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Integral Higher Order Sliding Mode and Singular Optimal Stabilization

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Abstract— **This paper explores the possibilities of the so-called Integral Higher-Order Sliding-Mode (IHOSM) control framework to solve singular optimal stabilization (SOS) problems of arbitrary order for uncertain systems. We connect the order of singularity of SOS with the co-dimension of the set of singular optimal manifold. IHOSM is designed allowing to reach singular optimal manifold in prescribing time moment and ensures SOS. Moreover, this controller provides the insensitivity of trajectory w.r.t. matched bounded uncertainties.**

Keywords. **Singular Optimal Control, Variable Structure System.**

I. INTRODUCTION

A. Background

Singular *LQ* control is one of the classical field of optimal control motivated by applications in aeronautics [1], [2], economy [3], servo theory [4], etc.

The main results in singular optimal control were received in 60th and 70th and could be summarized as follows:

- the higher-order singular control problem leads to reduction of the dimension of the singular optimal manifold (SOM) (see [5], [3]);

- to reach the singular optimal surface the impulse controllers are proposed (see, for example [5], [6]);

- in the paper [6], the combination of impulses and first order sliding modes controls are used to reach higher order singular optimal surface;

- it is suggested the concept of cheap control converting the singular optimization stabilization (SOS) problem into singularly perturbed *LQ* optimal problems ensuring the fast convergence of the solutions to the SOM (see, for example [7], [8], [9]).

The main disadvantage of solutions of *LQ* optimal problem is that they are not robust with respect to perturbations and parameter variations (see, for example [10], [11], [12]).

From the other hand LQ singular control is used for the conventional (first order) sliding surface design (see, for example [13] chapter 9, [14] chapter 4).

The design work flow can be summarized as:

- design a sliding surface based on SOM;
- a conventional sliding mode controller is designed to maintain the solutions on the designed SOM.

However, some issues was not covered by the conventional (first order) sliding modes approach in [13]:

- the case of singular optimal problems of higher order [5];
- the starting time at the SOM is not specified;
- the robustness of the proposed solution w.r.t. perturbation.

Contributions

In this paper we propose an Integral Higher Order Sliding Mode (IHOSM) approach [15] to arbitrary order singular LQ problems for uncertain systems. In this paper:

- a notion of order of singularity of LQ problem is introduced;
- based on the order of singularity the integral quasi-continuous HOSM is designed allowing to:
  - reach the SOM in a desired time instant;
  - maintain the system solution on the SOM;
  - ensure the insensitivity of the system trajectory with respect to the bounded matched uncertainties.

Generally the initial condition does not belong to SOM. So the difference in the approaches to solve SOS problem consists in the reaching phase to approach the SOM. The main approach to reach SOM is impulse jump from any initial conditions to the SOM(see, for example, [5], [3]);

For the case when it is not possible to realize impulses it is reasonable to start the SOS from some time moment.

In this situation:

- the first order sliding mode surface is suggested and the impulse jumps are using to reach sliding surface ([6]);

- the concept of the cheap control is used ensuring fast asymptotic convergence of solution to SOM which in this case is a slow motions manifold [7], [8], [9].

It is necessary to remark that the solution of SOS is not robust with respect to perturbations and parameter variations [10].

The goal of this paper is to design a trajectory for reaching of SOM in desired time and ensure the insensibility of predesigned trajectory w.r.t. bounded matching uncertainties.

II. PROBLEM STATEMENT

Consider the system

$$\dot{x} = Ax + B(u+\xi), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ is the scalar control input, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$,the perturbation

$$|\xi(t,x)| \leq L, \ L > 0. \tag{3}$$

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is Lebesgue-measurable and we will suppose that here and always bellow the solution of the system (1)-(2) is unique in the sense of Filippov ([16]). The SOS problem for dynamics (1) could not be not well posed because the system (1) is uncertain.

That is why together with system (1) consider the nominal singular LQ problem with dynamics:

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0, \tag{4}$$

We consider a quadratic in the states and a *“control-free”* cost as cost function, i.e,

$$J = \frac{1}{2}\int_{t_1}^{\infty} x^{\mathsf{T}}(t)\, Q x(t)\, dt, \tag{5}$$

where $Q = Q^{\mathsf{T}} \geq 0$, a semipositive definite matrix $t_1 > t_0$ and t_0 is the initial time instant. The cost function does not depend on control that is why this problem leads to the solution of the SOS problem.

The solution of nominal SOS problem (4) ,(5) lies on SOM. In this paper we introduce the order of singularity of SOS problem.

It is clear that initial condition (2) can not belong to SOM for SOS problem (4), (5). That is why we will design a trajectory starting with initial conditions (2) and arriving in SOM at the time moment t_1.

The third objective of this paper is to ensure the insensibility of proposed solution for nominal SOS problem (4),(5) with respect to uncertainties (3) in dynamics (1). With this aim a connection between the order of singularity SOS and order of sliding mode is established and IHOSM ensuring desired insensibility is designed.

III. SINGULAR OPTIMAL STABILIZATION FOR NOMINAL SYSTEM

A. Transformation to Brunovsky canonical form

Suppose that the pair (A,B) of the system (4) is controllable. Given the characteristic polynomial for the matrix A

$$\det(sI - A) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n, \tag{6}$$

there exist a non-singular matrix T transforming (4) to controllable canonical form [17]. Define $z := Tx$. Taking the derivative with respect to time, it can be rewritten as

$$\dot{z} = \tilde{A}z + \tilde{B}u, \quad z(t_0) = z_0 \tag{7}$$

where $\tilde{A} := TAT^{-1}$, $\tilde{B} := TB$ and

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \vdots & 1 \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{8}$$

The index (5) in the new variables has the form

$$J = \frac{1}{2}\int_{t_1}^{\infty} z^{\mathsf{T}}(t)\, \tilde{Q} z(t)\, dt; \quad \tilde{Q} := TQT^{-1} \tag{9}$$

where $\tilde{Q}$ has the following block structure:

$$\tilde{Q} = \left(\underbrace{\begin{matrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix}}_{n-k\ columns} \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}}_{k-columns} \right), \tag{10}$$

$$\tilde{Q}_{22} > 0, \quad \tilde{Q} \geq 0$$

the involved block matrices have dimension $\tilde{Q}_{22} \in \mathbb{R}$, $\tilde{Q}_{11} \in \mathbb{R}^{(n-k-1)\times(n-k-1)}$, $\tilde{Q}_{12}^{T} = \tilde{Q}_{21} \in \mathbb{R}^{1\times(n-k-1)}$.

Definition 1: For a given singular control problem, the Order of Singularity is defined as $i = k+1$, and k is the number of zero columns in (10).

Remark 2: The order of singularity can take any value from 1 to n. This corresponds to all the possible higher order singular control problems for the index (9), up to the point to minimize a functional depending only on the first coordinate.

Remark 3: It should be noticed that in [13] *LQ* singular optimal stabilization is used for conventional (first order) sliding mode surface design.

B. Transformation of the cost function

Let us use the connection between the order of singularity and the appropriate order of sliding mode. Define a new auxiliary variable v_i to avoid the cross terms in the same manner as [13]:

$$v_i = z_2 + L_i \mathbf{z}_1; \quad L_i := \left(\tilde{Q}_{22}\right)^{-1} \tilde{Q}_{12}^{T} \tag{11}$$

For variables $(\mathbf{z}_1, v_i)$ the index (9) takes the form:

$$J = \frac{1}{2}\int_{t_1}^{\infty} \left(\mathbf{z}_1^{\mathsf{T}}(t)\hat{Q}_{11}\mathbf{z}_1(t) + v_i^{\mathsf{T}}(t)\tilde{Q}_{22}v_i(t)\right) dt. \tag{12}$$

where $\hat{Q}_{11} = \tilde{Q}_{11} - \tilde{Q}_{12}\left(\tilde{Q}_{22}\right)^{-1}\tilde{Q}_{12}^{\mathsf{T}}$.

C. Design of the SOM

Now we can state the general higher order optimal sliding mode problem as follows: for any order of singularity i, minimize the cost function (12), subject to the dynamics (7). To find out the optimality conditions for each problem, we take again the last coordinate as the virtual control variable and minimize (12), subjected to the partial dynamics

$$\dot{\mathbf{z}}_1 = \left(A_i - B_i\left(\tilde{Q}_{22}\right)^{-1}\tilde{Q}_{12}^{\mathsf{T}}\right)\mathbf{z}_1 + B_i v_i \tag{13}$$

where the involved matrices have the form and dimension:

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{(n-i)\times(n-i)}; \tag{14}$$

$$B_i = \left.\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right\} \; n-i \text{ rows } .$$

$$\text{with } \mathbf{z}_1 \in \mathbb{R}^{n-i}, \; \tilde{Q}_{22} \in \mathbb{R}, \quad \tilde{Q}_{12} \in \mathbb{R}^{n-i}$$

The next theorem is a higher order extension of the results concerning the first-order SOS problem [13]. The assumption that (A,B) is controllable implies that $(\tilde{A}_i,\tilde{B}_i)$ is also controllable.

Lemma 4: The optimal value acting as virtual minimizing control in (12) is

$$z_2 = -\left(\tilde{Q}_{22}\right)^{-1}\left(B_i^T P_i + \tilde{Q}_{12}^{\top}\right)\mathbf{z}_1 \tag{15}$$

where P_i, is the solution of the matrix Riccati equation

$$P_i\tilde{A}_i + \tilde{A}_i^{\top} P_i + \hat{Q}_{11} - P_i B_i \left(\tilde{Q}_{22}\right)^{-1} B_i^{\top} P_i = 0 \tag{16}$$

with

$$\tilde{A}_i = A_i - B_i\left(\tilde{Q}_{22}\right)^{-1}\tilde{Q}_{12}^T; \quad \hat{Q}_{11} = \tilde{Q}_{11} - \tilde{Q}_{12}\left(\tilde{Q}_{22}\right)^{-1}\tilde{Q}_{12}^{\top}$$

Lemma 5: The order of singularity in (12), is equal to the order of sliding mode necessary to keep the solution in the SOM.

D. Design of Transient Trajectory to Reach SOM

In the general case the solution of (4) does not start on the SOM, therefore specification of the starting point of the SOM is required.

As it was shown in the previous section the optimal singular surface has the form

$$S_i(z) = z_2 + \left(\tilde{Q}_{22}\right)^{-1}\left[\left(B_i^T P_i + \tilde{Q}_{12}^{\top}\right)\mathbf{z}_1\right] \tag{17}$$

Suppose that at least for some $j, 0 \le j \le i-1$ we have $S_i^{(j)}(z_0) \neq 0$.We would like for the system trajectories reaches the ith order sliding mode set $S_i(z(t_1)) = \dot{S}_i(z(t_1)) = ... = S_i^{(i-1)}(z(t_1)) = 0$ at the reaching time t_1.

Let define the transient trajectory $\mu_i(t)$ for the system (4) solution during the time interval $t_0 \le t \le t_1$ as a polynomial of the form:

$$\begin{gathered} \mu_i(t) = (t-t_1)^i \times \\ \left(c_0 + c_1(t-t_0) + ... + c_{i-1}(t-t_0)^{i-1}\right). \end{gathered} \tag{18}$$

satisfying the initial conditions

$$\begin{gathered} \mu_i(t_0) {=} S_i(z_0), \; \dot{\mu}_i(t_0) {=} \dot{S}_i(z_0), ..., \mu_i^{(i-1)}(t_0) \\ {=} S_i^{(i-1)}(z_0). \end{gathered} \tag{19}$$

At the arrival time on SOM t_1 we have $\mu_i(t_1){=}\ \dot{\mu}_i(t_1),...,\mu_i^{(i-1)}(t_1){=}0$. The parameters c_i could be uniquely defined from (19). Define the function $t_1 - t_0$ as a positive-defined function of the initial conditions as

$$t_1 - t_0 = T_i\left(S_i(z_0), \dot{S}_i(z_0), ..., S_i^{(i-1)}(z_0)\right); \tag{20}$$

For any $\lambda, p =$const> 0 the function T_i could be uniquely defined as [15] :

$$T_i = \lambda\left(|S_i(z_0)|^{p/i} + \left|\dot{S}_i(z_0)\right|^{p/(i-1)} + \cdots + \left|S_i^{(i-1)}(z_0)\right|^p\right)^{1/p} \tag{21}$$

The function $\mu_i(t)$ is uniquely determined by (19), (18) and (20).

IV. CONTROL DESIGN

A. Nominal Equivalent Control

Based on (15), let us rewrite the sliding surface as:

$$\begin{gathered} S_i = z_2 + M_i\mathbf{z}_1 \\ M_i := \left(\tilde{Q}_{22}\right)^{-1}\left(B_i^T P_i + \tilde{Q}_{12}^T\right) \end{gathered} \tag{22}$$

Let us use $\left(\mathbf{z}_1^{\top}, S_i, \; \dot{S}_i, ..., S_i^{(i-1)}\right)$ as state variables. From equation (13) and (11), we obtain $\dot{\mathbf{z}}_1 = (A_i - B_iM_i)\mathbf{z}_1 + B_iS_i$. and deriving S_i i-times we will have $S_i^{(i)} = u_{eq0}\left(\mathbf{z}_1^{\top}, S_i, \; \dot{S}_i, ..., S_i^{(i-1)}\right) + \tilde{u} + \tilde{\xi}$.where u_{eq0} is a nominal (known) value of equivalent control. Resting the value of u_{eq0} we can rewrite the system in the form and, consequently,

$$\begin{gathered} \dot{\mathbf{z}}_1 = (A_i - B_iM_i)\mathbf{z}_1 + B_iS_i, \; S_i^{(i)} = \upsilon + \tilde{\xi}., \upsilon \\ = \tilde{u} - u_{eq0} \end{gathered} \tag{23}$$

B. IHOSM Design

Now from [18] we can conclude following theorem:

Theorem 6: The controller

$$\upsilon = \Phi_i\Psi_{i-1,i}(\Sigma_i, \dot{\Sigma}_i, ..., \Sigma_i^{(i-1)}), \Phi_i \ge ||\tilde{\xi}||, \tag{24}$$

where

$$\begin{gathered} \varphi_{0,i} = \Sigma_i; \; N_{0,i} = |\Sigma_i|, \\ \Psi_{0,i} = \varphi_{0,i}/N_{0,i} = sign\Sigma_i; \varphi_{l,i} \\ = \Sigma_i^{(l)} + \beta_i N_{l-1,i}^{(i-l)/(i-l+1)}\Psi_{l-1,i} \\[1em] N_{l,i} = \left|\Sigma_i^{(l)}\right| + \beta_i N_{l-1,i}^{(i-l)/(i-l+1)} \\ \Psi_{l-1,i} = \varphi_{l,i}/N_{l,i}, \\ \Sigma_i(t,z) {=} \begin{cases} S_i(z) - \mu_i(t), & t_0 \le t \le t_1 \\ S_i(z), & t \ge t_1 \end{cases} \end{gathered} \tag{25}$$

for sufficiently big $\Phi_i \ge ||\tilde{\xi}||$ established the finite-time stable r-sliding mode $S_i(z_0) = \dot{S}_i(z(t)) = ... = S_i^{(i-1)}(z(t)) \equiv 0$ for $t \ge t_1$. The equality $S_i(z(t)) = \mu_i(t)$, is kept during the transient process $t_1 \ge t \ge t_0$.

Remark 7: To adjust the parameters of the controllers we follow [18].The quasi-continuous higher order controller ([19],[18]) for $i-1,2,3,4$ takes the form:

$$\upsilon_1 = -\Phi_1 sign\Sigma_1,$$

1) $\upsilon_2 = -\Phi_2\left(\dot{\Sigma}_2 + |\Sigma_2|^{1/2} sign\Sigma_2\right) / \left(\left|\dot{\Sigma}_2\right| + |\Sigma_2|^{1/2}\right)$,

$$\upsilon_3 = -\Phi_3 \frac{\left[\ddot{\Sigma}_3 + 2\left(\left|\dot{\Sigma}_3\right| + |\Sigma_3|^{2/3}\right)^{-1/2}\left(\dot{\Sigma}_3 + |\Sigma_3|^{2/3} sign\Sigma_3\right)\right]}{\left[\left|\ddot{\Sigma}_3\right| + 2\left(\left|\dot{\Sigma}_3\right| + |\Sigma_3|^{2/3}\right)^{1/2}\right]},$$

$$\upsilon_4 = -\Phi_4\varphi_{3,4}/N_{3,4},$$

$$\begin{gathered} \varphi_{3,4} = \dddot{\Sigma}_4 + 3\left[\ddot{\Sigma}_4 {+} \left(\left|\dot{\Sigma}_4\right| + 0.5|\Sigma_4|^{3/4}\right)^{-1/3}\right. \\ \left.\times\left(\dot{\Sigma}_4 + 0.5|\Sigma_4|^{3/4} sign\Sigma_4\right)\right] \\ \times\left[\left|\ddot{\Sigma}_4\right| + \left(\left|\dot{\Sigma}_4\right| + 0.5|\Sigma_4|^{3/4}\right)^{2/3}\right]^{1/2}, \end{gathered} \tag{26}$$

and

$$N_{3,4} = |\dddot{\Sigma}_4| + 3\left[\ddot{\Sigma}_4 + \left(|\dot{\Sigma}_4| + 0.5\,|\Sigma_4|^{3/4}\right)^{-2/3}\right]^{1/2}, \tag{27}$$

V. DESCRIPTION OF THE ALGORITHM

To summarize the procedure design we have the following algorithm:

Step 1: Transformation of the system (4) into controllability canonical form (7)

Step 2: Transformation of the functional (10) into canonical form (12) .

Step 3: Design the surface S_i solving the corresponding algebraic Riccati equation (16).

Step 4: Calculation of nominal equivalent control (23) compensating the known part of the system.

Step 5: Design the polynomial (18) corresponding to the reaching phase.

Step 6: Design the corresponding quasi-continuous IHOSM control (24).

VI. CONCLUSION

This paper shows natural connection between order of singularity of the LQ and order of sliding modes. The SOM is considered as the sliding manifold for HOSM of corresponding order. The Integral quasi-continuous HOSM is suggested.

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