

# Closed-loop Reference Models for Output-Feedback Adaptive Systems

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**Abstract**—Closed-loop reference models have recently been proposed for states accessible adaptive systems. They have been shown to have improved transient response over their open loop counter parts. The results in the states accessible case are extended to single input single output plants of arbitrary relative degree.

## I. INTRODUCTION

Recently a class of adaptive controllers with *Closed-loop Reference Models* (CRM) for states accessible control has been proposed [1]–[7]. The main feature of this class is the inclusion of a feedback gain in the reference model. Without the feedback gain the CRM reduces to the *Open-loop Reference Model* (ORM) which is used in classical adaptive control [8], [9]. The main advantage of the CRM-adaptive systems is their ability to shape and improve the transient response of the adaptive system. This was demonstrated in [3]–[5] through the use of  $\mathcal{L}_2$  norms of the model following error, the derivative of the adaptive parameter and the rate of control input, and improved further in in [3]–[5] with an analytical justification for the reduction in high frequency oscillations which are conspicuously absent in CRM systems. References [6], [7] also addressed a peaking phenomenon that occurs in CRM systems. In [3]–[7], it was shown that the extra design freedom in the adaptive system in the form of the feedback gain in the reference model allowed this improvement. Other recent works on states accessible CRM adaptive control can be found in [10], [11].

This paper addresses the next step in the design of adaptive systems, which is the case when only outputs are available for measurement rather than the entire state. It is shown that even with output feedback, the resulting CRM-based adaptive systems are first and foremost stable, and exhibit an improved transient response. As in the case when states are accessible, it is shown that this improvement is possible due to the suitable choice of the feedback gain in the reference model. Unlike the approach in [12], the classical model reference adaptive control structure is used here, with the focus on single-input single-output systems.

Similar to the states-accessible case, the CRM-based adaptive systems with output feedback presented here have the advantage of an improved transient response. This is made possible by the introduction of an extra degree of freedom in the reference model. The introduction of this degree of freedom reduces the burden on the adaptive controller by

allowing the reference model to meet the plant “half-way”. Also, this degree of freedom allows the trade-off between speed of adaptation and the size of the parametric uncertainty to be relaxed and enables reduced oscillations. In the specific context of output feedback, the advantage of the CRM over the ORM manifests in the form of judicious selections of the reference model, filters, and the feedback gain. These allow the underlying transfer function in the error model to not only be strictly positive real, but also to have poles and zeros that are arbitrarily fast. In addition, the CRM allows the analysis and design of a minimal representation of the adaptive system, and removes the restriction that ORM-based adaptive systems often possess, which is due to the location of the eigenvalues of the nonobservable states of the underlying adaptive system. The CRM-based design and overall analysis of stability and transient response are presented in this paper. Both cases when the relative degree is unity and arbitrary are analyzed.

This paper is organized as follows. Section II contains the notation. In Section III the control problem is defined. Section IV contains the analysis of the ORM with relative degree 1. Section V contains the analysis of the CRM with relative degree 1. Section VI analysis the arbitrary relative degree case, and Section VII closes with our conclusions.

## II. NOTATION

All norms unless otherwise stated are the Euclidean norm and induced Euclidean norm. Let  $\mathcal{PC}_{[0,\infty)}$  denote the set of all bounded piecewise continuous signal.

*Definition 1:* Let  $x, y \in \mathcal{PC}_{[0,\infty)}$ . The big O-notation,  $y(t) = O[x(t)]$  is equivalent to the existence of constants  $M_1, M_2 > 0$  and  $t_0 \in \mathbb{R}^+$  such that  $|y(t)| \leq M_1|x(t)| + M_2 \forall t \geq t_0$ .

*Definition 2:* Let  $x, y \in \mathcal{PC}_{[0,\infty)}$ . The small o-notation,  $y(t) = o[x(t)]$  is equivalent to the existence of constants  $\beta(t) \in \mathcal{PC}_{[0,\infty)}$  and  $t_0 \in \mathbb{R}^+$  such that  $|y(t)| = \beta(t)|x(t)| \forall t \geq t_0$  and  $\lim_{t \rightarrow \infty} \beta(t) = 0$ .

*Definition 3:* Let  $x, y \in \mathcal{PC}_{[0,\infty)}$ . If  $y(t) = O[x(t)]$  and  $x(t) = O[y(t)]$ . Then  $x$  and  $y$  are said to be equivalent and denoted as  $x(t) \sim y(t)$ .

*Definition 4:* Let  $x, y \in \mathcal{PC}_{[0,\infty)}$ .  $x$  and  $y$  are said to grow at the same rate if  $\sup_{t \leq \tau} |x(\tau)| \sim \sup_{t \leq \tau} |y(\tau)|$ .

*Definition 5:* The prime notation is an operator that removes the high frequency gain from a transfer function

$$\mathcal{W}(s) \triangleq k \frac{s^{m-1} + b_1 s^{m-2} + \dots + b_{m-1}}{s^m + a_1 s^{m-1} + \dots + a_m}$$

so that

$$\mathcal{W}'(s) \triangleq \frac{\mathcal{W}(s)}{k}$$

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### III. THE CONTROL PROBLEM

Consider the *Single Input Single Output* (SISO) system of equations

$$y(t) = W(s)u(t) \quad (1)$$

where  $u \in \mathbb{R}$  is the input,  $y \in \mathbb{R}$  is the measurable output, and  $s$  the differential operator. The transfer function of the plant is parameterized as

$$W(s) \triangleq k_p \frac{Z(s)}{P(s)} \triangleq k_p W'(s) \quad (2)$$

where  $k_p$  is a scalar, and  $Z(s)$  and  $P(s)$  are monic polynomials with  $\deg(Z(s)) < \deg(P(s))$ . The following assumptions will be made throughout.

*Assumption 1:*  $W(s)$  is minimum phase.

*Assumption 2:* The sign of  $k_p$  is known.

*Assumption 3:* The relative degree of  $W(s)$  is known.

The goal is to design a control input  $u$  so that the output  $y$  in (1) tracks the output  $y_m$  of the reference system

$$y_m(t) = W_m(s)r(t) \triangleq k_m \frac{Z_m(s)}{P_m(s)}r(t) \quad (3)$$

where  $k_m$  is a scalar and  $Z_m(s)$  and  $P_m(s)$  are monic polynomials with  $W_m(s)$  relative degree  $n^*$ . In Sections IV through VI, we present the control designs for  $n^* = 1$  and for an arbitrary  $n^*$ .

### IV. $n^* = 1$ AND ORM

*Assumption 4:*  $W'_m(s)$  is *Strictly Positive Real* (SPR).

The structure of the adaptive controller is now presented:

$$\dot{\omega}_1(t) = \Lambda\omega_1 + b_\lambda u(t) \quad (4)$$

$$\dot{\omega}_2(t) = \Lambda\omega_2 + b_\lambda y(t) \quad (5)$$

$$\omega(t) \triangleq [r(t), \omega_1^T(t), y(t), \omega_2^T(t)]^T \quad (6)$$

$$\theta(t) \triangleq [k(t), \theta_1^T(t), \theta_0(t), \theta_2^T(t)]^T \quad (7)$$

$$u = \theta^T(t)\omega \quad (8)$$

where  $\Lambda \in \mathbb{R}^{(n-1) \times (n-1)}$  is Hurwitz,  $b_\lambda \in \mathbb{R}^{n-1}$ ,  $\hat{k} \in \mathbb{R}$ ,  $\omega_1, \omega_2 \in \mathbb{R}^{n-1}$ , and  $\theta \in \mathbb{R}^{2n}$  is adaptive gain vector with  $k(t) \in \mathbb{R}$ ,  $\theta_1(t) \in \mathbb{R}^{n-1}$ ,  $\theta_2(t) \in \mathbb{R}^{n-1}$  and  $\theta_0(t) \in \mathbb{R}$ . The update law for the adaptive parameter is then defined as

$$\dot{\theta}(t) = -\gamma \text{sign}(k_p) e_y \omega, \quad (9)$$

where  $e_y = y - y_m$ .

Before stability is proved, a discussion on parameter matching is needed. Let  $\theta_c \triangleq [k_c, \theta_{1c}^T, \theta_{0c}, \theta_{2c}^T]^T$  be a constant vector. When  $\theta(t) = \theta_c$  the forward loop and feedback loop take the form

$$\frac{\lambda(s)}{\lambda(s) - C(\theta_c; s)} \quad \text{and} \quad \frac{D(\theta_c; s)}{\lambda(s)}.$$

For simplicity we choose  $\lambda(s) = Z_m(s)$ , but note that this is not necessary and the stability of the adaptive system will still hold. The closed loop system is now of the form

$$y(t) = W_{cl}(\theta_c; s)r(t)$$

with

$$W_{cl}(\theta_c; s) \triangleq \frac{k_c k_p Z(s) Z_m(s)}{(Z_m(s) - C(\theta_c; s))P(s) - k_p Z(s)D(\theta_c; s)}.$$

From the Bezout Identity, a  $\theta^{*T} \triangleq [k^*, \theta_1^{*T}, \theta_0^*, \theta_2^{*T}]^T$  exists such that  $W_{cl}(\theta^*; s) = W_m(s)$ .

Therefore,

$$y(t) = k_p W'_m(s) (\phi^T(t)\omega(t) + k^* r(t)) \quad (10)$$

and

$$e_y(t) = k_p W'_m(s) \phi(t)\omega(t), \quad (11)$$

where  $\phi(t) = \theta(t) - \theta^*(t)$  and  $k^* = k_m/k_p$ .

#### A. Stability for $n^* = 1$

The plant in (2) can be represented by the unknown quadruple,  $(A_p, b_p, c_p, k_p)$

$$\dot{x} = A_p x + b_p u; \quad y = k_p c_p^T x \quad (12)$$

where  $k_p c_p^T (sI - A_p) b_p = W(s)$ .

In general one does not need to keep the high frequency gain as a separate variable when writing the transfer function dynamics in state space form. In the context of adaptive control however, the sign of  $k_p$  is important in proving stability and is therefore always singled out from the rest of the dynamics. Using (12), the dynamics in (10) can be represented as

$$\dot{x} = A_{mn} x + b_{mn} (\phi^T(t)\omega + k^* r); \quad y = k_p c_{mn}^T x \quad (13)$$

where

$$A_{mn} = \begin{bmatrix} A_p + b_p \theta_0^* k_p c_p^T & b_p \theta_1^{*T} & b_p \theta_2^{*T} \\ b_\lambda \theta_0^* k_p c_p^T & \Lambda + b_\lambda \theta_1^{*T} & b_\lambda \theta_2^{*T} \\ b_\lambda k_p c_p^T & 0 & \Lambda \end{bmatrix}$$

$$b_{mn} = \begin{bmatrix} b_p \\ b_\lambda \\ 0 \end{bmatrix}, \quad c_{mn} = \begin{bmatrix} c_p \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x \triangleq \begin{bmatrix} x_p \\ \omega_1 \\ \omega_2 \end{bmatrix}$$

with the reference model having an equivalent non-minimal representation

$$\dot{x}_{mn} = A_{mn} x_{mn} + b_{mn} k^* r; \quad y_m = k_p c_{mn}^T x_{mn}$$

with the property that

$$k_p c_{mn}^T (sI - A_{mn}) b_{mn} = k_p W'_m(s).$$

The non-minimal error vector is defined as  $e_{mn} = x - x_{mn}$  and satisfies the following dynamics

$$\dot{e}_{mn} = A_{mn} e_{mn} + b_{mn} \phi^T \omega; \quad e_y = k_p c_{mn}^T e_{mn}. \quad (14)$$

*Theorem 1:* Following Assumptions 1-4, the plant in (1) with the reference model in (3), controller in (8) and the update law in (9) are globally stable with the model following error asymptotically converging to zero.

*Proof:* See [8, §5.3]. ■

## V. $n^* = 1$ AND CRM

Unlike the ORM, in this case, the reference model is chosen as

$$\dot{x}_m = A_m x_m + b_m k_m r + \ell(y - y_m), \quad y_m = c_m^T x_m \quad (15)$$

where  $(A_m, b_m, c_m^T)$  is an  $m$  dimensional system in observer canonical form with  $c_m^T = [0 \dots 0 \ 1]$  and satisfying

$$c_m^T (sI - A_m) b_m k_m = W_m(s).$$

The feedback gain  $\ell$  makes the reference model somewhat similar to the Luenberger observer.  $y_m(t)$  is now related to the reference command  $r(t)$  and model following error  $e_y(t)$  as

$$y_m(t) = W_m(s)r(t) + W_\ell(s)(y(t) - y_m(t)) \quad (16)$$

where

$$W_\ell(s) \triangleq k_\ell \frac{Z_\ell(s)}{P_m(s)}, \quad (17)$$

and  $k_\ell \in \mathbb{R}$  along with the  $m - 1$  order monic polynomial  $Z_\ell(s)$  are a function of  $\ell$  and free to choose. Subtracting (16) from (10) results in the following differential relation

$$e_y = k_p W_e'(s) \phi^T \omega \quad (18)$$

where

$$W_e'(s) \triangleq \frac{Z_m(s)}{P_m(s) - k_\ell Z_\ell(s)}. \quad (19)$$

*Lemma 2:* An  $\ell$  can be chosen such that  $W_e'(s)$  is SPR for any  $n^* = 1$  and minimum phase transfer function  $W_m'(s)$ .

*Proof:* The product  $k_\ell Z_\ell(s)$  a polynomial of order  $n - 1$  with  $n - 1$  degrees of freedom through  $\ell$ .  $P_m(s)$  is a monic polynomial of degree  $n$ . Therefore,  $P_m(s) - k_\ell Z_\ell(s)$  is a monic polynomial of order  $n$  with  $n - 1$  degrees of freedom determined by  $\ell$ . Thus for any  $Z_m(s)$  the roots of  $W_e'(s)$  can be placed freely in the closed left-half plane such that  $W_e'(s)$  is SPR. ■

Let

$$A_e = A_{mn} + G \ell k_p c_{mn}^T \quad (20)$$

where  $G$  transforms  $x_m$  to the controllable subspace in  $x_{mn}$ , which always exist [13]. The non-minimal error dynamics therefore take the form

$$\dot{e}_{mn}(t) = A_e e_{mn}(t) + b_{mn} \phi(t) \omega(t). \quad (21)$$

*Remark 1:* It is worth noting that in the construction of the minimal and non-minimal systems the location of the gains  $k_p$  and  $k_m$  switch from being located at the input to the output. The non-minimal systems is never created and thus need not be realized. For the case of the minimal reference model in (15) it is critical however that  $k_m$  appears at the input of the system. This is done so that given the canonical form of  $c_m$  the  $\ell$  in (15) completely determines the zeros and high frequency gain of  $W_\ell(s)$  in (17).

*Theorem 3:* Following Assumptions 1-3 and  $\ell$  chosen as in Lemma 2, the plant in (1) with the reference model in (15), controller in (8) and the update law in (9) are globally stable

with the model following error asymptotically converging to zero.

*Proof:* Given that  $W_e'(s)$  is SPR, there exists a  $P_e = P_e^T > 0$  such that

$$A_e^T P_e + P_e A_e = -Q_e \text{ and } P_e b_{mn} = c_{mn}. \quad (22)$$

where  $Q_e = Q_e^T > 0$ . Thus  $V = e_{mn}^T P_e e_{mn} + \frac{\phi^T \phi}{\gamma |k_p|}$  is a Lyapunov function with derivative  $\dot{V} = -e_{mn}^T Q_e e_{mn}$ . Barbalat Lemma ensures the asymptotic convergence of  $e_{mn}$  to zero. ■

### A. Performance

Now that we have proved stability we can return to a minimal representation of the error dynamics in (18) which is

$$\dot{e}_m = A_\ell e_m + b_m k_p \phi^T \omega, \quad e_y = c_m^T e_m; \quad (23)$$

where the all the eigen-values of  $A_\ell$  are the roots to  $P_m(s) - k_\ell Z_\ell(s)$ , as can be seen from (19). Recall the Anderson version of KY Lemma;

$$A_\ell^T P + P A_\ell = -g g^T - 2\mu P; \quad P b_m = c_m \quad (24)$$

where

$$\mu \triangleq \min_i |\lambda_i(A_\ell)|, \quad i = 1 \text{ to } m. \quad (25)$$

The following performance function

$$V_p = e_m^T P e_m + \frac{\phi^T \phi}{\gamma |k_p|} \quad (26)$$

has a time derivative

$$\dot{V}_p \leq -2\mu e_m^T P e_m. \quad (27)$$

From (27) it directly follows that

$$\|e_y(t)\|_{L_2}^2 \leq \frac{1}{2\mu} \left( \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|e(0)\|^2 + \frac{1}{\gamma |k_p|} \frac{\|\phi(0)\|^2}{\lambda_{\min}(P)} \right). \quad (28)$$

*Example 1:* The transfer function  $W_e'(s)$  must be SPR, therefore, the poles of  $W_e'(s)$  are limited by the location of its zeros. The order of  $A_m$  however is free to choose so long as  $m \geq 1$ , thus we can choose  $m = 1$ . Therefore making

$$W_m(s) = k_m \frac{1}{s + a_m}$$

where  $b_m = k_m$  and  $A_m = -a_m$ . The closed loop reference model transfer function therefore is

$$W_e(s) = k_m \frac{1}{s + a_m + l} \quad (29)$$

where  $\ell = -l$ ,  $l > 0$ . From (29), it is clear that there are no zeros limiting the location of the closed loop pole.

Further more, the Anderson Lemma reduces to the trivial solution of  $P = 1$ ,  $g = 0$ , and  $\mu = a_m + l$ . Since there are no zeros to worry about,  $W_e'(s)$  is SPR for all  $l$ . Therefore,  $\mu$  can be chosen arbitrarily. The bound in (28) for this example simplifies to

$$\|e_y(t)\|_{L_2} \leq \frac{1}{2(a_m + l)} \left( \|e(0)\|^2 + \frac{\|\phi(0)\|^2}{\gamma |k_p|} \right). \quad (30)$$

*Remark 2:* The use of CRMs has two advantages compared to the use of ORMs. The first is that the reference model need not be SPR a priori, but only needs to be of appropriate relative degree. There are several methods of dealing with non-SPR reference models for  $n^* = 1$ , but these methods require the use of pre-filters [14], or augmented error approaches (see [8], and Section VI).

The second advantage is illustrated in Example 1. Using this approach, a reference model can be chosen such that it has no zeros. When this is done and a CRM is used, the location of the slowest pole of the error model dynamics is free to choose. When using ORMs, the location of the slowest eigenvalue of the closed-loop error model is not free to choose, as speeding up the reference model eigenvalues without the use of CRMs will require the use of high-gain feedback which is equivalent to  $\|\theta^*\|$  being large if the open-loop plant has slow eigenvalues.

In the next section, we will show that the above advantages help in realizing a bound on the derivatives of the adaptive parameters which leads to a reduction in the oscillations that adaptive systems often exhibit.

## VI. ARBITRARY $n^*$ AND CRM

For higher relative degrees it is common to use an augmented error approach, where by the original model following error  $e_y$  is not used to adjust the adaptive parameter, but an augmented error signal which does satisfy the SPR conditions needed for stability. The augmented error method used in this result is Error Model 2 as presented in [8, §5.4], with some changes to the notation.

For ease of exposition and clarity in presentation we present the  $k_p$  known. For the case when  $k_p$  unknown we refer the reader to [15].

### A. Stability for known high frequency gain

We begin by replacing Assumption 2 with:

*Assumption 2':*  $k_p$  is known.

Without loss of generality we choose  $k_m = k_p = 1$  and the control input for the generic relative degree case reduces to

$$u(t) = r(t) + \bar{\theta}^T(t)\bar{\omega}(t) \quad (31)$$

where  $\bar{(\cdot)}$  denotes the vectors,

$$\bar{\omega}(t) \triangleq [\omega_1^T(t), y(t), \omega_2^T(t)]^T \quad (32)$$

$$\bar{\theta}(t) \triangleq [\theta_1^T(t), \theta_0(t), \theta_2^T(t)]^T. \quad (33)$$

A feedforward time varying adaptive gain  $k(t)$  is no longer needed and thus  $r(t)$  has been removed from the regressor vector do to the fact that  $k_p = k_m = 1$ . The model following error then, satisfies the following differential relation

$$e_y = W_e'(s)\bar{\phi}^T\bar{\omega} \quad (34)$$

where the reader is reminded that the prime notation removes the high frequency gain from transfer functions, and since  $k_m = k_p = 1$ ,  $W_e'(s) = W_e(s)$ . A stable minimally realized filter  $F(s)$  with no zeros is used to generate the filtered regressor

$$\bar{\zeta} = F(s)I\bar{\omega} \quad (35)$$

where  $I$  is the  $2n-1$  by  $2n-1$  identity matrix,  $F(s)$  designed with unity high frequency gain, and  $F(s)$  and  $\ell$  chosen so that

$$W_f'(s) \triangleq W_e'(s)F^{-1}(s) \quad (36)$$

is SPR.

*Lemma 4:* For any stable  $F(s)$  an  $\ell$  can be chosen such that  $W_f'(s)$  is SPR.

*Proof:* The proof follows the same arguments as in Lemma 2. ■

The tuning law for the arbitrary relative degree case uses an augmented error  $e_a$ , which is generated from the model following error  $e_y$  and an auxiliary error  $e_\chi$ . Using the CRM in (15), the augmented and auxiliary error are defined as:

$$e_a \triangleq e_y + W_f'(s)(e_\chi - e_a\bar{\zeta}^T\bar{\zeta}) \quad (37)$$

$$e_\chi \triangleq \bar{\theta}^T\bar{\zeta} - F(s)\bar{\theta}^T\bar{\omega}. \quad (38)$$

A stable tuning law for the system is then defined as

$$\dot{\bar{\theta}} = -\gamma e_a\bar{\zeta}. \quad (39)$$

*Theorem 5:* Following Assumptions 1, 2' and 3, with  $\ell$  chosen such that  $W_f'(s)$  is SPR, the plant in (1) with the reference model in (15), controller in (31) and update law in (39) are globally stable with the model following error  $e_y$  asymptotically converging to zero.

*Proof:* The proof proceeds in 4 steps. First it is shown that  $\bar{\theta}(t)$  and  $e_a$  are bounded and that  $e_a, \dot{\bar{\theta}} \in \mathcal{L}_2$ . Second, treating  $\bar{\theta}(t)$  as a bounded time-varying signal, then all signals in the adaptive system can grow at most exponentially. Third, if it is assumed that the signals grow in an unbounded fashion, then it can be shown that  $y, \omega_1, \omega_2, \bar{\omega}, \bar{\zeta}$  and  $u$  grow at the same rate. Finally, from the fact that  $\dot{\bar{\theta}} \in \mathcal{L}_2$  it is shown that  $\omega_2$  and  $\bar{\omega}$  do not grow at the same rate. This results in a contradiction and therefore, all signals are bounded and furthermore,  $e_y(t)$  asymptotically converges to zero. Steps 1 and 4 are detailed below. Steps 1-3 follow directly from [8, §5.5] with little changes. Step 4 does involve a modification to the analysis which is addressed in detail next.

*Step 1:* Expanding the error dynamics in (37) and canceling like terms of  $W_e'(s)\bar{\theta}^{*T}\bar{\omega}$  we have

$$e_a = -W_e'(s)\bar{\theta}^{*T}\bar{\omega} + W_f'(s)(\bar{\theta}^T\bar{\zeta} - e_a\bar{\zeta}^T\bar{\zeta}).$$

Adding and subtracting  $W_f'(s)\bar{\theta}^{*T}\bar{\zeta}$  the equation becomes

$$e_a = W_f'(s)(\bar{\phi}^T\bar{\zeta} - e_a\bar{\zeta}^T\bar{\zeta}) + \delta(t) \quad (40)$$

where  $\delta(t)$  is an exponentially decaying term do to initial conditions and defined as

$$\delta(t) = W_f'(s)(\bar{\theta}^{*T}\bar{\zeta}(t) - F(s)\bar{\theta}^{*T}\bar{\omega}(t)). \quad (41)$$

Breaking apart  $\bar{\zeta}$  from its definition in (35) and noting that  $\bar{\theta}^*$  now commutes with  $F(s)$  we have that

$$\delta(t) = W_f'(s)(\bar{\theta}^{*T}(F(s) - F(s))I\bar{\omega}). \quad (42)$$

Therefore, if the filter  $F(s)$  is chosen to have the same initial conditions when constructing  $\bar{\zeta}$  and  $e_\chi$  then,  $\delta = 0$

for all time. For this reason we ignore the affect of choosing different filter initial conditions. The interested reader can see how one can prove stability in augmented error approaches where  $\delta(0) \neq 0$  [8, pg. 213], with the addition of an extra term in the Lyapunov function.

A non-minimal representation of  $e_a$  is given as

$$\dot{e}_{an} = A_e e_{an} + b_{an} (\bar{\phi}^T \bar{\zeta} - e_a \bar{\zeta}^T \bar{\zeta}), \quad e_a = c_{an}^T e_{an} \quad (43)$$

where

$$c_{an}^T (sI - A_e)^{-1} b_{an} \triangleq W_f'(s). \quad (44)$$

Given that  $W_f(s)$  is SPR, there exists a  $P_a = P_a^T > 0$  such that

$$A_e^T P_a + P_a A_e = -Q_a \text{ and } P_a b_{an} = c_{an}. \quad (45)$$

where  $Q_a = Q_a^T > 0$ .

Consider the Lyapunov candidate  $V = e_{an}^T P_a e_{an} + \frac{\bar{\phi}^T \bar{\phi}}{\gamma}$ . Differentiating along the system dynamics in (43) and substitution of the tuning law from (39) results in  $\dot{V} \leq -e_{an}^T Q_a e_{an} - 2e_a^2 \bar{\zeta}^T \bar{\zeta}$ . Therefore,  $e_{an}, \bar{\theta} \in \mathcal{L}_\infty$  and  $e_{an}, \dot{\bar{\theta}} \in \mathcal{L}_2$

*Step 2:* The plant dynamics can be expressed as

$$\dot{x} = A_{mn} x + b_{mn} (\bar{\phi}^T(t) \omega + r); \quad y = c_{mn}^T x \quad (46)$$

where with an appropriate choice of a  $C$  can be expressed as

$$\dot{x} = (A_{mn} + b_{mn} \bar{\phi}^T(t) C) x + b_{mn} r \quad (47)$$

From Step 1 it is known that  $\bar{\phi}$  is bounded, and therefore  $x$  grows at most exponentially. Futhermore, for  $r$  piecewise continuous,  $x$  and  $\bar{\zeta}$  are both piecewise continuous as well.

*Step 3:* If it is assumed that all signals grow in an unbounded fashion then it can be shown that

$$\begin{aligned} \sup_{\tau \leq t} |y(\tau)| &\sim \sup_{\tau \leq t} \|\omega_1(\tau)\| \sim \sup_{\tau \leq t} \|\omega_2(\tau)\| \dots \\ &\sim \sup_{\tau \leq t} \|\bar{\omega}\| \sim \sup_{\tau \leq t} \|\bar{\zeta}\| \sim \sup_{\tau \leq t} |u(\tau)| \end{aligned} \quad (48)$$

[8, §5.5]

*Step 4:* Rewriting (38) in terms of  $\bar{\omega}$  we have that

$$e_\chi \triangleq \bar{\theta}^T F(s) I \bar{\omega} - F(s) \bar{\theta}^T \bar{\omega} \quad (49)$$

and given that  $\dot{\bar{\theta}} \in \mathcal{L}_2$  and  $F(s)$  is stable the following holds

$$e_\chi(t) = o \left[ \sup_{\tau \leq t} \|\bar{\omega}(\tau)\| \right]. \quad (50)$$

The above bound follows from the *Swapping Lemma* [8, Lemma 2.11]. From (39) and the fact that  $\dot{\bar{\theta}} \in \mathcal{L}_2$  we have that  $e_a \bar{\zeta} \in \mathcal{L}_2$ . Given that  $W_f'(s)$  is asymptotically stable, [8, Lemma 2.9] can be applied and it follows that

$$W_f'(s) ((e_a \bar{\zeta})^T \bar{\zeta}) = o \left[ \sup_{\tau \leq t} \|\bar{\zeta}(\tau)\| \right] \quad (51)$$

The plant output can be written in terms of the reference model and model following error as

$$\begin{aligned} y(t) &= y_m(t) + e_y(t) \\ &= W_m'(s) r(t) + (1 + W_\ell'(s)) e_y(t). \end{aligned}$$

Using (37),  $e_y(t) = e_a - W_f'(s) (e_\chi - e_a \bar{\zeta}^T \bar{\zeta})$  and the above equation expands as

$$\begin{aligned} y(t) &= W_m'(s) r(t) + (1 + W_\ell'(s)) e_a \\ &\quad - (1 + W_\ell'(s)) W_f'(s) (e_\chi - e_a \bar{\zeta}^T \bar{\zeta}). \end{aligned}$$

Using (50) (51) and noting that  $1 + W_\ell'(s)$  is asymptotically stable [8, Lemma 2.9] can be applied again and

$$\begin{aligned} y(t) &= W_m'(s) r(t) + (1 + W_\ell'(s)) e_a \\ &\quad + o \left[ \sup_{\tau \leq t} \|\bar{\zeta}(\tau)\| \right] + o \left[ \sup_{\tau \leq t} \|\bar{\omega}(\tau)\| \right]. \end{aligned}$$

Given that  $r$  and  $e_a$  are piecewise continuous and bounded we finally have that

$$y(t) = o \left[ \sup_{\tau \leq t} \|\bar{\omega}(\tau)\| \right]. \quad (52)$$

This contradicts (48) and therefore all signals are bounded. Furthermore, from (43) it now follows that  $\dot{e}_{an}$  is bounded and given that  $e_{an} \in \mathcal{L}_2$ , from Step 1, it follows that  $e_{an}$  asymptotically converges to zero and therefore  $\lim_{t \rightarrow \infty} e_a(t) = 0$ . From (50) it follows that  $e_\chi$  asymptotically converges to zero. Therefore,  $\lim_{t \rightarrow \infty} e_y(t) = 0$ . The above analysis differs from the analysis for the ORM output feedback adaptive control do to the fact that one can not a priori assume that  $y_m(t)$  is bounded, do to the feedback of  $e_y$  into the reference model. ■

## B. Performance when $k_p$ known

Just as in the  $n^* = 1$  case, with stability proved a Lyapunov performance function can be studied that uses a minimal representation of the dynamics. That being said, consider the minimal representation of the dynamics in (40)

$$\dot{e}_{am} = A_\ell e_{am} + b_{am} (\bar{\phi}^T \bar{\zeta} - e_a \bar{\zeta}^T \bar{\zeta}), \quad e_y = c_{am}^T e_{am} \quad (53)$$

in observer canonical form so that  $c_{am}^T = [0 \dots 0 \ 1]$  and

$$c_{am}^T (sI - A_\ell)^{-1} b_{am} \triangleq W_f'(s)$$

Recall the Anderson version of KY Lemma;

$$A_\ell^T P_p + P_p A_\ell = -g g^T - 2\mu P_p; \quad P_p b_{am} = c_{am} \quad (54)$$

where  $\mu$  is defined in (25). The following performance function  $V_p = e_{am}^T P_p e_{am} + \frac{\bar{\phi}^T \bar{\phi}}{\gamma}$  has a time derivative  $\dot{V}_p \leq -2\mu e_{am}^T P_p e_{am} - 2e_a^2 \bar{\zeta}^T \bar{\zeta}$ . From (VI-B) it directly follows that

$$\|e_a(t)\|_{L_2}^2 \leq \frac{1}{2\mu} \left( \frac{\lambda_{\max}(P_p)}{\lambda_{\min}(P_p)} \|e(0)\|^2 + \frac{1}{\gamma} \frac{\|\bar{\phi}(0)\|^2}{\lambda_{\min}(P_p)} \right) \quad (55)$$

and

$$\|\dot{\bar{\theta}}(t)\|_{L_2}^2 \leq \frac{1}{2} (\gamma^2 \lambda_{\max}(P_p) \|e(0)\|^2 + \gamma \|\bar{\phi}(0)\|^2). \quad (56)$$

Ultimately we would like to compute the  $\mathcal{L}_2$  norm of  $e_\chi$  and  $e_y$ . Given that these norms will depend explicitly on the specific values of the filter and reference model, we perform that analysis in the following example.

*Example 2:* In this example we consider a relative degree 2 plant. The reference model is chosen as

$$W_m(s) = \frac{1}{s^2 + b_1s + b_2} \quad (57)$$

and the filter is chosen as

$$F(s) = \frac{1}{s + f_1}. \quad (58)$$

The reference model gain is expanded as  $\ell = [-l_1 \quad -l_2]^T$ . Then

$$W_e(s) = \frac{1}{s^2 + (b_1 + l_1)s + (b_2 + l_2)} \quad (59)$$

and

$$W_f(s) = \frac{s + f_1}{s^2 + (b_1 + l_1)s + (b_2 + l_2)}. \quad (60)$$

Since,  $k_p = k_m = 1$ , then  $W_m(s) = W'_m(s)$ ,  $W_e(s) = W'_e(s)$  and  $W_f(s) = W'_f(s)$ . For stability to hold  $W'_f(s)$  must be SPR and from (60) it is clear that the SPR condition can be satisfied by choosing  $\ell$  and  $f_1$  appropriately. More importantly though, we see that the slowest eigenvalue of  $W_f(s)$  can be arbitrarily placed and thus the  $\mu$  in (25) can be arbitrarily increased.

$$\|e_\chi(t)\|_{\mathcal{L}_2}^2 \leq 3 \left( \frac{e_\chi^2(0)}{2f_1} + \left( \frac{e_\chi^2(0)}{4f_1^2} + \frac{\|\bar{\omega}(t)\|_\infty^2}{f_1^3} \right) \|\dot{\bar{\theta}}(t)\|_{\mathcal{L}_2}^2 \right) \quad (61)$$

A detailed proof of this expression is given in [15, Appendix A]. Furthermore, we have the following bound for the model following error

$$\|e_y(t)\|_{\mathcal{L}_2}^2 \leq 2\|e_a(t)\|_{\mathcal{L}_2}^2 + 2\|e_\zeta(t)\|_{\mathcal{L}_2}^2 \quad (62)$$

where  $e_\zeta(t) \triangleq W_f(s)e_\chi(t)$  can be bounded as

$$\|e_\zeta\|_{\mathcal{L}_2}^2 \leq 3m^2 \left( \frac{e_\zeta^2(0)}{2\mu} + \left( \frac{e_\chi(0)^2}{4\mu f_1} + \frac{\|\bar{\omega}(t)\|_\infty^2}{\mu f_1^2} \right) \|\dot{\bar{\theta}}(t)\|_{\mathcal{L}_2}^2 \right). \quad (63)$$

The bound in (63) is given in [15, Appendix B].

*Remark 3:* Now we compare the norms in (61) and (63) for an ORM and CRM system and note that increasing both  $f_1$  and  $\mu$  decreases the two norms. For the ORM system  $\ell = 0$ , therefore  $\mu$  is solely a function of  $b_1$  and  $b_2$  in (60). The coefficients  $b_1$  and  $b_2$  can not be arbitrarily changed without affecting the matching parameter vector  $\bar{\theta}^*$ . In the presence of persistence of excitation,  $\bar{\theta}(t) \rightarrow \bar{\theta}^*$  and large  $\bar{\theta}^*$  will directly imply a large control input. Furthermore, one can not arbitrarily change the reference model poles, as the reference model is a target behavior for the plant, in which case the control engineer may not want to track a reference system with arbitrarily fast poles. Therefore, given that  $b_1$  and  $b_2$  are not completely free to choose this also limits the value of  $f_1$  as  $W_f(s)$  must always be SPR. In the CRM case  $b_1$  and  $b_2$  can be held fixed and  $l_1$ ,  $l_2$  and  $f_1$  can be adjusted so that the poles of  $W_f(s)$  are arbitrarily fast and  $W_f(s)$  is still SPR. Therefore, the added degree of freedom through  $\ell$  in the CRM adaptive systems allows more flexibility in decreasing the  $\mathcal{L}_2$  norm of  $e_y$ .

*Remark 4:* In the above, we have derived bounds on the  $\mathcal{L}_2$  norm of the tracking error. That the same error has finite  $\mathcal{L}_\infty$  bounds is easily shown using Lyapunov function arguments and the fact that projection algorithms ensure exponential convergence of the error to a compact set, similar to the analysis in [3]–[6].

## VII. CONCLUSION

This work shows that with the introduction of CRMs the adaptive system can have improved transient performance in terms of reduction of the  $\mathcal{L}_2$  norm of the model following error and the derivative of adaptive perimeter. The analysis techniques used followed from the states accessible CRM adaptive systems in [3]–[5]. Not reported in this work are techniques used to analytically support the reduction in oscillations when CRMs are used, as performed in [6], [7]. Extending the results from [6], [7] to the output feedback case are the subject of ongoing investigation.

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