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On the Structure of Linear Behaviors over Quaternions*

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***Abstract*— Some results on the time-domain structure of linear, time-invariant systems over quaternions are presented, both in the continuous and in the discrete-time case. Within the behavioral approach, a system is defined as a set of trajectories (functions or sequences). In this paper, the trajectories are solutions of linear differential or difference equation with constant coefficients which belong to the skew-field of quaternions.**

As in the real and complex case, the equations may be represented by polynomials whose roots are related to the solutions. However, the properties of the roots and the structure of the corresponding solutions, which are analyzed in the paper, differ in many aspects from the standard commutative case.

I. INTRODUCTION

Quaternions are the simplest example of skew-field. Being related to rotations in space, they have a strong connection with mechanics and, therefore, with control of rigid bodies [2]. At the same time, the quaternionic formalism is used in quantum mechanics, leading to possible applications in quantum control [9], [4]. Recently, quaternions have been used also in coding theory and wireless communications [5], [6].

However, to the author's knowledge, a study of the solutions of simple linear differential and difference equations with quaternionic coeficients is still missing. Some works have been published (for instance, [1]) but, apparently, only particular cases have been considered.

This paper deals only with the scalar case, using just the point of view of polynomial operators. Relations with the quaternionic eigenvalue problem or with complex and real representations will be studied, along with more generic behaviors defined by matrix equations, in a forthcoming paper.

II. QUATERNIONS

The set of quaternions, introduced by Hamilton in 1843 and denoted by $\mathbb{H}$, may be defined in different ways. The most simple, perhaps, is the following: $\mathbb{H}$ is the real algebra generated by two elements i and j which satisfy

$$\mathsf{i}^2 = \mathsf{j}^2 = (\mathsf{ij})^2 = -1.$$

A. Standard notation

By introducing $\mathsf{k} = \mathsf{ij}$, it is possible to prove that these three *imaginary units* are pairwise different, satisfy the famous conditions

$$\mathsf{ij} = \mathsf{k} = -\mathsf{ji}, \qquad \mathsf{jk} = \mathsf{i} = -\mathsf{kj}, \qquad \mathsf{ki} = \mathsf{j} = -\mathsf{ik} \tag{1}$$

and generate the quaternions as follows: for every $\alpha \in \mathbb{H}$ there exist four unique real numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha = \alpha_0 + \alpha_1\mathsf{i} + \alpha_2\mathsf{j} + \alpha_3\mathsf{k}$. Starting from this representation, the *real part* of α is $\operatorname{Re}\alpha = \alpha_0$, its *imaginary part* is $\operatorname{Im}\alpha = \alpha_1\mathsf{i} + \alpha_2\mathsf{j} + \alpha_3\mathsf{k}$, its *conjugate* is $\overline{\alpha} = \operatorname{Re}\alpha - \operatorname{Im}\alpha = \alpha_0 - \alpha_1\mathsf{i} - \alpha_2\mathsf{j} - \alpha_3\mathsf{k}$, and its *norm* is $|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{\alpha\overline{\alpha}} = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. Note that $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$ but, as in the complex case, the following relations hold

$$\alpha + \overline{\alpha} = 2\operatorname{Re}\alpha \qquad\qquad \alpha - \overline{\alpha} = 2\operatorname{Im}\alpha.$$

By the above mentioned properties, it follows that $\mathbb{H}$ is a skew-field. Actually, for any $\alpha \in \mathbb{H}$, whenever $\alpha \neq 0$, also $|\alpha| \neq 0$ and therefore α has inverse $\alpha^{-1} = \frac{1}{|\alpha|^2}\overline{\alpha}$.

B. Vector notation

Alternatively, any quaternion q may be described as $q = (q_0, q_I)$, where the components of the vector $q_I \in \mathbb{R}^3$ are q_1, q_2, and q_3. In this way, the product of $\alpha, \beta \in \mathbb{H}$, which can be simplified by applying equalities (1), is equal to

$$\alpha\beta = (\alpha_0\beta_0 - \alpha_I \cdot \beta_I, \alpha_0\beta_I + \beta_0\alpha_I + \alpha_I \times \beta_I), \tag{2}$$

where '$\cdot$' and '$\times$' are the dot and, respectively, cross product of three-dimensional real vectors. Observe that the cross product is the only non commutative (anti-commutative) operation.

Remark 2.1: When no ambiguity arises, the notation q_I will be used both for the vector containing the real coefficients of the imaginary part and for the imaginary part itself. For instance, in the expression $q = q_0 + q_I$ it is clear that $q_I = \operatorname{Im} q$ while the equality $q = \alpha + \beta_I \times \gamma_I$ would be equivalent to $q = (\alpha_0, \alpha_I + \beta_I \times \gamma_I)$.

Mixing the two notations, by (2) it follows that

$$\alpha_I^2 = -\overline{\alpha_I}\alpha_I = -\alpha_I \cdot \alpha_I = -|\alpha_I|^2 \leq 0. \tag{3}$$

In other words, the square of the imaginary part is always a real, non positive number.

C. Commuting quaternions

Quaternions do not commute, in general, and so it would be interesting to know under what condition $\alpha\beta = \beta\alpha$, with $\alpha, \beta \in \mathbb{H}$. Using the commutator brackets '$[\cdot,\cdot]$' and equation (2), it is easy to check that

$$[\alpha, \beta] = \alpha\beta - \beta\alpha = 2\alpha_I \times \beta_I = [\alpha_I, \beta_I]. \tag{4}$$

Hence, $[\alpha, \beta] = 0 \Leftrightarrow \alpha_I \times \beta_I = 0$, which is a condition for parallelism of vectors α_I and β_I. The following statements resume the situation and add some important details.

Lemma 2.2: Given $\alpha \in \mathbb{H} \setminus \mathbb{R}$ and $\nu \in \mathbb{H}$, then

$$[\alpha, \nu] = 0 \;\Leftrightarrow\; \exists a, b \in \mathbb{R} : \nu = a + b\alpha.$$

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Lemma 2.3: Given $\alpha \in \mathbb{H} \setminus \{0\}$ and $n \in \mathbb{Z}$, there exist $a, b \in \mathbb{R}$ such that $\alpha^n = a + b\alpha$ (or $\overline{\alpha}^n = a + b\alpha$).

Proof: Note that $\alpha^{-n} = |\alpha|^{-2n}\,\overline{\alpha}^n$. Hence, we shall consider $n > 0$ without loss of generality. Let $\alpha = (\alpha_0, \alpha_I)$. Since $\alpha_I^2 \in \mathbb{R}$ by equation (3), then α_I^n is real for n even and a real multiple of α_I for n odd. Once α_0 and α_I obviously commute, the result follows by applying the binomial formula to $\alpha^n = (\alpha_0 + \alpha_I)^n$ or $\overline{\alpha}^n = (\alpha_0 - \alpha_I)^n$. ■

Theorem 2.4: Integer powers of $\alpha \in \mathbb{H}$ and $\overline{\alpha}$, polynomials with real coefficients in α and $\overline{\alpha}$, and functions defined by power series, e.g., $e^{\alpha t}$ and $e^{\overline{\alpha} t}$, commute with α.

D. Similarity

In some sense, which will be explained in Remark 4.6, conjugacy of quaternions is not the best generalization of conjugacy of complex numbers, and another concept is needed.

Definition 2.5: Two quaternions $\alpha, \beta \in \mathbb{H}$ are similar, denoted by $\alpha \sim \beta$, if there exists $0 \neq \eta \in \mathbb{H}$ such that $\alpha = \eta^{-1}\beta\eta$.

Similarity, which is clearly an equivalence relation, may be expressed in different ways.

Theorem 2.6: Given $\alpha, \beta \in \mathbb{H}$, the following statements are equivalent:

1) $\alpha \sim \beta$;
2) $\alpha_0 = \beta_0$ and $|\alpha| = |\beta|$ (or $|\alpha_I| = |\beta_I|$);
3) $\alpha(\alpha - \overline{\beta}) = (\alpha - \overline{\beta})\beta$ and $(\alpha - \overline{\beta})\alpha = \beta(\alpha - \overline{\beta})$.

Proof: The equivalence $1 \Leftrightarrow 2$ is well-known [11]. We shall prove only $2 \Leftrightarrow 3$ (just the first equality). Calculate

$$\begin{aligned} \alpha(\alpha - \overline{\beta}) = (\alpha - \overline{\beta})\beta \quad &\Leftrightarrow \\ \alpha^2 - \alpha\overline{\beta} = \alpha\beta - |\beta|^2 \quad &\Leftrightarrow \\ \alpha^2 + |\beta|^2 = \alpha(\beta + \overline{\beta}) \quad &\Leftrightarrow \\ \alpha_0^2 + 2\alpha_0\alpha_I + \alpha_I^2 + \beta_0^2 - \beta_I^2 = 2\alpha_0\beta_0 + 2\beta_0\alpha_I & \end{aligned} \tag{5}$$

and then observe that equation (5) contains real and purely imaginary parts, thus being equivalent, by (3), to the system

$$\begin{cases} (\alpha_0 - \beta_0)\alpha_I = 0 \\ (\alpha_0 - \beta_0)^2 = |\alpha_I|^2 - |\beta_I|^2 \end{cases}$$

The implication $2 \Rightarrow 3$ is now obvious. Vice versa, suppose that the system is satisfied. If $\alpha_I = 0$, then $|\beta_I| \leq 0$, by the second equation, hence $\beta_I = 0$ and $\alpha = \alpha_0 = \beta_0 = \beta$. If $\alpha_I \neq 0$, then $\alpha_0 = \beta_0$, by the first equation, and also $|\alpha_I|^2 = |\beta_I|^2$, i.e., $|\alpha| = |\beta|$. The other case is similar. ■

III. DYNAMICAL SYSTEMS AND BEHAVIORS

In this paper we will follow J. C. Willems' behavioral approach to dynamical system (see, for instance [10], [8] for a more comprehensive exposition of the notions that are here just outlined): a *behavior* $\mathcal{B}$ is a set of functions, called *trajectories*, having the same domain $\mathbb{T}$, called *time set*, and the same codomain W. In formulas,

$$\mathcal{B} \subseteq W^{\mathbb{T}} = \{w : \mathbb{T} \to W\}.$$

In this paper, behaviors are solution sets of linear difference or differential equations with constant quaternionic coefficients. More specifically, we will consider discrete-time systems, with $\mathbb{T} = \mathbb{Z}$ and

$$\mathcal{B} = \left\{ w : \mathbb{Z} \to \mathbb{H}^n \text{ such that} \sum_{l=M}^{N} R_l w(t + l) = 0, \forall t \in \mathbb{Z} \right\}, \tag{6}$$

and continuous-time systems, with $\mathbb{T} = \mathbb{R}$ and

$$\mathcal{B} = \left\{ w : \mathbb{R} \to \mathbb{H}^n \text{ such that} \sum_{l=0}^{N} R_l w^{(l)}(t) = 0, \forall t \in \mathbb{R} \right\}. \tag{7}$$

Note that, since $R_l \in \mathbb{H}^{g \times n}$ are constant matrices, the systems are time-invariant: if we define the *backward shift operator* by $(\sigma^\tau w)(t) = w(t + \tau)$, with $t, \tau \in \mathbb{T}$, this means that $w \in \mathcal{B} \Rightarrow \sigma^\tau w \in \mathcal{B}$ for every $\tau \in \mathbb{T}$. Equivalently, $\mathcal{B}$ is shift-invariant, i.e., $\sigma^\tau \mathcal{B} \subseteq \mathcal{B}$.

In the continuous case, where $w^{(l)}$ is the l-th order derivative of w, trajectories are supposed to be sufficiently smooth, otherwise equations have to be intended in a distributional sense (see [8]).

It is possible to treat discrete and continuous linear systems in a unified 'algebraic' way by means of polynomial operators.

In the discrete-time case, $\mathcal{B}$ in (6) is defined by equation $\sum_{l=M}^{N} R_l w(t + l) = \sum_{l=M}^{N} R_l \sigma^l w(t) = R(\sigma)w = 0$, where $R(s) = \sum_{l=M}^{N} R_l s^l \in \mathbb{H}^{g \times n}[s, s^{-1}]$ is a quaternionic Laurent polynomial matrix (i.e., a polynomial with both positive and negative powers of s) that can act on w as a *linear difference operator*.

Without loss of generality, we shall suppose that $R \in \mathbb{H}^{g \times n}[s]$. Indeed, being $\mathcal{B}$ time-invariant, $w \in \mathcal{B}$ if and only if $\sigma^\tau w \in \mathcal{B}$, i.e., $R(\sigma)\sigma^\tau w = 0$, for any $\tau \in \mathbb{Z}$. So, if we take $\tau = -M$, the behavior $\mathcal{B}$ can be equivalently defined by $R(s)s^{-M}$, which is a 'simple' polynomial (see also [7]).

Analogously, if $R(s) = \sum_{l=0}^{N} R_l s^l \in \mathbb{H}^{g \times n}[s]$, the condition in (7) can be written in the operator form $R\left(\frac{d}{dt}\right) w(t) = \sum_{l=0}^{N} R_l \frac{d^l}{dt^l} w(t) = 0$.

Finally, both in the discrete and in the continuous case, the behavior is the kernel of the operator R, where $R(\sigma)$ is a difference operator when $\mathbb{T} = \mathbb{Z}$ and $R(\frac{d}{dt})$ is a differential operator when $\mathbb{T} = \mathbb{R}$. For this reason, $R(s)$ is called *kernel representation* of $\mathcal{B}$.

Even if a behavior can be defined by other representations as, for instance, *image* and *input/output representations*, only the kernel representations (6) and (7) will be analyzed in this paper. In particular, for the sake of simplicity, we will deal just with the scalar case: the coefficients are scalars and, therefore, the system is defined by a polynomial.

Example 3.1: Let $\alpha \in \mathbb{H}$. Then, the kernel representation given by the polynomial $r(s) = s - \alpha$ corresponds to the equation

$$r(\sigma)w(t) = (\sigma - \alpha)w(t) = 0 \;\Leftrightarrow\; w(t + 1) = \alpha w(t)$$

in the discrete case and to

$$r(\tfrac{d}{dt})w(t) = \left(\tfrac{d}{dt} - \alpha\right) w(t) = 0 \;\Leftrightarrow\; w'(t) = \alpha w(t)$$

in the continuous case.

A straightforward calculation shows that, as in the commutative case, the solutions are $w(t)=\alpha^t q$, when $t\in\mathbb{Z}$, and $w(t)=e^{\alpha t}q$, when $t\in\mathbb{R}$, being $q\in\mathbb{H}$ an arbitrary constant.

The previous example is, maybe, misleading, since the general case is not so simple. For instance, in the real case, the linearity of the equations induces a vector space structure on $\mathcal{B}$. Here, due to non-commutativity, the solutions cannot be multiplied by constants on the left because, in general, $\alpha^t q \neq q\alpha^t$ and $e^{\alpha^t} q \neq q e^{\alpha^t}$ (see Theorem 2.4). However, it can be easily proved that

Theorem 3.2: A behavior $\mathcal{B}$ defined by equation (6) or (7) is a right $\mathbb{H}$-vector space:

$$w_1, w_2 \in \mathcal{B},\ q_1, q_2 \in \mathbb{H} \ \Rightarrow\ w_1 q_1 + w_2 q_2 \in \mathcal{B}.$$

In order to further clarify how the structure of the behavior is defined by its kernel representation, it is necessary to explain the *strange* nature of quaternionic polynomials.

IV. QUATERNIONIC POLYNOMIALS AND THEIR ROOTS

Polynomials over noncommutative rings may be defined in several ways. Here, following [3, §16], the polynomial indeterminate 's' shall commute with the coefficients. For example, $\alpha s\beta s^2\gamma = \alpha\beta\gamma s^3$, maintaining the correct order of the coefficients. Therefore, we may now define formally

$$\mathbb{H}[s] = \left\{ r(s) = \sum_{l=0}^{n} r_l s^l,\ n\in\mathbb{N},\ r_l\in\mathbb{H}, l = 0,\dots,n \right\}.$$

Observe that, due to its *operatorial* meaning, 's' had to commute with constants. In the discrete-time case, for instance, let $w:\mathbb{Z}\to\mathbb{H}$ be a trajectory and $\alpha\in\mathbb{H}$. Then αs and $s\alpha$ act in the same way on w, being

$$(\alpha\sigma)w(t) = \sigma\big(\alpha w(t)\big) = \alpha w(t+1).$$

Therefore, the product of polynomials corresponds to the composition of the (differential or difference) operators they represent: if $r = r_1 r_2 \in \mathbb{H}[s]$, then

$$r(\sigma)w = r_1(\sigma)\circ r_2(\sigma)w = r_1(\sigma)\big(r_2(\sigma)w\big).$$

Nevertheless, there is a great disadvantage: evaluation is not as 'natural' as in the commutative case. In fact, given a quaternionic value $\alpha\in\mathbb{H}$, $r(\alpha)\neq r_1(\alpha)r_2(\alpha)$.

Theorem 4.1 ([3, 16.3]): Let $r\in\mathbb{H}[s]$ and $\alpha\in\mathbb{H}$. If $r = r_1 r_2$ and $\nu = r_2(\alpha)\neq 0$, then

$$r(\alpha) = r_1(\nu\alpha\nu^{-1})r_2(\alpha).$$

The first fundamental consequence of this fact is that, even if every polynomial of degree n can be factorized into n linear factors, this does not provide its roots — just the rightmost factor corresponds to a root of the polynomial.

Example 4.2: Consider the factorized quaternionic polynomial $r(s) = (s-\beta)(s-\alpha)\in\mathbb{H}[s]$. Since

$$r(s) = s^2 - \beta s - s\alpha + \beta\alpha = s^2 - (\alpha+\beta)s + \beta\alpha,$$

we can check immediately that α is a root of r:

$$r(\alpha) = \alpha^2 - (\alpha+\beta)\alpha + \beta\alpha = 0,$$

while β, in general, is not a root:

$$r(\beta) = \beta^2 - (\alpha+\beta)\beta + \beta\alpha = [\beta,\alpha]\neq 0.$$

A general rule relating roots and factorizations is given by the following result.

Theorem 4.3: If $\alpha\in\mathbb{H}$ is a root of $r\in\mathbb{H}[s]$, there exists $r_1\in\mathbb{H}[s]$ such that $r(s) = r_1(s)(s-\alpha)$.

The proof of the theorem is just an iterative application of the following lemma.

Lemma 4.4: Let $\alpha,\beta\in\mathbb{H}\setminus\mathbb{R}$ such that $\alpha\neq\overline{\beta}$ and define $\nu = \alpha-\overline{\beta}$, $\tilde{\alpha} = \nu^{-1}\alpha\nu\sim\alpha$ and $\tilde{\beta} = \nu^{-1}\beta\nu\sim\beta$. Then

$$(s-\beta)(s-\alpha) = (s-\tilde{\alpha})(s-\tilde{\beta}). \tag{8}$$

Proof: We show that the coefficients are equal:

$$\alpha+\beta = \tilde{\alpha}+\tilde{\beta} = \nu^{-1}(\alpha+\beta)\nu \Leftrightarrow \nu(\alpha+\beta) = (\alpha+\beta)\nu,$$
$$\beta\alpha = \tilde{\alpha}\tilde{\beta} = \nu^{-1}\alpha\beta\nu \Leftrightarrow \nu\beta\alpha = \alpha\beta\nu.$$

By substituting ν and simplifying the expressions, we get $\alpha(\beta+\overline{\beta}) = (\beta+\overline{\beta})\alpha$ from the first equation and $\alpha|\beta|^2 = |\beta|^2\alpha$ from the second one. ■

Different factorizations may exist, since an order alteration changes the factors too, and their number may be infinite. Actually, consider in Example 4.2 the case $\beta = \overline{\alpha}$, which is excluded by the previous lemma.

Definition 4.5: The *minimal polynomial* of $\alpha\in\mathbb{H}$ is

$$\psi_\alpha(s) = (s-\overline{\alpha})(s-\alpha) = s^2 - 2\alpha_0 s + |\alpha|^2.$$

The minimal polynomial has real coefficients, thus commutes with other factors. Moreover, by Theorem 2.6.2, $\psi_\beta = \psi_\alpha$ for every $\beta\sim\alpha$: this means that it has an infinite number of factorizations and roots, which are all the elements of the equivalence class of α, called *conjugacy class*,

Remark 4.6: Both in the complex and in the quaternionic case, the conjugacy class of a non-real number is associated with a polynomial defined by two conjugated factors. Nevertheless, only in the first case 'conjugated elements' and 'conjugacy class' are equivalent concepts.

Finally, consider again Lemma 4.4. When $\alpha\sim\beta$, by Theorem 2.6.3, $\tilde{\beta} = \alpha$ and $\tilde{\alpha} = \beta$. In other words, the left and the right side of equation (8) are *identical*.

This situation is very important and generalizes to quaternionic polynomials the concept of multiplicity.

Theorem 4.7: Let

$$r(s) = (s-\alpha_n)(s-\alpha_{n-1})\cdots(s-\alpha_2)(s-\alpha_1)\in\mathbb{H}[s].$$

If α_l, $l = 1,\dots,n$, belong to the same equivalence class and consecutive elements are not conjugated, i.e., $\alpha_l\neq\overline{\alpha}_{l+1}$, $l = 1,\dots,n-1$, then α_1 is the only root of r and the given factorization is unique: the factors *and* their order cannot change.

Example 4.8: First, we will show that the only root of the polynomial $r(s) = (s - \mathsf{j})(s - \mathsf{i}) = s^2 - (\mathsf{i} + \mathsf{j})s - \mathsf{k}$ is i. Indeed, by Theorem 4.1, a second root would be $\nu \sim \mathsf{j}$, thus satisfying $\nu^2 = \nu_I^2 = -1$. Therefore,

$$\begin{aligned} \nu^2 - (\mathsf{i}+\mathsf{j})\nu - \mathsf{k} = -1 - (\mathsf{i}+\mathsf{j})\nu - \mathsf{k} = 0 \Leftrightarrow \\ -(\mathsf{i}+\mathsf{j})\nu = 1 + \mathsf{k} \Leftrightarrow \\ \nu = -(\mathsf{i}+\mathsf{j})^{-1}(1+\mathsf{k}) = \tfrac{1}{2}(\mathsf{i}+\mathsf{j})(1+\mathsf{k}) = \mathsf{i}. \end{aligned}$$

Now it is easy to prove that r does not have other factorizations: if $r(s) = (s - \beta)(s - \alpha)$, as in Example 4.2, it follows that $\alpha + \beta = \mathsf{i} + \mathsf{j}$ and $\beta\alpha = -k$. Multiplying the first equality by α on the right, we get

$$\alpha^2 + \beta\alpha = (\mathsf{i}+\mathsf{j})\alpha \Leftrightarrow \alpha^2 - (\mathsf{i}+\mathsf{j})\alpha - \mathsf{k} = r(\alpha) = 0,$$

whose unique solution is $\alpha = \mathsf{i}$, whence $\beta = \mathsf{j}$.

V. STRUCTURE OF LINEAR QUATERNIONIC BEHAVIORS

In this section it will be shown that, as in the commutative case, the factors of $r \in \mathbb{H}[s]$ are associated with trajectories which are the basis of the behavior $\mathcal{B} = \ker r$, as a right $\mathbb{H}$-vector space. Therefore, the analysis of the possible factorizations of r reveals the structure of the behavior.

A. Powers of a single factor – discrete case

Consider first the case $r(s) = (s - \alpha)^n$, with $\alpha \in \mathbb{H}$, and discrete-time trajectories. The basis of $\mathcal{B}$ is given by the functions $\varphi_\alpha^l(t) = \frac{t(t-1)\cdots(t-l+1)}{l!}\alpha^{t-l}$, $l = 0, \ldots, n-1$, $t \in \mathbb{Z}$. Actually, if we first apply the operator $\sigma - \alpha$,

$$\begin{aligned} (\sigma - \alpha)\varphi_\alpha^l(t) &= (\sigma - \alpha)\tfrac{t(t-1)\cdots(t-l+1)}{l!}\alpha^{t-l} \\ &= \tfrac{(t+1)t\cdots(t-l+2) - t(t-1)\cdots(t-l+1)}{l!}\alpha^{t+1-l} \\ &= \tfrac{(t+1-t+l-1)t\cdots(t-l+2)}{l!}\alpha^{t+1-l} \\ &= \tfrac{t(t-1)\cdots(t-(l-1)+1)}{(l-1)!}\alpha^{t-(l-1)} = \varphi_\alpha^{l-1}(t). \end{aligned}$$

So, in particular, $(\sigma - \alpha)^l\varphi_\alpha^l(t) = \varphi_\alpha^0(t)$. Hence,

$$\begin{aligned} (\sigma - \alpha)^{l+1}\varphi_\alpha^l(t) = (\sigma - \alpha)\varphi_\alpha^0(t) = (\sigma - \alpha)\alpha^t \\ = \alpha^{t+1} - \alpha\alpha^t = 0. \end{aligned}$$

B. Powers of a single factor – continuous case

As in the previous the case, the basis of the continuous-time behavior $\mathcal{B} = \ker(\frac{d}{dt} - \alpha)^n$, with $\alpha \in \mathbb{H}$, is given by a family of functions: $\varphi_\alpha^l(t) = \frac{t^l}{l!}e^{\alpha t}$, $l = 0, \ldots, n-1$, $t \in \mathbb{R}$. The operator $\frac{d}{dt} - \alpha$ acts on φ_α^l as $\sigma - \alpha$ before:

$$\begin{aligned} (\tfrac{d}{dt} - \alpha)\varphi_\alpha^l(t) &= (\tfrac{d}{dt} - \alpha)\tfrac{t^l}{l!}e^{\alpha t} \\ &= \tfrac{lt^{l-1}}{l!}e^{\alpha t} + \tfrac{t^l}{l!}\alpha e^{\alpha t} - \alpha\tfrac{t^l}{l!}e^{\alpha t} \\ &= \tfrac{t^{l-1}}{(l-1)!}e^{\alpha t} = \varphi_\alpha^{l-1}(t). \end{aligned}$$

So, once again, $(\frac{d}{dt} - \alpha)^l\varphi_\alpha^l(t) = \varphi_\alpha^0(t)$, hence

$$\begin{aligned} (\frac{d}{dt} - \alpha)^{l+1}\varphi_\alpha^l(t) = (\tfrac{d}{dt} - \alpha)\varphi_\alpha^0(t) = (\tfrac{d}{dt} - \alpha)e^{\alpha t} \\ = \alpha e^{\alpha t} - \alpha e^{\alpha t} = 0. \end{aligned}$$

Remark 5.1: Even if the functions are different, the same notation was used in the discrete and in the continuous case. In fact, the operators behave in the same way. So, in what follows, the indeterminate 's' will be also used as an operator, representing both σ and $\frac{d}{dt}$.

C. The minimal polynomial

If one of the factors of r is a minimal polynomial ψ_α, then every $\beta \sim \alpha$ is a root of r and, therefore, φ_β^0 is a solution. If the factor is ψ_α^n, then φ_β^l is a solution for every $l = 0, \ldots, n-1$.

The fact that infinite functions are solutions of the equation $\psi_\alpha(s)w = 0$ may be somewhat surprising. However, it makes perfectly sense because, according to the following theorem, all the solutions are just linear combinations of a basis given by two (i.e., the degree of ψ_α) functions.

Theorem 5.2: Let $\alpha \in \mathbb{H} \setminus \mathbb{R}$. Then for every $\beta \sim \alpha$,

$$\varphi_\beta^l = \varphi_\alpha^l\mu + \varphi_{\overline{\alpha}}^l\nu,\ l \in \mathbb{N},$$

where $\mu = (\alpha - \overline{\alpha})^{-1}(\alpha - \overline{\beta})$ and $\nu = (\alpha - \overline{\alpha})^{-1}(\alpha - \beta)$.

Before proving the theorem, a very useful extension of Theorem 2.6.3 will be presented.

Lemma 5.3: Let $\alpha \sim \beta$. Then the following hold:

$$\alpha - \overline{\beta} = \beta - \overline{\alpha} = \alpha_I + \beta_I, \tag{9}$$

$$\alpha - \beta = \overline{\beta} - \overline{\alpha} = \alpha_I - \beta_I, \tag{10}$$

$$\alpha(\alpha_I + \beta_I)^{\pm 1} = (\alpha_I + \beta_I)^{\pm 1}\beta, \tag{11}$$

$$\alpha(\alpha_I - \beta_I)^{\pm 1} = (\alpha_I - \beta_I)^{\pm 1}\overline{\beta}, \tag{12}$$

(formulas with negative power, only if it is possible).

Proof: The first two formulas are obvious, being $\alpha_0 = \beta_0$, and the last two can be proved as in Theorem 2.6.3.

As for the negative powers, observe that, for instance,

$$\alpha(\alpha_I+\beta_I)^{-1} = \alpha(\alpha_I+\beta_I)(\alpha_I+\beta_I)^{-2} = \tfrac{1}{(\alpha_I+\beta_I)^2}\alpha(\alpha_I+\beta_I)$$

where the fraction is real, by (3), and so commutes. ■

Remark 5.4: Identities (11) and (12) admit many *variations*: α and β are exchanged when *passing* from one side to the other of $\alpha_I + \beta_I$; with $\alpha_I - \beta_I$, also a conjugation is added or removed. For example, considering also (9) and (10),

$$\overline{\alpha}(\beta - \overline{\alpha}) = (\alpha - \overline{\beta})\overline{\beta}, \quad (\beta - \alpha)\alpha = \overline{\beta}(\overline{\alpha} - \overline{\beta}).$$

Proof: (Theorem 5.2) Since φ_α^l is a power (series) of α, it commutes with (any power of) $\alpha - \overline{\alpha} = 2\alpha_I$, by Theorem 2.4, and so equation (11) can be applied, obtaining

$$\varphi_\alpha^l\mu = \varphi_\alpha^l(\alpha - \overline{\alpha})^{-1}(\alpha - \overline{\beta}) = (\alpha - \overline{\alpha})^{-1}(\alpha - \overline{\beta})\varphi_\beta^l.$$

Similarly, applying equation (12), we get

$$\varphi_{\overline{\alpha}}^l\nu = \varphi_{\overline{\alpha}}^l(\alpha - \overline{\alpha})^{-1}(\alpha - \beta) = (\alpha - \overline{\alpha})^{-1}(\alpha - \beta)\varphi_\beta^l.$$

So, the result is proved, since

$$\begin{aligned} \varphi_\alpha^l\mu + \varphi_{\overline{\alpha}}^l\nu &= (\alpha - \overline{\alpha})^{-1}(\beta - \overline{\alpha})\varphi_\beta^l + (\alpha - \overline{\alpha})^{-1}(\alpha - \beta)\varphi_\beta^l \\ &= (\alpha - \overline{\alpha})^{-1}(\beta - \overline{\alpha} + \alpha - \beta)\varphi_\beta^l = \varphi_\beta^l. \end{aligned}$$

■

D. Other kinds of multiplicity

By Theorem 4.7, one last case is missing: the product of factors which differ, but have roots in the same conjugacy class. The main result is the following.

Theorem 5.5: Let $r \in \mathbb{H}[s]$ have degree n and only the root α. A basis of $\ker r$ is given by the n trajectories

$$\begin{aligned} w_1 &= \varphi_\alpha^0 \\ w_2 &= \varphi_\alpha^1 + \nu_1 \varphi_\alpha^0 \\ w_3 &= \varphi_\alpha^2 + \nu_1 \varphi_\alpha^1 + \nu_2 \varphi_\alpha^0 \\ &\vdots \\ w_n &= \varphi_\alpha^{n-1} + \nu_1 \varphi_\alpha^{n-2} + \cdots + \nu_{n-1} \varphi_\alpha^0 \end{aligned}$$

with suitable coefficients $\nu_l \in \mathbb{H}$, $l = 1, \ldots, n-1$.

A direct proof of the theorem will not be given here. Instead, it will be shown, through examples, how the coefficients ν_l can be constructed. To this aim, new notation and results about a *generalized commutator* will be introduced.

Definition 5.6: Given $\alpha, \beta, \nu \in \mathbb{H}$, let

$$\llbracket \beta, \nu \rrbracket_\alpha = \beta\nu - \nu\alpha.$$

The shorthand $\llbracket \gamma, \beta, \nu \rrbracket_\alpha = \llbracket \gamma, \llbracket \beta, \nu \rrbracket_\alpha \rrbracket_\alpha$ will be also used.

Lemma 5.7: Let $\alpha, \beta, \gamma, \nu, \mu \in \mathbb{H}$ and $a, b \in \mathbb{R}$. Then:

1) $\llbracket \alpha, \nu \rrbracket_\alpha = [\alpha, \nu]$; in particular, $\llbracket \alpha, a + b\alpha \rrbracket_\alpha = 0$;
2) $\llbracket \beta, a\nu + b\mu \rrbracket_\alpha = a\llbracket \beta, \nu \rrbracket_\alpha + b\llbracket \beta, \mu \rrbracket_\alpha$;
3) $\llbracket \beta, \nu\mu \rrbracket_\alpha = \begin{cases} \nu\llbracket \beta, \mu \rrbracket_\alpha & \text{if } [\beta, \nu] = 0, \\ \llbracket \beta, \nu \rrbracket_\alpha \mu & \text{if } [\alpha, \mu] = 0; \end{cases}$
4) $\llbracket \beta, \nu \rrbracket_\alpha - \llbracket \overline{\beta}, \nu \rrbracket_\alpha = (\beta - \overline{\beta})\nu$;
5) $\beta \sim \gamma \Rightarrow \llbracket \gamma, (\gamma - \overline{\beta})^{\pm 1}\nu \rrbracket_\alpha = (\gamma - \overline{\beta})^{\pm 1}\llbracket \beta, \nu \rrbracket_\alpha$;
6) $\alpha \sim \beta \Rightarrow \llbracket \gamma, \beta, \nu \rrbracket_\alpha = (\gamma - \overline{\beta})\llbracket \beta, \nu \rrbracket_\alpha \Rightarrow \llbracket \overline{\beta}, \beta, \nu \rrbracket_\alpha = 0$;
7) $\llbracket \overline{\beta}, \beta, \nu \rrbracket_\alpha = \nu\psi_\beta(\alpha)$; if $\nu = 1$, $\llbracket \overline{\beta}, \beta - \alpha \rrbracket_\alpha = \psi_\beta(\alpha)$.

Proof: 1) By Lemma 2.2; 2,3,4) by definition; 5) by Lemma 5.3; 6) the equality is equivalent to $-(\beta + \overline{\beta})\nu\alpha + \nu\alpha^2 + \overline{\beta}\beta\nu = -\nu(\alpha + \overline{\alpha})\alpha + \nu\alpha^2 + \nu\overline{\alpha}\alpha = 0$, since $\overline{\beta}\beta = \overline{\alpha}\alpha \in \mathbb{R}$ and $\beta + \overline{\beta} = \alpha + \overline{\alpha} \in \mathbb{R}$; 7) as before, $\overline{\beta}\beta, \beta + \overline{\beta} \in \mathbb{R}$ are the coefficients of the minimal polynomial of β; note that $\llbracket \beta, 1 \rrbracket_\alpha = \beta - \alpha$. ■

Before showing another important property of the generalized commutator, the following result justifies its definition within our context.

Theorem 5.8: Let $\alpha, \nu \in \mathbb{H}$, $\beta \sim \alpha$ and $l \in \mathbb{N}$. Then

$$(s - \beta)\nu\varphi_\alpha^l = \nu\varphi_\alpha^{l-1} - \llbracket \beta, \nu \rrbracket_\alpha \varphi_\alpha^l,$$

where $\varphi_\alpha^{-1} = 0$ is the zero function.

Proof: As it was shown in Sections V-A and V-B, $(s - \alpha)\varphi_\alpha^l = \varphi_\alpha^{l-1} \Leftrightarrow s\varphi_\alpha^l = \varphi_\alpha^{l-1} + \alpha\varphi_\alpha^l$. Therefore,

$$(s - \beta)\nu\varphi_\alpha^l = \nu\varphi_\alpha^{l-1} + \nu\alpha\varphi_\alpha^l - \beta\nu\varphi_\alpha^l = \nu\varphi_\alpha^{l-1} - \llbracket \beta, \nu \rrbracket_\alpha \varphi_\alpha^l.$$

■

Observe that, to find the trajectories mentioned in Theorem 5.5, it is necessary to determine the coefficients 'ν', while the application of Theorem 5.8 gives $\llbracket \beta, \nu \rrbracket_\alpha$. So, the the question is: when, and how, is it possible to solve the equation $\llbracket \beta, \nu \rrbracket_\alpha = \mu$ in the variable ν?

Theorem 5.9: Given $\alpha, \beta, \mu \in \mathbb{H}$, the equation $\llbracket \beta, \nu \rrbracket_\alpha = \mu$

1) has the unique solution $\nu = \llbracket \overline{\beta}, \mu \rrbracket_\alpha \psi_\beta(\alpha)^{-1}$, if $\beta \not\sim \alpha$;
2) has a solution if and only if $\llbracket \overline{\beta}, \mu \rrbracket_\alpha = 0$, if $\beta \sim \alpha$; in this case, when the solution exists, it is given by

$$\nu = (2\alpha_I)^{-2}\llbracket \beta, \mu \rrbracket_\alpha + \nu_h,$$

where ν_h is any solution of $\llbracket \beta, \nu \rrbracket_\alpha = 0$.

Proof: By Lemma 5.7.7, applying $\llbracket \overline{\beta}, \cdot \rrbracket_\alpha$ we get that

$$\llbracket \beta, \nu \rrbracket_\alpha = \mu \Rightarrow \nu\psi_\beta(\alpha) = \llbracket \overline{\beta}, \beta, \nu \rrbracket_\alpha = \llbracket \overline{\beta}, \mu \rrbracket_\alpha. \qquad (13)$$

If $\beta \not\sim \alpha$, then $\psi_\beta(\alpha) \neq 0$ and the formula for the solution, if it exists, is obtained. To check that the equation always admits the given solution, it suffices to replace it into the equation and apply Lemma 5.7.3 (since $[\alpha, \psi_\beta(\alpha)^{-1}] = 0$) and again Lemma 5.7.7.

Suppose now that $\beta \sim \alpha$. Since $\psi_\beta(\alpha) = 0$, by equation (13) we deduce that, if a solution exists, $\llbracket \overline{\beta}, \mu \rrbracket_\alpha = 0$. Vice versa, assume that $\llbracket \overline{\beta}, \mu \rrbracket_\alpha = 0$ and let $\tilde{\nu} = c\llbracket \beta, \mu \rrbracket_\alpha$, $c \in \mathbb{R}$. Then, by Lemma 5.7.2&4&7

$$\begin{aligned} \llbracket \beta, \tilde{\nu} \rrbracket_\alpha &= c\llbracket \beta, \beta, \mu \rrbracket_\alpha = c(\beta - \overline{\beta})\llbracket \beta, \mu \rrbracket_\alpha \\ &= c(\beta - \overline{\beta})\big(\llbracket \overline{\beta}, \mu \rrbracket_\alpha + (\beta - \overline{\beta})\mu\big) = c(2\beta_I)^2\mu. \end{aligned}$$

Hence, $\tilde{\nu}$ is a solution if and only if $c = (2\beta_I)^{-2} = (2\alpha_I)^{-2}$. Since the given equation is $\mathbb{R}$-linear, by Lemma 5.7.2, the general solution is $\nu = \tilde{\nu} + \nu_h$, being ν_h a solution of the homogeneous equation. ■

To complete the statement of the previous theorem, the solutions of the homogeneous equation $\llbracket \beta, \nu \rrbracket_\alpha = 0$ have to be characterized.

Corollary 5.10: Let $\alpha, \beta \in \mathbb{H}$ and $V = \{a + b\alpha,\ a, b \in \mathbb{R}\}$. Define $V_1 = (\alpha - \overline{\beta})V$ and be V_2 the set of quaternions whose vector representation in $\mathbb{R}^4$ is orthogonal to the vector representation of V, i.e., to $(1, 0)$ and $(0, \alpha_I)$. Then

$$\llbracket \beta, \nu \rrbracket_\alpha = 0 \Leftrightarrow \begin{cases} \nu = 0 & \text{if } \beta \not\sim \alpha; \\ \nu \in V_1 & \text{if } \beta \sim \alpha, \beta \neq \overline{\alpha}; \\ \nu \in V_2 & \text{if } \beta = \overline{\alpha} \notin \mathbb{R}; \\ \nu \in \mathbb{H} & \text{if } \beta = \alpha \in \mathbb{R}. \end{cases}$$

Proof: By Theorem 5.9.1, $\nu = 0$ is the unique solution when $\beta \not\sim \alpha$ and, obviously, $\llbracket \beta, \nu \rrbracket_\alpha = \beta\nu - \nu\alpha = 0$ for any $\nu \in \mathbb{H}$ when $\alpha = \beta \in \mathbb{R}$. If $\beta \sim \alpha$, but $\beta \neq \overline{\alpha}$, by Lemma 5.7.1&5 and Lemma 2.2,

$$\begin{aligned} \llbracket \beta, \nu \rrbracket_\alpha = 0 &\Leftrightarrow (\alpha - \overline{\beta})^{-1}\llbracket \beta, \nu \rrbracket_\alpha = 0 \\ &\Leftrightarrow \llbracket \alpha, (\alpha - \overline{\beta})^{-1}\nu \rrbracket_\alpha = [\alpha, (\alpha - \overline{\beta})^{-1}\nu] = 0 \\ &\Leftrightarrow (\alpha - \overline{\beta})^{-1}\nu \in V \Leftrightarrow \nu \in (\alpha - \overline{\beta})V = V_1. \end{aligned}$$

When $\beta = \overline{\alpha}$, by Theorem 5.9.2 and equation (4), $\llbracket \overline{\alpha}, \nu \rrbracket_\alpha = 0$ if and only if there exist $\xi \in \mathbb{H}$ such that $\nu = \llbracket \alpha, \xi \rrbracket_\alpha = [\alpha, \xi] = 2\alpha_I \times \xi_I$. This means that ν has zero real part, i.e., its vector representation is orthogonal to $(1, 0)$, and by the properties of the cross product, $\nu_I \cdot \alpha_I = 0$. Thus $v \in V_2$. ■

Example 5.11: Let $\alpha = 1 + \mathsf{i} \sim \beta = 1 + \mathsf{j}$. Using the notation of Corollary 5.10 and considering $a, b \in \mathbb{R}$, $V = \{a + b\mathsf{i}\}$, $V_1 = (\mathsf{i} + \mathsf{j})V = \{a(\mathsf{i} + \mathsf{j}) + b(1 + \mathsf{k})\}$ and $V_2 = \{a\mathsf{j} + b\mathsf{k}\}$.

Remark 5.12: The results presented in Theorem 5.9 and Corollary 5.10 can be partially found (with completely different notations) in [3, 10&11, p. 274].

In the following examples, the coefficients ν_1 and ν_2 of Theorem 5.5 will be calculated for $r(s) = (s-\gamma)(s-\beta)(s-\alpha)$, with $\alpha \sim \beta \sim \gamma$ and $\alpha \neq \overline{\beta} \neq \gamma$. The numerous reference to Lemma 5.7 will not be given and, to simplify the notation, the index α, the root of r, will be omitted, writing $\varphi^l = \varphi^l_\alpha$ and $\llbracket \beta, \nu \rrbracket = \llbracket \beta, \nu \rrbracket_\alpha$.

Example 5.13: To determine $w_2 = \nu_1\varphi^0 + \varphi^1$ that satisfies equation $(s-\beta)(s-\alpha)w_1 = 0$, start calculating

$$\begin{aligned} w_{21} &= (s-\alpha)w_1 = (s-\alpha)(\nu_1\varphi^0 + \varphi^1) \\ &= \nu_1\varphi^{-1} - \llbracket \alpha, \nu_1 \rrbracket \varphi^0 + \varphi^0 - \llbracket \alpha, 1 \rrbracket \varphi^1 \\ &= (-\llbracket \alpha, \nu_1 \rrbracket + 1)\varphi^0. \end{aligned}$$

Now, solve $(s-\beta)(s-\alpha)w_2 = (s-\beta)w_{21} = 0$:

$$\begin{aligned} (s-\beta)w_{21} &= (s-\beta)(-\llbracket \alpha, \nu_1 \rrbracket + 1)\varphi^0 \\ &= (-\llbracket \alpha, \nu_1 \rrbracket + 1)\varphi^{-1} - \llbracket \beta, -\llbracket \alpha, \nu_1 \rrbracket + 1 \rrbracket \varphi^0 \\ &= \llbracket \beta, \llbracket \alpha, \nu_1 \rrbracket - 1 \rrbracket \varphi^0 = 0 \Leftrightarrow \llbracket \beta, \llbracket \alpha, \nu_1 \rrbracket - 1 \rrbracket = 0. \end{aligned}$$

By Theorem 5.9.2, last equation is satisfied if and only if there exists μ_1 such that

$$\begin{aligned} \llbracket \overline{\beta}, \mu_1 \rrbracket = \llbracket \alpha, \nu_1 \rrbracket - 1 &\Leftrightarrow \llbracket \alpha, \nu_1 \rrbracket = \llbracket \overline{\beta}, \mu_1 \rrbracket + 1 \\ &\Leftrightarrow \llbracket \overline{\alpha}, \llbracket \overline{\beta}, \mu_1 \rrbracket + 1 \rrbracket = 0 \\ &\Leftrightarrow \llbracket \overline{\alpha}, \overline{\beta}, \mu_1 \rrbracket = -\llbracket \overline{\alpha}, 1 \rrbracket \\ &\Leftrightarrow (\overline{\alpha} - \beta)\llbracket \overline{\beta}, \mu_1 \rrbracket = -(\overline{\alpha} - \alpha) \\ &\Leftrightarrow \llbracket \overline{\beta}, \mu_1 \rrbracket = -(\alpha - \overline{\beta})^{-1}(\alpha - \overline{\alpha}) \\ &\Leftrightarrow \llbracket \alpha, \nu_1 \rrbracket = 1 - (\beta - \overline{\alpha})^{-1}(\alpha - \overline{\alpha}) \\ &\Leftrightarrow \llbracket \alpha, \nu_1 \rrbracket = (\beta - \overline{\alpha})^{-1}(\beta - \overline{\alpha} - \alpha + \overline{\alpha}) \\ &\Leftrightarrow \llbracket \alpha, \nu_1 \rrbracket = (\alpha - \overline{\beta})^{-1}(\beta - \alpha), \end{aligned} \tag{14}$$

using Theorem 5.9.2 many times and equation (10). Since we already know that the solution exists, it is equal to

$$\nu_1 = \frac{1}{(\alpha - \overline{\alpha})^2}\llbracket \alpha, (\alpha - \overline{\beta})^{-1}(\beta - \alpha) \rrbracket + a + b\alpha, \tag{15}$$

with arbitrary $a, b \in \mathbb{R}$. Using similar techniques, the solution can be further simplified, giving $\nu_1 = (\alpha - \overline{\beta})^{-1} + a + b\alpha$.

Remark 5.14: The 'arbitrary' part of (15) is irrelevant in this context. To show this, let $\nu_1 = \tilde{\nu}_1 + \nu_h$, with $\nu_h = a + b\alpha$ as in Theorem 5.9. By Theorem 5.5, the solution of $(s-\beta)(s-\alpha)w = 0$ is a linear combination of $w_1 = \varphi^0$ and $w_2 = \varphi^1 + \nu_1\varphi^0$, but w_1 and $\tilde{w}_2 = \varphi^1 + \tilde{\nu}_1\varphi^0$ are an equivalent basis. Indeed, since ν_h commutes with α, thus with φ^l, $\nu_1\varphi^0 = (\tilde{\nu}_1 + \nu_h)\varphi^0 = \tilde{\nu}_1\varphi^0 + \varphi^0\nu_h$. So, given $q_1, q_2 \in \mathbb{H}$, $w_1q_1 + w_2q_2 = w_1(q_1 + \nu_hq_2) + \tilde{w}_2q_2$.

Example 5.15: Also $r(s)w_3 = 0$, with $w_3 = \nu_2\varphi^0 + \nu_1\varphi^1 + \varphi^2$, will be solved by steps, as in the previous example. First, let

$$\begin{aligned} w_{31} &= (s-\alpha)w = (s-\alpha)(\nu_2\varphi^0 + \nu_1\varphi^1 + \varphi^2) \\ &= -\llbracket \alpha, \nu_2 \rrbracket \varphi^0 + \nu_1\varphi^0 - \llbracket \alpha, \nu_1 \rrbracket \varphi^1 + \varphi^1 \\ &= (-\llbracket \alpha, \nu_2 \rrbracket + \nu_1)\varphi^0 - \llbracket \overline{\beta}, \mu_1 \rrbracket \varphi^1, \end{aligned}$$

using (14) in the end. Then let $w_{32} = (s-\beta)w_{31}$, i.e.,

$$\begin{aligned} w_{32} &= \llbracket \beta, \llbracket \alpha, \nu_2 \rrbracket - \nu_1 \rrbracket \varphi^0 - \llbracket \overline{\beta}, \mu_1 \rrbracket \varphi^0 + \llbracket \beta, \llbracket \overline{\beta}, \mu_1 \rrbracket \rrbracket \varphi^1 \\ &= (\llbracket \beta, \llbracket \alpha, \nu_2 \rrbracket \rrbracket - \llbracket \beta, \nu_1 \rrbracket - \llbracket \overline{\beta}, \mu_1 \rrbracket)\varphi^0. \end{aligned}$$

Finally, from this it follows that $r(s)w_3 = (s-\gamma)w_{32} = 0$ $\Leftrightarrow \llbracket \gamma, \llbracket \beta, \alpha, \nu_2 \rrbracket - \llbracket \beta, \nu_1 \rrbracket - \llbracket \overline{\beta}, \mu_1 \rrbracket \rrbracket = 0$, and therefore, by Theorem 5.9.2, there exists μ_2 such that

$$\begin{aligned} \llbracket \overline{\gamma}, \mu_2 \rrbracket = \llbracket \beta, \alpha, \nu_2 \rrbracket - \llbracket \beta, \nu_1 \rrbracket - \llbracket \overline{\beta}, \mu_1 \rrbracket &\Leftrightarrow \\ \llbracket \beta, \alpha, \nu_2 \rrbracket = \llbracket \overline{\gamma}, \mu_2 \rrbracket + \llbracket \beta, \nu_1 \rrbracket + \llbracket \overline{\beta}, \mu_1 \rrbracket &\Leftrightarrow \\ \llbracket \overline{\beta}, \llbracket \overline{\gamma}, \mu_2 \rrbracket + \llbracket \beta, \nu_1 \rrbracket + \llbracket \overline{\beta}, \mu_1 \rrbracket \rrbracket = 0 &\Leftrightarrow \\ \llbracket \overline{\beta}, \overline{\gamma}, \mu_2 \rrbracket = -\llbracket \overline{\beta}, \overline{\beta}, \mu_1 \rrbracket &\Leftrightarrow \\ (\overline{\beta} - \gamma)\llbracket \overline{\gamma}, \mu_2 \rrbracket = -(\overline{\beta} - \beta)\llbracket \overline{\beta}, \mu_1 \rrbracket &\Leftrightarrow \\ \llbracket \overline{\gamma}, \mu_2 \rrbracket = -(\gamma - \overline{\beta})^{-1}(\beta - \overline{\beta})\llbracket \overline{\beta}, \mu_1 \rrbracket. & \end{aligned} \tag{16}$$

Since the three quantities on the right side of equation (16) are known and, on the left, $\llbracket \beta, \alpha, \nu_2 \rrbracket = (\beta - \overline{\alpha})\llbracket \alpha, \nu_2 \rrbracket$, from equation (16) we get

$$\llbracket \alpha, \nu_2 \rrbracket = (\beta - \overline{\alpha})^{-1}(\llbracket \overline{\gamma}, \mu_2 \rrbracket + \llbracket \beta, \nu_1 \rrbracket + \llbracket \overline{\beta}, \mu_1 \rrbracket)$$

which, by Theorem 5.9.2, determines ν_2. After some calculations, using $\tilde{\nu}_1 = (\alpha - \overline{\beta})^{-1}$ as explained by Remark 5.14, $\tilde{\nu}_2 = (\alpha - \overline{\beta})^{-1}(\gamma - \overline{\beta})^{-1}(\gamma - \beta)(\alpha - \overline{\beta})^{-1}$ is obtained.

VI. CONCLUSIONS

After introducing the skew-field of quaternions, the properties of quaternionic polynomials were investigated. The relation between their roots and the trajectories of the dynamical systems they define, when acting as operators, was clarified. In this way, the structure of the solutions of scalar linear differential and difference equations with constant quaternionic coefficients was completely described.

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