{aligned} \tag{10}$$

due to $\mathsf{E}[e_k]\mathsf{E}[e_k^{\mathrm{T}}] \succeq 0$. From the solution (7) of $e_k$, $V_k$ is bounded by

$$\begin{aligned} V_k &\preceq \mathsf{E}\left[\left(\Phi(k,1)e_1 - \sum_{j=1}^{k-1}\Phi(k,j+1)\delta_j\frac{1}{2c_j(n+j)}\varepsilon_j\right)\right. \\ &\qquad \left.\left(\Phi(k,1)e_1 - \sum_{j=1}^{k-1}\Phi(k,j+1)\delta_j\frac{1}{2c_j(n+j)}\varepsilon_j\right)^{\mathrm{T}}\right] \\ &= \mathsf{E}\left[\Phi(k,1)e_1e_1^{\mathrm{T}}\Phi^{\mathrm{T}}(k,1)\right] \\ &\quad + \mathsf{E}\left[\sum_{j=1}^{k-1}\frac{\varepsilon_j^2}{c_j^2(n+j)^2}\Phi(k,j+1)\delta_j\delta_j^{\mathrm{T}}\Phi^{\mathrm{T}}(k,j+1)\right] \\ &= \mathsf{E}\left[\Phi(k,1)e_1e_1^{\mathrm{T}}\Phi^{\mathrm{T}}(k,1)\right] \\ &\quad + \mathsf{E}\left[\sum_{j=1}^{k-1}\frac{\varepsilon_j^2}{c_j^2(n+j)^2}\Phi(k,j+1)\Phi^{\mathrm{T}}(k,j+1)\right]. \end{aligned} \tag{11}$$

*First term:* Let us consider the first term of the left hand side of the inequality (11). By using the maximum eigenvalue $\|e_1\|_2^2$ of a rank one matrix $e_1e_1^{\mathrm{T}}$, the first term is rewritten as

$$\mathsf{E}\left[\Phi(k,1)e_1e_1^{\mathrm{T}}\Phi^{\mathrm{T}}(k,1)\right] \preceq \|e_1\|_2^2\mathsf{E}\left[\Phi(k,1)\Phi^{\mathrm{T}}(k,1)\right].$$

Employing the definition of the transition matrix, for any $\ell = 1,2,\ldots,k-1$,

$$\begin{aligned} &\mathsf{E}\left[\Phi(k,\ell)\Phi^{\mathrm{T}}(k,\ell)\right] \\ &= \mathsf{E}\left[\Phi(k,\ell+1)\left(I - \frac{1}{n+\ell}\delta_\ell\delta_\ell H - \frac{1}{n+\ell}H\delta_\ell\delta_\ell^{\mathrm{T}}\right.\right. \\ &\qquad \left.\left. + \frac{1}{(n+\ell)^2}\delta_\ell\delta_\ell^{\mathrm{T}}H^2\delta_\ell\delta_\ell^{\mathrm{T}}\right)\Phi^{\mathrm{T}}(k,\ell+1)\right] \end{aligned}$$

holds. Noticing that

$$\mathsf{E}[\delta_\ell\delta_\ell^{\mathrm{T}}] = I, \qquad \mathsf{E}[\delta_\ell\delta_\ell^{\mathrm{T}}\delta_\ell\delta_\ell^{\mathrm{T}}] = nI, \qquad \frac{I}{2} \preceq H \preceq I,$$

we obtain a recursive inequality

$$\begin{aligned} &\mathsf{E}\left[\Phi(k,\ell)\Phi^{\mathrm{T}}(k,\ell)\right] \\ &\preceq \left(1 - \frac{1}{n+\ell} + \frac{n}{(n+\ell)^2}\right)\mathsf{E}\left[\Phi(k,\ell+1)\Phi^{\mathrm{T}}(k,\ell+1)\right] \\ &= \left(1 - \frac{\ell}{(n+\ell)^2}\right)\mathsf{E}\left[\Phi(k,\ell+1)\Phi^{\mathrm{T}}(k,\ell+1)\right]. \end{aligned} \tag{12}$$

A solution of this inequality leads to

$$\begin{aligned} \mathsf{E}\left[\Phi(k,1)e_1e_1^{\mathrm{T}}\Phi^{\mathrm{T}}(k,1)\right] &\preceq \|e_1\|_2^2\prod_{\ell=1}^{k-1}\left(1 - \frac{\ell}{(n+\ell)^2}\right)I \\ &\preceq \|e_1\|_2^2\exp\left(-\sum_{\ell=1}^{k-1}\frac{\ell}{(n+\ell)^2}\right)I \\ &\preceq \|e_1\|_2^2\frac{n+1}{n+k}\exp(1)I, \end{aligned} \tag{13}$$

that is, the above is an upper bound of the first term of the left-hand side of (11).

*Second term:* Now, we are going to consider the second term of the left-hand side of the inequality (11). From the recursive inequality (12), we similarly obtain

$$\mathsf{E}\left[\Phi(k,j+1)\Phi^{\mathrm{T}}(k,j+1)\right] \preceq \frac{n+j+1}{n+k}\exp(1)I.$$

The inequality implies

$$\begin{aligned} &\mathsf{E}\left[\frac{\sigma^2}{c_j^2(n+j)^2}\sum_{j=1}^{k-1}\Phi(k,j+1)\delta_j\delta_j^{\mathrm{T}}\Phi^{\mathrm{T}}(k,j+1)\right] \\ &\preceq \frac{\sigma^2\exp(1)}{n+k}\sum_{j=1}^{k-1}\left(\frac{1}{n+j}\right)^{5/3}(n+j+1)I \\ &\preceq \frac{2^{5/3}\sigma^2\exp(1)}{n+k}\sum_{j=1}^{k-1}\left(\frac{1}{n+j+1}\right)^{2/3}I \\ &\preceq \frac{2^{5/3}\sigma^2\exp(1)}{n+k}\int_{n+1}^{n+k}\left(\frac{1}{x}\right)^{2/3}\mathrm{d}xI \\ &\preceq \left(\frac{2}{n+k}\right)^{2/3}6\sigma^2\exp(1)I, \end{aligned} \tag{14}$$

that is, this provides an upper bound of the second term, where we utilize an inequality

$$\begin{aligned} \frac{1}{n+j} &\le \frac{2}{n+j+1}, \\ \sum_{j=1}^{k-1}\left(\frac{1}{n+j+1}\right)^{2/3} &\le \int_{n+1}^{n+k}\left(\frac{1}{x}\right)^{2/3}\mathrm{d}x. \end{aligned}$$

An upper bound of the covariance matrix $V_k$ is given by

$$V_k \preceq \|e_1\|_2^2\frac{n+1}{n+k}\exp(1)I + \left(\frac{2}{n+k}\right)^{2/3}6\sigma^2\exp(1)I$$

from (13) and (14).

Now, Markov inequality (see, Lemma 4) leads to a probabilistic inequality

$$\begin{aligned}&\mathsf{P}^\infty\left(\|e_k-\mathsf{E}[e_k]\|_2\le\sqrt{\frac{1}{\gamma}\mathrm{trace}V_k}\right)\\&=\mathsf{P}^\infty\left(\|e_k-\mathsf{E}[e_k]\|_2^2\le\frac{1}{\gamma}\mathrm{trace}V_k\right)\\&\le\gamma.\end{aligned}$$

We employ our evaluation of of $V_k$, an upper bound of $\|e_k-\mathsf{E}[e_k]\|_2$ is given by

$$\begin{aligned}&\sqrt{\frac{1}{\gamma}\mathrm{trace}V_k}\\&\le\sqrt{\frac{\exp(1)}{\gamma}\mathrm{trace}\left(\|e_1\|_2^2\frac{n+1}{n+k}I+\left(\frac{2}{n+k}\right)^{2/3}6\sigma^2I\right)}\\&\le\sqrt{\frac{n(n+1)\exp(1)}{\gamma(n+k)}}\|e_1\|_2+\sqrt{\left(\frac{2}{n+k}\right)^{2/3}\frac{6n\sigma^2\exp(1)}{\gamma}}.\end{aligned}$$

We therefore see the statement of Lemma 2. ■

Now, let us prove Theorem 1 from these lemmas. When the number of iterations reaches $\bar{k}$, the probability which satisfies

$$\begin{aligned}\|x_{\bar{k}}-x^*\|&=\|e_{\bar{k}}\|_2\\&\le\|\mathsf{E}[e_{\bar{k}}]\|_2+\|e_{\bar{k}}-\mathsf{E}[e_{\bar{k}}]\|_2\\&\le\left(1+\sqrt{\frac{n\exp(1)}{\gamma}}\right)\sqrt{\frac{n+1}{n+\bar{k}}}\|x_1-x^*\|_2\\&\quad+\sqrt{\left(\frac{2}{n+\bar{k}}\right)^{2/3}\frac{6n\sigma^2\exp(1)}{\gamma}}\end{aligned}\tag{15}$$

is greater than $1-\gamma$. We therefore conclude that

$$\|x_{\bar{k}}-x^*\|_2=\|e_{\bar{k}}\|_2\le\alpha\|x_1-x^*\|_2+\beta$$

holds with probability $1-\gamma$ from (4) and (5).

## IV. Some Remarks

In this section, we discuss some properties on our stopping rule. We first compare our stopping rule with the rate of convergence [2], [9]. It is well known that the norm $\|x_k-x^*\|_2$ of the estimation error of SPSA goes to zero proportional to $k^{-1/3}$ for large $k$ when we set $c_k=(n+k)^{-1/6}$. On the other hand, the equation (15) also states that the rate at which $\tilde{x}_k$ goes to zero is proportional to $k^{-1/3}$ when $k$ becomes large enough. This result is the same as the conventional result.

Secondly, let us compare our stopping rule with analysis of a noise-free SPSA [25], [26]. If we can obtain a measurement without any noise, the estimation error is bounded by

$$\|x_k-x^*\|_2\le\left(1+\sqrt{\frac{n\exp(1)}{\gamma}}\right)\sqrt{\frac{n+1}{n+k}}\|x_1-x^*\|_2$$

from (15). This inequality implies that the estimation error of noise-free SPSA goes to zero proportional to $k^{-1/2}$ for large $k$ when we set $c_k=(n+k)^{-1/6}$, then, this result corresponds to that in [25], [26]. Furthermore, a stopping rule of noise-free SPSA is the following.

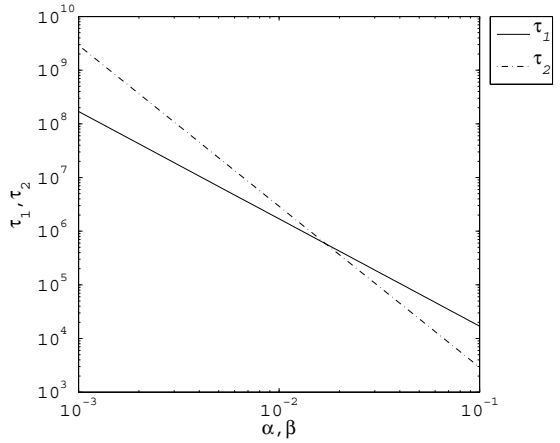


Fig. 1. The necessary number of iterations of the stochastic approximation with $\gamma=0.01$.

*Corollary 2:* For given constants $\alpha\in(0,\infty)$ and $\gamma\in(0,1)$, select $\bar{k}\in\mathbb{N}$ which satisfies

$$\bar{k}\ge\frac{n+1}{\alpha^2}\left(1+\sqrt{\frac{n\exp(1)}{\gamma}}\right)-n.$$

Then, the $\bar{k}$-th candidate $x_{\bar{k}}$ of the solution satisfies

$$\mathsf{P}^\infty\left(\|x_{\bar{k}}-x^*\|_2\le\alpha\|x_1-x^*\|_2\right)\ge1-\gamma$$

for any initial candidate $x_1\in\mathbb{R}^n$ of the solution.

From these discussions, we can see that our stopping rule for SPSA also provides detailed analysis on transient behavior.

## V. Numerical Examples

In this section, we examine the number $\bar{k}$ of iterations in SPSA. We first prepare some parameters of this algorithm. We set the dimension $n=4$ of $x$ and the variance as $\sigma=0.025$. Setting $\gamma=0.01$, we plot $\tau_1$, $\tau_2$ for each $\alpha$ and $\beta$ on Fig. 1. The solid line and the dashed one denote $\tau_1$ and $\tau_2$, respectively. We secondary show $\tau_1$ and $\tau_2$ at fixed $\alpha=\beta=0.01$. We plot $\tau_1$, $\tau_2$ for each $\gamma$ on Fig. 2. Other parameters $n$ and $\sigma$ is the same as the previous settings.

From these figures, $\tau_1$ does not become too large for each $\alpha$ or each $\gamma$. That is, we can specify small $\alpha$. Since the parameter $\alpha$ determines the expectation of the residual $x_{\bar{k}}-x^*$, this is a desirable property. On the other hand, $\tau_2$ becomes large for small $\beta$ or small $\gamma$. However, we notice that Remark 1 states the objective function value at the solution candidate goes to the optimal value and the difference between its value and the optimal value is proportional to $k^2$. We therefore see that we do not have to select too small $\beta$.

## VI. Concluding Remarks

We have developed a stopping rule for SPSA. We have provided a relationship between the number of iterations

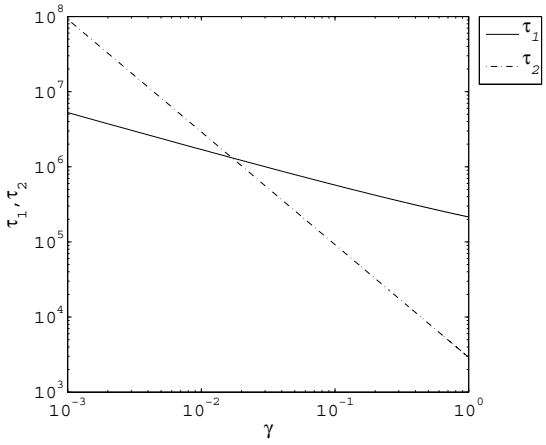


Fig. 2. The necessary number of iterations of the stochastic approximation with $\alpha = \beta = 0.01$.

and parameters which specify accuracy of the solution in a probabilistic sense. Note that we can specify the number of iterations which has been derived in this paper is polynomial of parameters of the problem. Furthermore, we confirmed the number of iterations is natural by comparing our stopping rule and the conventional results on the rate of asymptotic convergence analysis.

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## APPENDIX

*Lemma 3:* [13] For given constants $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, and $\gamma \in (0,1)$, select $\bar{k} \in \mathbb{N}$ satisfying

$$\bar{k} \geq \max\{\tau_1, \tau_2\},$$
$$\tau_1 \doteq \frac{n+1}{\alpha^2} - n,$$
$$\tau_2 \doteq \left(\frac{3n\sigma^2}{2\beta^2\gamma}\right)^{3/2} - n.$$

Then, the $\bar{k}$-th candidate of the solution satisfies

$$\mathsf{P}^\infty\left(\|x_{\bar{k}} - x^*\|_2 \leq \alpha\|x_1 - x^*\|_2 + \beta\right) \geq 1 - \gamma$$

for any initial candidate $x_1 \in \mathbb{R}^n$ of the solution.
In fact, when the number of iterations reaches $\bar{k}$,

$$\|x_{\bar{k}} - x^*\|_2 \leq \sqrt{\frac{n+1}{n+\bar{k}}}\|x_1 - x^*\|_2 + \sqrt{\frac{3n\sigma^2}{2\gamma(n+\bar{k})^{2/3}}} \qquad (16)$$

holds.

*Lemma 4 (Markov inequality [27]):* For any $\alpha > 0$,

$$\mathsf{P}\left(x \leq \frac{\mathsf{E}[x]}{\alpha}\right) \geq 1 - \alpha$$

holds, where $x \in [0,\infty)$ is a nonnegative random variable and there exists its expectation $\mathsf{E}[x] < \infty$.