

# Stability of Discrete-Time Systems with Stochastically Delayed Feedback

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**Abstract**—This paper investigates the stability of linear systems with stochastic delay in discrete time. Stability of the mean and second moment of the non-deterministic system is determined by a set of deterministic discrete-time equations with distributed delay. A theorem is provided that guarantees convergence of the state with convergence of the second moment, assuming that delays are identically independently distributed. The theorem is applied to a scalar equation where the stability of the equilibrium is determined.

## I. INTRODUCTION

Noise and delays are often sources of concern for control engineers and, most recently, a concern in efforts to progress the field of synthetic biology. In genetic regulatory networks, delays arise in the transcription and translation processes [11]. Often the production of a protein induced by its activating transcription factor is modeled as an instantaneous process. In fact, there is a delay in the process with some stochastic variation.

Much investigation has been done on linear systems with a stochastic state matrix [10] but little has been done on analysis of systems with stochastically varying delays and even less analyzing the effects of stochastic delay variations. The latter has been addressed in [6] using queuing theory. Most results on stability use Lyapunov approaches which result in theorems that require the existence of positive definite matrices satisfying linear matrix inequalities; see [1,9,12]. However, these usually provide conservative conditions for stability and it is difficult to evaluate how conservative they are. Similarly, taking the worst case scenario (e.g. largest delay) can lead to unnecessary conservativeness or may simply give erroneous results. Finally, calculating stability for each delay and taking the intersection of the stable regimes in parameter space do not necessarily give the stability of the stochastic system [2].

In this paper we present a method for investigating the effects of stochastic delays in discrete-time dynamical systems. We derive a set of deterministic systems containing distributed delays which describe the time evolution of the mean and second moment of the stochastic system. Under certain conditions, we can guarantee with probability one (w.p.1) the stability of the equilibrium. We apply this method

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to a simple scalar example and demonstrate that it is possible to obtain noise-induced stability in some cases. That is, a system with a single deterministic delay may be unstable but upon introducing stochasticity in the delay, the system can become asymptotically stable.

## II. PROBLEM SET UP

In this paper, we consider the scalar stochastic system

$$X(k+1) = aX(k) + bU(k), \quad (1)$$

where  $X(k) \in \mathbb{R}$  is a stochastic variable at time  $k \in \mathbb{Z}$  and  $U(k)$  represents the stochastic delayed feedback

$$U(k) = X(k - \tau(k)). \quad (2)$$

The delay  $\tau(k)$  takes finite positive integer values  $\tau(k) \in [1, \dots, N]$  and  $N$  denotes the maximum delay. The initial condition includes the state values in the past  $N$  time steps. We can generalize the problem to include uncertainty in the initial condition, where  $X(0), X(-1), \dots, X(-N)$  are selected from known distributions.

We consider the following probability density function for  $U(k)$

$$p_{U(k)}(u) = \int_0^\infty p_{X(k-\tau(k))|\tau(k)}(u|\sigma) p_{\tau(k)}(\sigma) d\sigma, \quad (3)$$

where the density function  $p_{\tau(k)}$  for the delay is

$$p_{\tau(k)}(\sigma) = \sum_{i=1}^N w_i \delta(\sigma - \tau_i), \quad (4)$$

with

$$\sum_{i=1}^N w_i = 1. \quad (5)$$

The function  $\delta(\ast)$  denotes the Dirac delta function. The discrete stochastic variable  $\tau(k)$  has finite support, because  $N$  is a finite integer. All the possible delays are given by positive integers  $\tau_i$ , and  $w_i$  represents their associated weights, or likelihood of occurring. It is important to note that the delays are identically independently distributed at each time step  $k$ . For ease of notation we take  $\tau_i = i$  and  $w_i \geq 0$ . The integral in (3) is considered along the positive axis because the delays are positive. If we evaluate (3) using (4) we obtain

$$p_{U(k)}(u) = \sum_{i=1}^N w_i p_{X(k-\tau_i)}(u). \quad (6)$$

With this we can proceed to analyze the statistical properties of system (1,2).

### III. TIME EVOLUTION OF THE MEAN AND SECOND MOMENT

First, we derive equations governing the time evolution of the expected value,  $\mathbb{E}[X(k)]$ . Taking the expectation on both sides of (1) we obtain

$$\mathbb{E}[X(k+1)] = a \mathbb{E}[X(k)] + b \mathbb{E}[U(k)], \quad (7)$$

where the expected value of the feedback term  $U(k)$  is

$$\begin{aligned} \mathbb{E}[U(k)] &= \int_0^\infty u p_{U(k)}(u) du \\ &= \int_0^\infty u \left[ \sum_{i=1}^N w_i p_{X(k-\tau_i)}(u) \right] du \\ &= \sum_{i=1}^N w_i \int_0^\infty u p_{X(k-\tau_i)}(u) du \\ &= \sum_{i=1}^N w_i \mathbb{E}[X(k-\tau_i)]. \end{aligned} \quad (8)$$

Let us define the deterministic variable

$$y(k) = \mathbb{E}[X(k)]. \quad (9)$$

Substituting this into (7) and (8), the dynamics of the expectation is described by the deterministic system with distributed delay

$$y(k+1) = a y(k) + b \sum_{i=1}^N w_i y(k-\tau_i). \quad (10)$$

Since the system is a discrete-time system with finite maximum delay, the state space is finite dimensional. By defining the state vector

$$\vec{X}(k) = \begin{bmatrix} X(k) \\ X(k-1) \\ \vdots \\ X(k-N) \end{bmatrix}, \quad (11)$$

equation (1) can be rewritten as

$$\vec{X}(k+1) = \mathbf{A}(k) \vec{X}(k), \quad (12)$$

where  $\mathbf{A}(k) \in \mathbb{R}^{(N+1) \times (N+1)}$  is a stochastic variable whose probability distribution is independent of  $\vec{X}(k)$ . So we have

$$\begin{aligned} p_{\vec{X}(k), \mathbf{A}(k)}(\vec{X}, \mathbf{A}) &= p_{\mathbf{A}(k) | \vec{X}(k)}(\mathbf{A} | \vec{X}) p_{\vec{X}(k)}(\vec{X}) \\ &= p_{\mathbf{A}(k)}(\mathbf{A}) p_{\vec{X}(k)}(\vec{X}). \end{aligned} \quad (13)$$

Notice, that the sequence  $\{\vec{X}(k)\}$  is a Markov chain and the sequence  $\{\mathbf{A}(k)\}$  is mutually independent. Since the matrix  $\mathbf{A}(k)$  can only take on a finite set of values, its probability distribution becomes

$$p_{\mathbf{A}(k)}(\mathbf{A}) = \sum_{i=1}^N w_i \delta(\mathbf{A} - \mathbf{A}_i), \quad (14)$$

where

$$\mathbf{A}_i = \begin{bmatrix} a & bI_i(1) & bI_i(2) & \cdots & bI_i(N) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad (15)$$

with  $I_i$  being the indicator function

$$I_i(j) = \begin{cases} 0 & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases}$$

Indeed by taking the expected value of (12), one may also derive (10):

$$\begin{aligned} \mathbb{E}[\vec{X}(k+1)] &= \mathbb{E}[\mathbf{A}(k) \vec{X}(k)] \\ &= \int_{\mathbb{R}^{(N+1) \times (N+1)}} \int_{\mathbb{R}^{N+1}} \mathbf{A} \vec{X} p_{\vec{X}(k), \mathbf{A}(k)}(\vec{X}, \mathbf{A}) d\vec{X} d\mathbf{A} \\ &= \sum_{i=1}^N w_i \int_{\mathbb{R}^{N+1}} \mathbf{A}_i \vec{X} p_{\vec{X}(k)}(\vec{X}) d\vec{X} \\ &= \sum_{i=1}^N w_i \mathbf{A}_i \mathbb{E}[\vec{X}(k)]. \end{aligned} \quad (16)$$

Using the variable defined in (9), we define the deterministic state vector

$$\vec{y}(k) = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-N) \end{bmatrix}, \quad (18)$$

and obtain a deterministic system with distributed delay

$$\vec{y}(k+1) = \sum_{i=1}^N w_i \mathbf{A}_i \vec{y}(k), \quad (19)$$

where the state transition matrix is given by

$$\sum_{i=1}^N w_i \mathbf{A}_i = \begin{bmatrix} a & b w_1 & b w_2 & \cdots & b w_N \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (20)$$

Indeed, (10) and (19,20) are equivalent.

We now determine the governing equations for the second moment of  $X(k)$ . From (12) we have

$$\vec{X}(k+1) \vec{X}^T(k+1) = \mathbf{A}(k) \vec{X}(k) \vec{X}^T(k) \mathbf{A}^T(k). \quad (21)$$

Taking the expected value on both sides yields

$$\mathbb{E}[\vec{X}(k+1) \vec{X}^T(k+1)] = \mathbb{E}[\mathbf{A}(k) \vec{X}(k) \vec{X}^T(k) \mathbf{A}^T(k)], \quad (22)$$

where the expectation operator is taken element-wise, but we use the short-hand notation above. The right hand side

of (22) can be evaluated as

$$\begin{aligned}
& \mathbb{E} \left[ \mathbf{A}(k) \vec{X}(k) \vec{X}^T(k) \mathbf{A}^T(k) \right] \\
&= \int_{\mathbb{R}^{(N+1) \times (N+1)}} \int_{\mathbb{R}^{N+1}} \mathbf{A} \vec{X} \vec{X}^T \mathbf{A}^T p_{\vec{X}(k), \mathbf{A}(k)}(\vec{X}, \mathbf{A}) d\vec{X} d\mathbf{A} \\
&= \sum_{i=1}^N w_i \int_{\mathbb{R}^{N+1}} \mathbf{\Lambda}_i \vec{X} \vec{X}^T \mathbf{\Lambda}_i^T p_{\vec{X}(k)}(\vec{X}) d\vec{X} \\
&= \sum_{i=1}^N w_i \mathbf{\Lambda}_i \int_{\mathbb{R}^{N+1}} \vec{X} \vec{X}^T p_{\vec{X}(k)}(\vec{X}) d\vec{X} \mathbf{\Lambda}_i^T \\
&= \sum_{i=1}^N w_i \mathbf{\Lambda}_i \mathbb{E}[\vec{X}(k) \vec{X}(k)^T] \mathbf{\Lambda}_i^T. \tag{23}
\end{aligned}$$

Defining the deterministic matrix-valued variable

$$\mathbf{P}(k) = \mathbb{E}[\vec{X}(k) \vec{X}(k)^T], \tag{24}$$

and substituting this into (23) and (22) we obtain the deterministic system

$$\mathbf{P}(k+1) = \sum_{i=1}^N w_i \mathbf{\Lambda}_i \mathbf{P}(k) \mathbf{\Lambda}_i^T \tag{25}$$

for the time evolution of  $\mathbf{P}(k)$ . Note that  $p_{i,j}(k) = \mathbb{E}[X(k-i+1)X(k-j+1)]$  for  $i, j = 1, \dots, N+1$ . The second moment  $\mathbb{E}[X(k)^2]$  is given by the matrix element

$$p_{1,1}(k) = \mathbb{E}[X(k)^2] = \vec{C}^T \mathbf{P}(k) \vec{C}, \tag{26}$$

where  $\vec{C} = [1, 0, \dots, 0]^T$  but the time evolution of  $\mathbb{E}[X(k)^2]$  depends on other elements of the matrix  $\mathbf{P}(k)$ .

Exploiting that  $\mathbf{P}(k)$  is a symmetric matrix, i.e.

$$p_{i,j}(k) = \mathbb{E}[X(k-i+1)X(k-j+1)] = p_{j,i}(k),$$

we carry out the matrix multiplication in (25) and obtain a set of discrete time systems that describe the time evolution of the elements of  $\mathbf{P}(k)$ . This results in the distributed delay system

$$\begin{aligned}
p_{1,1}(k+1) &= a^2 p_{1,1}(k) + b^2 \sum_{i=1}^N w_i p_{1,1}(k-i) \\
&\quad + 2ab \sum_{i=1}^N w_i p_{1,i+1}(k), \\
p_{1,j}(k+1) &= a p_{1,j-1}(k) + b \sum_{i=1}^{j-3} w_i p_{1,j-i-1}(k-i) \\
&\quad + b \sum_{i=j-2}^N w_i p_{1,i-j+3}(k-j+2), \tag{27}
\end{aligned}$$

for  $j \in 2, \dots, N+1$ . If for a given  $j$  the subscript of  $w_j$  is less than one, then  $w_j = 0$  is considered and if the upper value on the sum is less than the lower value, then the sum is zero.

Now we show that one can obtain a Markovian structure for the system above. We define the state vector

$$\vec{P}(k) = \begin{bmatrix} p_{1,1}(k) \\ p_{1,2}(k) \\ \vdots \\ p_{i,N+1}(k) \end{bmatrix}, \tag{28}$$

and with this we define a super vector

$$\hat{P}(k) = \begin{bmatrix} \vec{P}(k) \\ \vec{P}(k-1) \\ \vdots \\ \vec{P}(k-N) \end{bmatrix}. \tag{29}$$

We can now represent (27) in state space form:

$$\hat{P}(k+1) = \hat{\mathbf{A}} \hat{P}(k), \tag{30}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_N \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{31}$$

The submatrices  $\mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_N \in \mathbb{R}^{(N+1) \times (N+1)}$  are given by

$$\mathbf{A} = \begin{bmatrix} a^2 & 2abw_1 & 2abw_2 & \cdots & 2abw_N \\ a & bw_1 & bw_2 & \cdots & bw_N \\ 0 & a & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a & 0 \end{bmatrix}, \tag{32}$$

$\mathbf{B}_i =$

$$\begin{bmatrix} b^2w_i & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ bw_i & bw_{i+1} & \cdots & bw_{N-1} & bw_N & 0 & \cdots & 0 \\ 0 & bw_i & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & bw_i & 0 & 0 & \cdots & 0 \end{bmatrix} \leftarrow \text{row } (i+2) \tag{33}$$

and  $\mathbf{I}$  is the  $(N+1)$ -dimensional identity matrix. Notice that  $\hat{\mathbf{A}} \in \mathbb{R}^{(N+1)^2 \times (N+1)^2}$ .

#### IV. STABILITY OF THE MEAN AND SECOND MOMENT

To determine the stability of a deterministic discrete time system one looks at the eigenvalues of the state transition matrix. The magnitude of all eigenvalues must be less than one for the system to be stable [5,8]. The stability of the mean is derived from the eigenvalues of the state transition

matrix  $\sum_{i=1}^N w_i \mathbf{A}_i$  in (19,20). The characteristic equation becomes

$$\det \left( s\mathbf{I} - \sum_{i=1}^N w_i \mathbf{A}_i \right) = 0 \Rightarrow (s-a) - b \sum_{i=1}^N w_i s^{-i} = 0. \quad (34)$$

To determine the stability boundaries in the parameter space  $(a, b)$ , we evaluate the characteristic equation so that  $|s| = 1$ . In particular,  $s = 1$ ,  $s = -1$  and  $s = e^{\pm i\theta}$  for  $\theta \in (0, \pi)$  are considered. Each of these provide a different set of stability curves [5,8]. Notice that for  $s = 1$ , one obtains a delay-independent condition.

Stability for the variance is determined in the same way using the state transition matrix in (31). At first glance, it appears that analyzing the stability of the second moment involves a matrix of dimension  $(N+1)^2$ , but it can be reduced to analyzing an  $(N+1)$ -dimensional matrix.

We denote the submatrices delimited by the lines in (31) as

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{bmatrix}, \quad (35)$$

which yields

$$\begin{aligned} \det(s\hat{\mathbf{I}} - \hat{\mathbf{A}}) &= \det(s\tilde{\mathbf{I}} - \tilde{\mathbf{D}}) \\ &\times \det \left( (s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\mathbf{I}} - \tilde{\mathbf{D}})^{-1}\tilde{\mathbf{C}} \right) \\ &= s^{N(N+1)} \det \left( (s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\mathbf{I}} - \tilde{\mathbf{D}})^{-1}\tilde{\mathbf{C}} \right), \end{aligned} \quad (36)$$

where  $\tilde{\mathbf{I}}$  denotes the  $N(N+1)$  dimensional identity matrix. We are left with determining the eigenvalues given by

$$\det \left( (s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\mathbf{I}} - \tilde{\mathbf{D}})^{-1}\tilde{\mathbf{C}} \right) = \det(\mathbf{M}_1 + \mathbf{M}_2) = 0, \quad (37)$$

where

$$\mathbf{M}_1 = \begin{bmatrix} s - a^2 & -2abw_1 & -2abw_2 & \dots & -2abw_N \\ -a & s - bw_1 & -bw_2 & \dots & -bw_N \\ 0 & -a & s & 0 & \dots & 0 \\ 0 & \frac{-bw_1}{s} & -a & s & 0 & \dots & 0 \\ 0 & \frac{-bw_2}{s^2} & \frac{-bw_1}{s} & -a & s & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \frac{-bw_{N-2}}{s^{N-2}} & \frac{-bw_{N-3}}{s^{N-3}} & \dots & \frac{-bw_1}{s} & -a & s \end{bmatrix} \quad (38)$$

and

$$\mathbf{M}_2 = \begin{bmatrix} -b^2 \sum_{i=1}^N \frac{w_i}{s^i} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-bw_1}{s} & \frac{-bw_2}{s} & \frac{-bw_3}{s} & \dots & \frac{-bw_N}{s} & 0 \\ \frac{-bw_2}{s^2} & \frac{-bw_3}{s^2} & \dots & \frac{-bw_N}{s^2} & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{-bw_{N-1}}{s^{N-1}} & \frac{-bw_N}{s^{N-1}} & 0 & \dots & 0 & 0 \end{bmatrix} \quad (39)$$

which give the characteristic equation for the second moment.

## V. NOTIONS OF STABILITY FOR STOCHASTIC SYSTEMS

We have provided deterministic discrete time equations whose stability determine the stability of the mean and second moment for the non-deterministic system (1,2). However, the first and second moments converging to zero does not guarantee that the state converges to zero with probability one (w.p.1) in all circumstances. We restate a theorem that can be found in [4]:

*The following implications hold*

$$\begin{array}{ccc} (X(k) \xrightarrow{w.p.1} X) & & \\ \downarrow & & \\ (X(k) \xrightarrow{P} X) & \Rightarrow & (X(k) \xrightarrow{D} X) \\ \uparrow & & \\ (X(k) \xrightarrow{r} X) & & \end{array}$$

for any  $r \geq 1$ . Also, if  $r > s \geq 1$  then

$$(X(k) \xrightarrow{r} X) \Rightarrow (X(k) \xrightarrow{s} X).$$

*No other implications hold in general.*

Here,  $X(k) \xrightarrow{r} X$  denotes that the sequence  $X(k)$  converges to a constant  $X$  in  $r^{\text{th}}$  order, for  $r \geq 1$ , which holds if  $\mathbb{E}[|X(k)|^r] < \infty$  for all  $k$  and

$$\mathbb{E}[|X(k) - X|^r] \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

while  $X(k) \xrightarrow{P} X$  and  $X(k) \xrightarrow{D} X$  denote convergence in probability and distribution [4]. Notice that convergence in  $r^{\text{th}}$  order only guarantees convergence in probability and distribution. Finally,  $X(k) \xrightarrow{w.p.1} X$  denotes convergence with probability one (w.p.1), that is, for every  $\epsilon > 0$ ,  $|X(k) - X| \geq \epsilon$  occurs only finitely often. Consequently, for each path  $\omega$ , there is a number  $k(\omega)$  so that  $|X(k) - X| \geq \epsilon$ , for all  $k > k(\omega)$ , see [7]. We may say that, with the exception of a finite set of sequences, all sequences  $\{X(k)\}$  converge pointwise towards  $X$ .

Convergence of the second moment  $X(k)^2$  is then equivalent to convergence in 2<sup>nd</sup> order since  $X(k)^2$  is positive definite. This is why convergence of the mean is insufficient but the convergence of the second moment may be enough.

Given the general vector case  $\vec{X}(k+1) = \mathbf{A}(k)\vec{X}(k)$ , where  $\{\mathbf{A}(k)\}$  are mutually independent random matrices, [7] provides the following theorem, using a Lyapunov function of the form  $\vec{X}^T \mathbf{Q} \vec{X}$ , where  $\mathbf{Q}$  is positive definite (denoted as  $\mathbf{Q} > 0$ ).

*Let  $\mathbf{Q} > 0$ ,  $\mathbf{C} \geq 0$  and*

$$\mathbb{E}[\mathbf{A}(k)^T \mathbf{Q} \mathbf{A}(k)] - \mathbf{Q} = -\mathbf{C}. \quad (40)$$

*Then  $\mathbb{E}[\vec{X}(k)^T \mathbf{C} \vec{X}(k)] \rightarrow 0$  and  $\vec{X}(k)^T \mathbf{C} \vec{X}(k) \rightarrow 0$  w.p.1. Let the  $\mathbf{A}(k)$  be identically distributed. If  $\{\vec{X}(k)\}$  is mean square stable (that is,  $\mathbb{E}[\vec{X}(k)^T \vec{X}(k)] \rightarrow 0$ ), then for any  $\mathbf{C} > 0$ , there is a  $\mathbf{Q} > 0$  satisfying (40).*

Given this theorem, if  $\{\mathbf{A}(k)\}$  are identically distributed and mutually independent in (12), there exists a solution  $\mathbf{Q}$  for (40) if we choose  $\mathbf{C} = \mathbf{I}$ . According to the theorem, the

existence of the solution implies  $\vec{X}(k)^T \vec{X}(k) \rightarrow 0$  w.p.1. This is a sufficient condition for w.p.1 stability when the delays  $\tau(k)$  are chosen independently of each other and from the same distribution at each  $k$  in (1,2).

### VI. EXAMPLES

Here we apply the stability conditions derived for mean and second moment to examples with different delay distributions. Figure 1 shows uniform delay distributions (left) and distributions with two equally probable delays (right), which we refer to as toggle distributions.  $E$  and  $V$  refer to the expected value and the variance of the delay distributions.

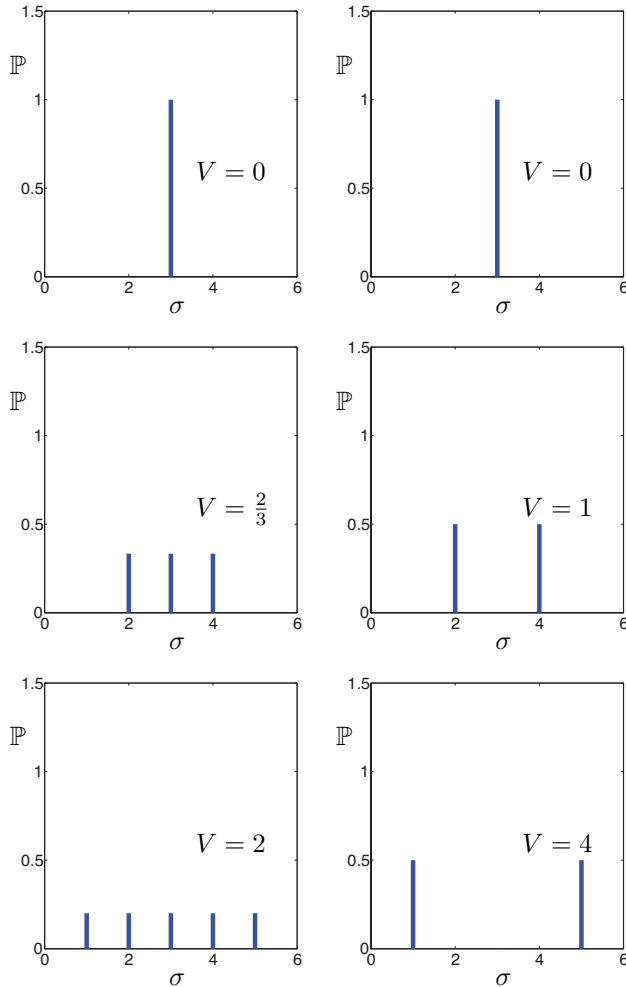


Fig. 1. Left: Discrete uniform delay distribution with expected value  $E = 3$ . Right: Discrete toggle distribution with  $E = 3$ . The variance  $V$  is listed in each panel.

Although stability of the second moment implies stability of the mean, it is interesting to take a look at the region of stability for the mean since it provides necessary conditions for stability. In [3] we showed that introducing additional delays to an already delayed continuous-time system may stabilize an unstable system. It is interesting to see that a similar result can be obtained for a discrete-time system.

Figure 2 shows the stability region for system (10) with uniform delay distribution (of expected value  $E$  and variance

$V$ ). The black (dash-dot) and red (dotted) curves indicate an eigenvalue crossings of the unit circle on the complex plane at 1 and  $-1$ . The green (solid) curves indicate a pair complex conjugate eigenvalues crossing the unit circle. One can see that as the variance is increased, the region of stability (shaded region) increases. It is important to point out the regions of stability for a single delay is not contained in the regions of stability for the distributed delays.

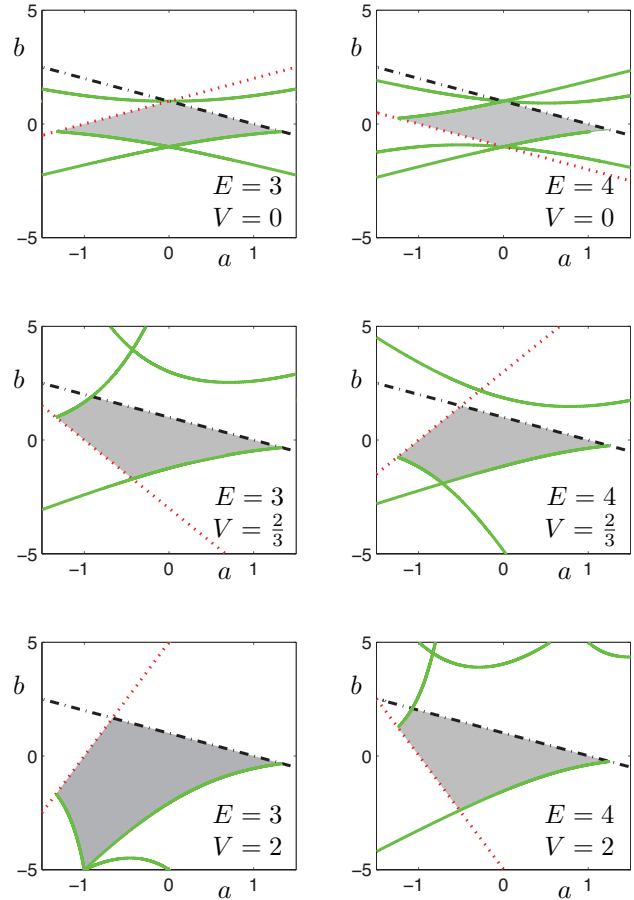


Fig. 2. Stability charts for the mean for uniform delay distributions. Shading indicates stability. When crossing a black (dash-dot) curve (from stable to unstable) an eigenvalue crosses the unit circle at 1 (outward), while crossing a red (dotted) curve indicates that an eigenvalue crosses the unit circle at  $-1$ . Crossing a green (solid) curve indicates that a pair of complex conjugate eigenvalues crosses the unit circle.

Next, we look at w.p.1 stability region. Recall that a system with identically independently distributed delays is stable w.p.1. if the second moment is stable. We first consider such systems with uniform delay distribution, then look at systems where the delay toggles between two values, each with equal probability.

The left panels in Fig. 3 show the stability boundaries of the non-deterministic system with uniform delay distribution. The curves indicate the stability losses of the mean as in Fig. 2 but here the shaded region indicates the region of w.p.1 stability (i.e. stability of the second moment). The shaded region was found by sweeping across the parameter space  $(a, b)$  and checking the eigenvalues of the system (30,31).

The right panels in Fig. 3 show stability charts for the toggle distribution. Again, we plot the mean stability curves and indicate the second moment stability regions by shading that imply w.p.1 stability. Here, the w.p.1 stability region is dominated by the region of stability for the mean of the system.

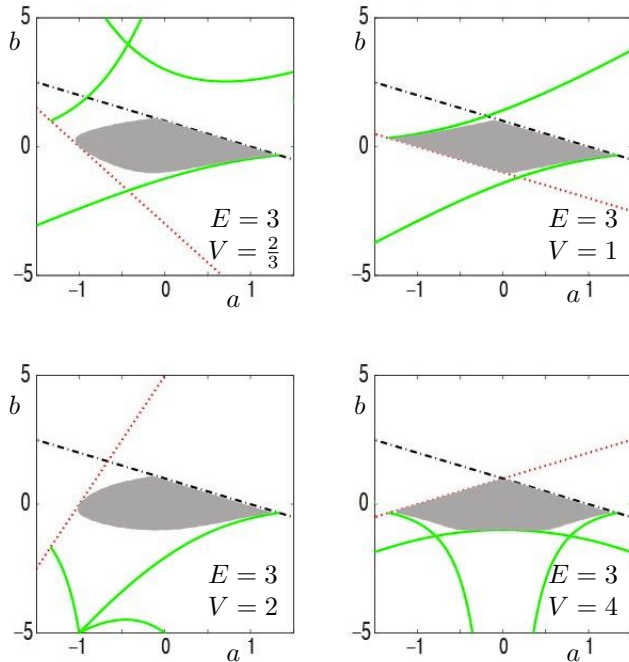


Fig. 3. Left: Stability boundaries for a uniform delay distribution. Right: Stability boundaries for toggle distribution. In all panels, the curves represent the stability losses of the mean as in Fig.2, while the shaded region indicates w.p.1 stability given by the second moment.

The introduction of stochasticity in the delay distorts the stability region when compared to the case of a single deterministic delay as can be seen in Fig. 4. Since some of the w.p.1 stability regions extend outside the stability bounds for the deterministic system we can stabilize the system by introducing uncertainty in the delay. We demonstrate this by numerical simulation in Fig. 4 where the parameters correspond to the mark “ $\times$ ” in the left panel.

## VII. CONCLUSION

In this paper, we defined a notion of stability and performed stability analysis for a class of linear system with stochastic delay. We investigated stability regions for scalar stochastically delayed feedback systems and made some interesting observations. We looked at two different types of delay distributions and found they had very different effects on the region of stability. In the case of the uniformly distributed delays, a worst case scenario would certainly be conservative. However, for the cases with two equally probable delays, the stability region of the mean seemed to provide a good approximation of the w.p.1 stability region. We also demonstrated that introducing stochasticity in the delay may stabilize the system that may look counter intuitive. Future work includes generalizing results for higher

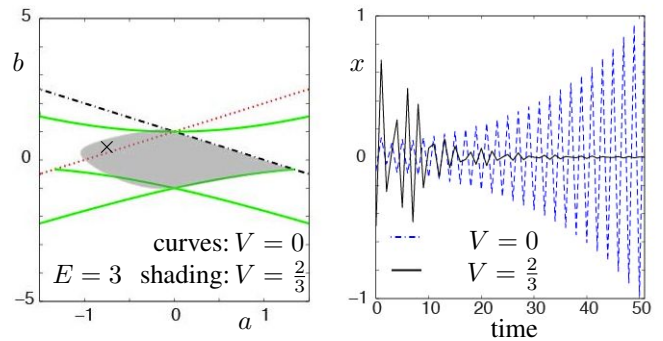


Fig. 4. Left: Comparing stability for the case of a deterministic single delay of  $\tau = 3$  (curves) with w.p.1 stability region (shaded) of a uniformly stochastically varying delay with mean  $E = 3$ . Right: Simulation comparing the cases deterministic and stochastic delay for  $a = -0.79$  and  $b = 0.3$  as indicated by “ $\times$ ”.

dimensional systems and finding general relationships between distribution types and size or shape of stability regions.

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