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Distributed Stabilization of 3D Circular Formations

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Abstract— **The paper presents distributed control design to stabilize circular formations of steered particles in three-dimensional space. In formation, the particles are required to follow equal radius circular paths with common orientation, but not necessarily common center. The formation is given by specifying desired separations of the centers of the circular paths and desired relative headings. The information exchange between the particles is modeled by a directed graph which is assumed to have a spanning tree. Control design is based on a hierarchical approach utilizing a reduction principle for asymptotic stability of closed sets.**

I. Introduction

Interest in modern systems of aerial, land and sea vehicles together with availability of modern robotic platforms prompt escalating interest in formation control. Influential research in this area includes the works [1], [2], [3], [4]. Formations in three dimension have received significant attention in this area [5], [6], [7], [8], and several results addressed circular formations. For example, in [9] the authors addressed constant bearing (CB) pursuit strategies for rectilinear, helical and circular formations for two kinematic particles. They also gave conditions for existence of relative equilibrium corresponding to planer circular formation under CB pursuit dynamics. In [10] a Lyapunov-based control design was used to design decentralized algorithms to stabilize formations in a flow field. The design led to relative equilibria corresponding to parallel and helical formations, and a special case of circular formations on the surface of a rotating sphere. In [11] a methodology to stabilize relative equilibria in a model of identical steered particles was presented. The stabilization problem was viewed as a consensus problem of the Lie algebra of the dynamical system, and the relative equilibria corresponded to parallel, helical and circular formations. The stabilizing feedbacks were first designed based on an all-to-all communications model and then extended to undirected time-varying setting using consensus estimators.

The work presented here addresses general circular formations for a model of steered particles similar to the ones used in the previous examples. The approach followed starts by posing the formation control problem as the requirement to asymptotically stabilize a goal set. Control design is then carried out to stabilize that goal set while taking the communication constraints into account. Different formations are obtained by specifying control parameters representing the relative positions of the centers of the circular paths, desired relative headings of the particles and the common orientation of the circular paths. The control design methodology relies on a hierarchical approach which breaks down the problem into three simpler, but nested, sub-problems. The feedbacks obtained are decentralized and static. The tools used here for stabilizing 3D circular formations, namely set stabilization and hierarchical design, has also been used to stabilize planar circular formations for dynamic unicycles in [12].

The work in [11] addressed a circular formation problem which bears certain similarities to the problem addressed here, but with a number of differences: in [11] the centers of the circular paths belong to the same axis, perpendicular to their planes. One cannot specify formations with arbitrary spacings, between the centers, or centers belonging to the same axis but not orthogonal to the particles planes. In addition, in [11], one cannot specify the ordering of the particles or their relative headings. Moreover, the feedback presented in [11] is static only if all-to-all communication applies. In the case of limited communication the authors used consensus filters which require each particles to broadcast the states of its controller to its neighbouring particles. Here, arbitrary formations can be obtained, using static feedbacks, with arbitrary spacing between the centers, arbitrary orientations of the planes of the circular paths, and arbitrary ordering and relative orientations of the particles.

The paper is organized as follows. In Section II, the three dimensional circular formation problem is posed. In Section III, this problem is formulated in the set stabilization framework. The hierarchical design methodology used to solve the problem is presented in Section IV. This section also presents a corollary to a reduction principle for asymptotic stability of closed sets. In Section V, control design is performed and distributed feedbacks are derived. Finally, simulation results are presented in Section VI, and Section VII closes with concluding remarks.

The following notation is used throughout the paper. If $a_1, \ldots, a_n$ are vectors or scalars then $\mathrm{col}(a_1, \ldots, a_n)$ denotes the vector $[a_1^\top, \ldots, a_n^\top]^\top$. If A, B are two matrices, $A \otimes B$ denotes the their Kronecker product. Given two sets $\mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{R}^n$, $\mathcal{X}_1 \times \mathcal{X}_2$ denotes their Cartesian product. The index set $\{1, \cdots, n\}$ is denoted by $\mathbf{n}$, and the n-vector of ones is denoted by $\mathbf{1}$. For $x \in \mathbb{R}$, $x \bmod 2\pi$ denotes its value modulo 2π. If $\theta, x \in \mathbb{R}$, then $x = \theta \bmod 2\pi$ states that $x \in \{\theta + 2\pi k, k\mathbb{Z}\}$. Similarly, if $\theta, x \in \mathbb{R}^n$, then $x = \theta \bmod 2\pi$ states that $x_i = \theta_i \bmod 2\pi$, $i = 1, \cdots, n$. $\mathsf{sat}(\mathbb{R})$ will be used to denote the class of C^1 functions $\phi : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \mathbb{R}$, $\phi(y) > 0$ and $|\phi(y)y| < 1$. Finally, for a vector $x \in \mathbb{R}^3$, $x^\times$ denotes the

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skew symmetric matrix

$$x^\times := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

II. Problem statement

Consider a collection of $n \geq 2$ kinematic particles, moving in three-dimensional Euclidean space and modeled by

$$\begin{aligned} \dot{x}^i &= u_i \mathbb{C}_1^i \\ \dot{\mathbb{C}}^i &= \mathbb{C}^i (w^i)^\times \end{aligned} \qquad i \in \mathbf{n}, \tag{1}$$

where $x^i \in \mathbb{R}^3$ is the position of the particle, $\mathbb{C}^i \in \mathrm{SO}(3)$, with columns $\mathbb{C}_1^i, \mathbb{C}_2^i, \mathbb{C}_3^i$, represents an orthonormal frame attached to each particle, $u_i \in \mathbb{R}$ is the forward speed and $w^i \in \mathbb{R}^3$ is the angular velocity in particle frame, refer to Figure 1. u_i and w^i will be seen as control inputs. Denote

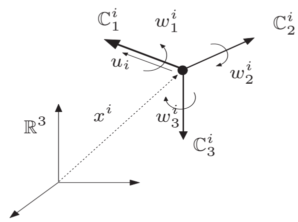

Fig. 1. Steered particle

the state of the i-th particle by $\chi^i = (x^i, \mathbb{C}^i) \in \mathcal{X}^i := \mathbb{R}^3 \times \mathrm{SO}(3)$, and let $\chi = (\chi^1, \cdots, \chi^n)$ denote the collective state of the n particles. The collective state space is $\mathcal{X} := \mathcal{X}^1 \times \cdots \times \mathcal{X}^n$. It is assumed here that each particle has access to the orientation of its attached frame, and that it can exchange relative positions and frame orientations with some other particles. The information flow shall be modeled by a sensor digraph $\mathcal{G}$. Each node of $\mathcal{G}$ represents a particle, where an edge from node i to node j of $\mathcal{G}$ means that particle i has access to its relative position and frame orientation with respect to particle j. The Laplacian of $\mathcal{G}$ is denoted by L, L^i denotes its i-th row and $L_{(3)} := L \otimes I_3$, $L_{(3)}^i = L^i \otimes I_3$, where I_3 is the 3×3 identity matrix. Refer to [13] for an overview on algebraic graph theory and digraphs. Here, it assumed that $\mathcal{G}$ is static, and that it has a globally reachable node, i.e. a node with arcs from every other node in the digraph. Equivalently, the graph has a spanning tree. A useful characterization of this property, which is used in the sequel, is given in [14] as follows.

Lemma II.1 (Lemma 2, [14])**.** *The digraph $\mathcal{G}$ has a globally reachable node if and only if* 0 *is a simple eigenvalue of* L.

By this lemma, if a digraph with Laplacian L has a globally reachable node then $\ker L = \operatorname{span} \mathbf{1}$ where $\ker$ denotes the kernel.

In this paper the following problem is investigated.

Circular Formation Stabilization Problem in Three Dimension (3D-CFP). Consider the n kinematic particles in (1). For a given sensor digraph $\mathcal{G}$ with a globally reachable node, design distributed feedbacks meeting the following specifications:

(i) *Circular path following.* For a unit vector $a \in \mathbb{R}^3$ and a suitable set of initial conditions, each particle should follow a circular path of radius $r > 0$, whose plane is orthogonal to a, and whose center is stationary but dependent on the initial conditions. The particles traverse the paths in a desired direction, clockwise or counter-clockwise, with respect to a, and at steady-state all particles should have forward speed $v > 0$.

(ii) *Formation stabilization.* The particles should converge to a formation given by desired relative positions of the centers of the circular paths in specification (i).

(iii) *Synchronization of headings.* At steady state, the particles should satisfy desired relative orientations given by desired differences of the headings $\mathbb{C}_1^i$'s.

Note that distributed feedbacks above means feedbacks that are consistent with the sensor digraph in the sense that, when computing its own feedback, particle i has only access to its own frame, and relative positions and frame orientations of particles that are visible to it according to $\mathcal{G}$,

This paper provides solution of 3D-CFP in the case where the information flow graph is bidirectional, which corresponds to the situation where the Laplacian L is symmetric. The control design provides circular path following in the counter-clockwise direction, with respect to a, but can be easily modified to achieve clockwise path following.

III. Problem formulation as set stabilization

The solution of 3D-CFP presented next relies on viewing the problem as a set stabilization one. 3D-CFP can be reformulated in this context as follows. First note that specification (i) is equivalent to making the set

$$\begin{aligned} \mathcal{C} = \{\chi \in \mathcal{X} : a \cdot \mathbb{C}_1^i = 0,\ u^i(\chi) = v, \\ w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i = \frac{v}{r} a,\ i \in \mathbf{n}\} \end{aligned}$$

attractive [1]. Notice that the angular velocity, in inertial frame, of particle i equals $w_1^i(\chi)\mathbb{C}_1^i + w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i$, hence on $\mathcal{C}$, where $a \cdot \mathbb{C}_1^i = 0$, the particle follows a circular path with stationary center irrespective of $w_1^i(\chi)$. Let

$$x_c^i(\chi^i) = x^i + r a \times \mathbb{C}_1^i \tag{2}$$

and denote $x_c(\chi) = \operatorname{col}(x_c^1(\chi^1), \cdots, x_c^n(\chi^n))$. Refer to Figure 2 for different formation variables. Note that if $a \cdot \mathbb{C}_1^i = 0$ then $x_c^i(\chi^i)$ lies at a distance r from x^i and the vector $x_c^i(\chi^i) - x^i$ is perpendicular to both a and $\mathbb{C}_1^i$, refer to Figure 2. From this it follows that $x_c^i(\chi^i)$ is the center of the circular path described in specification (i) of 3D-CFP. To specify the formation in specification (ii), one way is to specify vectors $p_1, \cdots, p_{n-1} \in \mathbb{R}^3$, and enforce the relations $x_c^i - x_c^{i+1} = p_i$, $i = 1, \cdots, n-1$. Using

[1]Definitions of stability and attractivity of sets, used in this paper, and their local and relative counterparts can be found in [15].

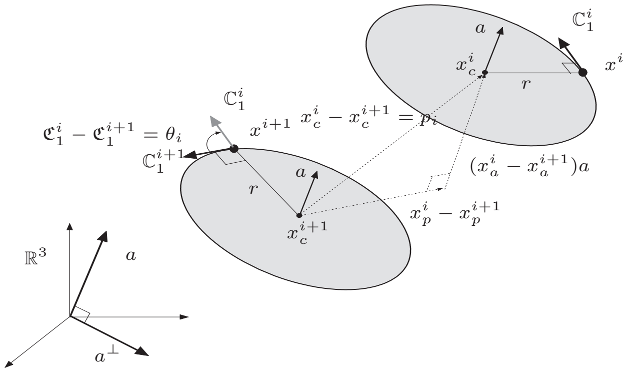

Fig. 2. Particles i and $i+1$ in formation

this, specification (ii) becomes equivalent to making the set $\{\chi \in \mathcal{C} : x_c^i - x_c^{i+1} = p_i,\ i = 1, \cdots, n-1\}$ attractive. Let $\alpha_n = 0$, $\alpha_i = \sum_{j=i}^{n-1} p_j$, $i = 1, \cdots, n-1$, and $\alpha = \mathrm{col}(\alpha_1, \cdots, \alpha_n)$. Using this, specification (ii) is then equivalent to making the set

$$\mathcal{F} = \{\chi \in \mathcal{C} :\ L_{(3)}(x_c(\chi) - \alpha) = 0\}.$$

attractive. The vector α will be called the *formation vector*. Note that α defines a formation modular a translation in $\mathbb{R}^3$. Let $\bar{\alpha} \in \mathbb{R}^3$ be a constant vector. It follows from Lemma II.1 that $L_{(3)}(x_c - \alpha - [\bar{\alpha}^\top \ \cdots \ \bar{\alpha}^\top]^\top) = L_{(3)}(x_c - \alpha)$.

Now, let $a^\perp$ be any unit vector perpendicular to a and $\mathfrak{C}_1 = [\mathfrak{C}_1^1 \ \cdots \ \mathfrak{C}_1^n]^\top$ where $\mathfrak{C}_1^i$ denotes the angle that $\mathbb{C}_1^i$ makes with $a^\perp$. To specify the desired relative orientations in specification (iii), one way is to specify desired angles $\theta_1, \cdots, \theta_{n-1}$ and enforce the relations $\mathfrak{C}_1^i - \mathfrak{C}_1^{i+1} = \theta_i$, $i = 1, \cdots, n-1$. Let $\beta_n = 0$, $\beta_i = \sum_{j=i}^{n-1} \theta_j$, $i = 1, \cdots, n-1$, and $\beta = [\beta_1 \ \cdots \ \beta_n]^\top$. Using this, it then follows that specification (iii) is equivalent to making the set

$$\Gamma = \{\chi \in \mathcal{F} : L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi\}$$

attractive. The vector of n angles β will be called the *synchronization vector*. Note that β defines unique relative orientations up to circular rotation. From the previous development, meeting specifications (i), (ii) and (iii) of 3D-CFP simultaneously is equivalent making the goal set

$$\Gamma = \{\chi \in \mathcal{X} : L_{(3)}(x_c(\chi) - \alpha) = 0, L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi,$$
$$a \cdot \mathbb{C}_1^i = 0, u_i(\chi) = v, w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i = \frac{v}{r}a, i \in \mathbf{n}\}. \tag{3}$$

attractive. To avoid large transients during disturbance recovery, one adds to this a requirement of stability. Hence 3D-CFP can be seen as the requirement to asymptotically stabilize the goal set Γ.

IV. Solution Methodology

The solution of 3D-CFP relies on a hierarchical approach which leverages recent results on reduction theorems for asymptotic stability of closed sets [15].

A. Design Hierarchy

The approach presented here relies on breaking down the solution of 3D-CFP into that of three simpler sub-problems addressing the stability of three nested sets $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$.

Prob 1. (*Vertical formation stabilization*) Make the particles approach planes orthogonal to a with separations consistent with the desired formation. This corresponds to stabilizing the set

$$\Gamma_1 = \{\chi \in \mathcal{X} : L(x_a - \alpha_a) = 0,\ a \cdot \mathbb{C}_1^i = 0,\ i \in \mathbf{n}\} \tag{4}$$

where $x_a = \mathrm{col}(x_a^1, \cdots, x_a^n)$, $x_a^i = x^i \cdot a$, $i \in \mathbf{n}$, (the vector of projections of x^i on a), $\alpha_a = [\alpha_1 \cdot a \ \cdots \ \alpha_n \cdot a]^\top$ (the vector of projections of α_i on a).

Prob 2. (*Horizontal formation stabilization*) On the planes in **Prob 1**, make the particles approach the formation in specification (ii) of 3D-CFP. This corresponds to stabilizing the set $\{\chi \in \Gamma_1 :\ L_{(3)}(x_c - \alpha) = 0\}$. Consider the centers of rotation defined in (2), define their projection orthogonal to a as

$$x_p^i = x_c^i - (x_c^i \cdot a)a,$$

and denote $x_p = \mathrm{col}(x_p^1, \cdots, x_p^n)$. Also, define the projection of α orthogonal to a by

$$\alpha_p = \mathrm{col}(\alpha_1 - (\alpha_1 \cdot a)a, \cdots, \alpha_n - (\alpha_n \cdot a)a).$$

Using this, **Prob 2** corresponds to stabilizing the set $\Gamma_2 \subset \Gamma_1$ where

$$\Gamma_2 = \{\chi \in \Gamma_1 :\ L_{(3)}(x_p - \alpha_p) = 0\} \tag{5}$$

Prob 3. (*Headings synchronization*) On Γ_2, make the particles acquire the desired relative headings forward speed. This corresponds to stabilizing the set $\Gamma_3 \subset \Gamma_2$ where

$$\begin{aligned}\Gamma_3 = \{\chi \in \Gamma_2 : L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi, u^i(\chi) = v,\\ w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i = \frac{v}{r}a,\ i \in \mathbf{n}\}\end{aligned} \tag{6}$$

Notice that $\Gamma_3 = \Gamma$ in (3). The hierarchical nature of the previous three problems stems from the fact that for $i = 2, 3$, **Prob** i is met (i.e. $\chi \in \Gamma_i$) only if **Prob** $i-1$ is met.

In the next section, the above sub-problems will be solved in the following three steps:

Step 1. Let

$$w^i(\chi) = \hat{w}^i(\chi) + \bar{w}^i(\chi) + \tilde{w}^i(\chi), \quad i \in \mathbf{n}. \tag{7}$$

First, pick $u^i(\chi)$, $\hat{w}^i(\chi)$, $\bar{w}^i(\chi)$, such that Γ_1, in (4), is invariant for the dynamics

$$\dot{x}^i = u^i(\chi)\mathbb{C}_1^i,\ \dot{\mathbb{C}}^i = \mathbb{C}^i(\hat{w}^i(\chi) + \bar{w}^i(\chi))^\times,\ i \in \mathbf{n}. \tag{8}$$

For any uniformly bounded functions $w_1^i(\chi)$, $i \in \mathbf{n}$, consider the following choice of $u^i(\chi)$, $\hat{w}^i(\chi)$, $\bar{w}^i(\chi)$

$$\begin{aligned}
u_i(\chi) &= v + \hat{k}\hat{u}_i(\chi)\\
\hat{w}_2^i(\chi) &= \frac{u_i(\chi)}{r}(a \cdot \mathbb{C}_2^i),\ \hat{w}_3^i(\chi) = \frac{u_i(\chi)}{r}(a \cdot \mathbb{C}_3^i)\\
\bar{w}_2^i(\chi) &= \frac{\delta_i(\chi)}{r}(a \cdot \mathbb{C}_2^i),\ \bar{w}_3^i(\chi) = \frac{\delta_i(\chi)}{r}(a \cdot \mathbb{C}_3^i)
\end{aligned} \tag{9}$$

where $\hat{k} > 0$ is a design constant and $\hat{u}_i, \delta_i : \mathcal{X} \to \mathbb{R}$, $i \in \mathbf{n}$, are smooth bounded functions to be specified in steps 2 and

3 respectively. Since on Γ_1, $a \cdot \mathbb{C}_1^i = 0$ for all $i \in \mathbf{n}$ then, using (9), it follows that

$$(\hat{w}_2^i(\chi) + \bar{w}_2^i(\chi))\mathbb{C}_2^i + (\hat{w}_3^i(\chi) + \bar{w}_3^i(\chi))\mathbb{C}_3^i = \frac{v + \hat{k}\hat{u}_i(\chi) + \delta_i(\chi)}{r} a, \quad i \in \mathbf{n},$$

on Γ_1. This means that the component of angular velocity perpendicular to $\mathbb{C}_1^i$ is in the direction of a, and so, the choice (9) makes Γ_1 invariant for (8).

It is then required to design $\tilde{w}_2^i(\chi)$, $\tilde{w}_3^i(\chi)$ to asymptotically stabilize Γ_1, and such that $\tilde{w}_2^i(\chi) = \tilde{w}_3^i(\chi) = 0$ on Γ_1.

Step 2. Design $\delta_i(\chi)$ such that Γ_2, in (5), is asymptotically stable for the dynamics (8), and such that $\delta_i(\chi) = 0$ on Γ_2, for all $i \in \mathbf{n}$. Notice that, by using (9), one has

$$\hat{w}_2^i(\chi)\mathbb{C}_2^i + \hat{w}_3^i(\chi)\mathbb{C}_3^i = \frac{v + \hat{k}\hat{u}_i(\chi)}{r} a, \quad i \in \mathbf{n}.$$

Therefore, since $u_i(\chi) = v + \hat{k}\hat{u}_i(\chi)$, $i \in \mathbf{n}$, on Γ_2 each particle is following a circular path with radius r and stationary center. It follows that Γ_2 is invariant for

$$\dot{x}^i = u^i(\chi)\mathbb{C}_1^i, \ \dot{\mathbb{C}}^i = \mathbb{C}^i(\hat{w}^i(\chi))^\times, \ i \in \mathbf{n}. \tag{10}$$

Step 3. Design $\hat{u}^i(\chi)$ such that Γ_3, in (6), is asymptotically stable relative to Γ_2, for the dynamics (10).

Following the previous three steps one gets the properties that Γ_1 is asymptotically stable, Γ_2 is asymptotically stable relative to Γ_1 and Γ_3 is asymptotically stable relative to Γ_2. The question then becomes whether these three properties imply that Γ_3 (the goal set) is asymptotically stable for the closed-loop system. The answer to this question relies on what is called the reduction theorem. This theorem has been studied for the stability of compact sets in [16], and recently, for the stability of non-compact sets in [15]. Using these results, the previous question is answered in the next section.

B. *Reduction Principle*

Consider a dynamical system described by

$$\dot{x} = f(x) \tag{11}$$

with state space a domain $\mathcal{X} \subset \mathbb{R}^n$ and f locally Lipschitz on $\mathcal{X}$, and let $\mathbb{R}^+ = [0, +\infty)$. Let $\phi(t, x_0)$ denote the solution of (11) for initial condition x_0, and consider also the following boundedness notion.

Definition IV.1 (Local uniform boundedness (LUB))**.** System Σ is *locally uniformly bounded near* Γ *(LUB)* if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$, where $B_\lambda(x)$, $B_m(x)$ denotes the open balls with radii λ, m centered at x.

Theorem III.2 in [15] addresses the following question. Consider the dynamical system (11), and suppose that two closed sets $\Gamma \subset \mathcal{O} \subset \mathcal{X}$ are positively invariant, and that Γ is asymptotically stable relative to $\mathcal{O}$. Under what conditions is Γ asymptotically stable relative to the state space $\mathcal{X}$? Using that result, the next corollary answers the question posed at the end of Section IV-A.

Corollary IV.2. *Consider three closed, and unbounded sets $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1$, subsets of $\mathcal{X} \subset \mathbb{R}^n$, that are positively invariant for* (11)*. If the following conditions are satisfied then Γ_3 is asymptotically stable for*(11)*.*

- *Γ_1 is asymptotically stable.*
- *For $i = 2, 3$, Γ_i is asymptotically stable relative to Γ_{i-1}.*
- *The system is LUB near Γ_3.*

V. CONTROL DESIGN

In this section the three hierarchical sub-problems presented in Section IV-A are solved using the three steps presented therein.

A. **Step 1.** *Asymptotic stabilization of* Γ_1

Recall Γ_1 in (4), $w^i(\chi)$ in (7), and the choice for $u_i(\chi), \hat{w}^i(\chi), \bar{w}^i(\chi)$ given in (9). Let

$$\begin{aligned}\tilde{w}_2^i &= \tilde{k}(a \cdot \mathbb{C}_1^i + k_\phi \phi(L^i(x_a - \alpha_a))(L^i(x_a - \alpha_a)))(a \cdot \mathbb{C}_3^i)\\ \tilde{w}_3^i &= -\tilde{k}(a \cdot \mathbb{C}_1^i + k_\phi \phi(L^i(x_a - \alpha_a))(L^i(x_a - \alpha_a)))(a \cdot \mathbb{C}_2^i)\end{aligned} \tag{12}$$

where $\phi \in \mathsf{sat}(\mathbb{R})$ and $\tilde{k} > 0$, $1 > k_\phi > 0$ are design constants. Using this, the following result follows.

Proposition V.1. *There exists $\hat{k}, \tilde{k}^* > 0$ such that for all $\tilde{k} \in (0, \tilde{k}^*)$ the feedbacks* (9)*,* (12)*, with $w_1^i(\chi)$, $\hat{u}_i(\chi)$, $\delta_i(\chi)$, $i \in \mathbf{n}$, any uniformly bounded C^1 functions, render Γ_1 asymptotically stable for* (1)*. In addition, for all initial conditions near Γ_1 the particles approach fixed planes.*

The proof is omitted here due to space limitations.

B. **Step 2.** *Asymptotic stabilization of Γ_2 relative to Γ_1*

The objective here is to design the functions $\delta_i(\chi)$, $i \in \mathbf{n}$, to stabilize the set Γ_2, in (5), relative to Γ_1. Let

$$\delta_i(\chi) = \bar{k}\phi((L^i_{(3)}(x_p - \alpha_p))^\top \mathbb{C}_1^i)(L^i_{(3)}(x_p - \alpha_p))^\top \mathbb{C}_1^i, \ i \in \mathbf{n} \tag{13}$$

where $\phi \in \mathsf{sat}(\mathbb{R})$ and $\bar{k} > 0$. Proposition V.2 below shows that this feedback solves the objective above.

Proposition V.2. *There exists $\hat{k}, \bar{k}^* > 0$ such that for all $\bar{k} \in (0, \bar{k}^*)$, the feedbacks* (9)*,* (12)*,* (13) *with $w_1^i(\chi)$, $\hat{u}_i(\chi)$ any uniformly bounded C^1 functions and $\tilde{k}$ as in Proposition V.1, render Γ_2 asymptotically stable, for* (1)*, relative to Γ_1. In addition, for all initial conditions in some neighbourhood of Γ_2, and which belongs to Γ_1, the particles approach fixed formations.*

The proof is omitted here due to space limitations.

C. **Step 3.** *Asymptotic stabilization of Γ_3 relative to Γ_2*

Given a synchronization vector β, one is now left with the objective of designing $\hat{u}_i(\chi)$, $i \in \mathbf{n}$, so as to stabilize the set

$$\Gamma_3 = \{\chi \in \Gamma_2 : \ L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi, \ u^i(\chi) = v, \\ w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i = \frac{v}{r} a, \ i \in \mathbf{n}\},$$

relative to Γ_2. Remember that $\mathfrak{C}_1$ was defined as $\mathfrak{C}_1 = [\mathfrak{C}_1^1 \ \cdots \ \mathfrak{C}_1^n]^\top$ where $\mathfrak{C}_1^i$ denotes the angle that $\mathbb{C}_1^i$ makes with $a^\perp$. Notice that, from the previous step, on Γ_2 all particles follow circles of radius r and fixed center. Their motion, therefore, can be completely characterized by $\dot{\mathfrak{C}}^i$. Since on Γ_2, $a \cdot \mathbb{C}_1^i = 0$, for all $i \in \mathbf{n}$, it is straight forward to see that

$$\dot{\mathfrak{C}}_1^i = \frac{v + \hat{k}\hat{u}_i(\chi)}{r}, \quad i \in \mathbf{n}. \tag{14}$$

Thus, to stabilize Γ_3 relative to Γ_2 one needs to design C^1 feedbacks $\hat{u}_i(\chi)$ that stabilize the set $\mathcal{S} = \{\mathfrak{C}_1 : L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi\}$ for (14), and in addition on $\{\chi \in \Gamma_3 : \mathfrak{C}_1 \in \mathcal{S}\}$ it must hold that $\hat{u}_i(\chi) = 0$ for all $i \in \mathbf{n}$. The asymptotic stability of Γ_3, relative to Γ_2, then follows by continuity. This objective is fulfilled using the feedback

$$\hat{u}_i(\chi) = -\sin(L^i(\mathfrak{C}_1 - \beta)), \quad i \in \mathbf{n}, \tag{15}$$

as shown in the following proposition.

Proposition V.3. *For any $\hat{k} > 0$, the control law* (15) *stabilizes the set $\mathcal{S} = \{\mathfrak{C}_1 : L(\mathfrak{C}_1 - \beta) = 0 \bmod 2\pi\}$ for* (14)*. Therefore, the feedbacks* (9), (12), (13), (15)*, with $w_1^i(\chi)$ any C^1 uniformly bounded functions and $\tilde{k}$, $\bar{k}$ as in Propositions V.1, V.2 respectively, render Γ_3, in* (6)*, asymptotically stable relative to Γ_2, for* (1).

The proof is omitted here due to space limitations.

D. Solution of 3D-CFP

From the previous developments, the following solution of 3D-CFP follows.

Theorem V.4. *Consider the system of n kinematic particles* (1)*, and assume that the sensor digraph $\mathcal{G}$ is bidirectional and has a globally reachable node. Let $\phi \in \mathsf{sat}(\mathbb{R})$. There exists $\hat{k}, \tilde{k}^*, \bar{k}^* > 0$ such that for all $\tilde{k} \in (0, \tilde{k}^*)$ and $\bar{k} \in (0, \bar{k}^*)$, the feedback*

$$u_i(\chi) = v - \hat{k}\sin(L^i(\mathfrak{C}_1 - \beta)) \tag{16}$$

and (17)*, for $i \in \mathbf{n}$, with $w_1^i(\chi)$ any uniformly bounded functions, renders the goal set Γ, in* (3)*, asymptotically stable for the closed-loop system* (1)-(16)-(17)*, hence solving 3D-CFP. In addition, for all initial conditions in some neighbourhood of Γ the particles approach fixed formations.*

Proof: The feedback (16)-(17) is uniformly bounded and hence all solutions of the closed-loop system are globally defined. Propositions V.1, V.2 and V.3 entails that Γ_1 is asymptotically stable, Γ_2 is asymptotically stable relative to Γ_1, and Γ_3 is asymptotically stable relative to Γ_2, for the closed-loop system. By this and Corollary IV.2, it follows that Γ_3, which is equal to Γ, is asymptotically stable if the closed-loop system is LUB near it. Since $\mathbb{C}^i \in$ SO(3), it is then sufficient to show that there exists $\mathfrak{X} > 0$ such that for all initial conditions in some neighbourhood of Γ_3, it holds that $\|x^i(t) - x^i(0)\| \leq \mathfrak{X}$, for all $i \in \mathbf{n}$. Using (2), to prove that property it is sufficient to show that $\|x_c^i(t) - x_c^i(0)\|$ has a uniform bound for all $x_c^i(0)$ in a neighbourhood of Γ_3. Notice that x_c^i can be expressed as $x_c^i = x_{ca}^i a + x_p^i$, where $x_{ca}^i = x_c^i \cdot a = x_a^i + r(a \times \mathbb{C}_1^i) \cdot a$. From Proposition V.1, it follows that $x_a^i(t)$, and hence $x_{ca}^i(t)$, are uniformly bounded for all initial conditions in some neighbourhood of Γ_1. Therefore, one only needs to prove that $\|x_p^i(t) - x_p^i(0)\|$ has a uniform bound for all $x_p^i(0)$ in a neighbourhood of Γ_2. Notice that this would imply that the closed-loop system is LUB near Γ_2, and hence also near Γ_3. This can be shown using the averaging theory [17] and some manipulations which are omitted here due to space limitations. From this it would follow that both Γ_2 and Γ_3 are asymptotically stable for the closed-loop system. In addition it can also be shown that on that neighbourhood $L_{(3)}(x_c(t) - \alpha)$ vanishes exponentially and so the particles approach fixed formations. ■

Using the fact that, in Theorem V.4, the closed-loop system is LUB near Γ_2, the following result follows directly.

Corollary V.5. *Consider the system of n kinematic particles* (1)*, and assume that the sensor digraph $\mathcal{G}$ is bidirectional and has a globally reachable node. Let $\phi \in \mathsf{sat}(\mathbb{R})$. There exists $\hat{k}, \tilde{k}^*, \bar{k}^* > 0$ such that for all $\tilde{k} \in (0, \tilde{k}^*)$ and $\bar{k} \in (0, \bar{k}^*)$, the feedback $u_i(\chi) = v$ and* (17)*, for $i \in \mathbf{n}$, with $w_1^i(\chi)$ any uniformly bounded functions, solves for specifications (i) and (ii) of 3D-CFP, simultaneously, by rendering the set $\{\chi \in \mathcal{X} : \ a \cdot \mathbb{C}_1^i = 0, \ u_i(\chi) = v, \ w_2^i(\chi)\mathbb{C}_2^i + w_3^i(\chi)\mathbb{C}_3^i = \frac{v}{r}a, \ i \in \mathbf{n}, L_{(3)}(x_c(\chi) - \alpha) = 0\}$ asymptotically stable for the closed-loop system.*

Remarks.

(1) The feedback in Theorem V.4 is distributed as required in 3D-CFP. This is obvious as $u^i(\chi)$ depends on $L^i(\mathfrak{C}_1 - \beta)$ and $w_2^i(\chi)$, $w_3^i(\chi)$ depend on $L^i(\mathfrak{C}_1 - \beta)$, $L^i(x_a - \alpha_a)$ and $L^i_{(3)}(x_p - \alpha_p)$, i.e., apart from $\mathbb{C}^i$, the feedback of particle i depends only on relative position and orientations of the particles visible to i according to $\mathcal{G}$.
(2) The feedback (16)-(17) renders the three sets $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$ asymptotically stable simultaneously. From this it follows that when the particles are in formation one can change β on the fly, in small increments if needed, to reconfigure the relative headings without the particles leaving the formation. Also if one changes α_p, for the same α_a the particles can be reconfigured on the parallel planes specified by α_a.
(3) The functions $w_1^i(\chi)$, $i \in n$, in Theorem V.4, provide an extra degree of freedom where they can be designed to provide arbitrary frame orientations around the desired $\mathbb{C}_1^i$ headings.
(4) Using the reduction theorem, Theorem III.2 in [15], the solution of 3D-CFP presented above can be directly extended to the corresponding version of the problem for a system of n fully actuated rigid bodies.

VI. Simulations

Consider 5 kinematic particles with Laplacian

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Case A. Consider the formation given by $a = [0 \ 0 \ 1]^\top$, $p_1 = p_2 = p_3 = p_4 = [1 \ 1 \ 1]^\top$, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, $r = 1$ and $v = 1$. The simulation result is given in Figure 3.

$$w_2^i(\chi) = \left(u_i(\chi) + \bar{k}\phi((L^i_{(3)}(x_p-\alpha_p))^\top \mathbb{C}_1^i)(L^i_{(3)}(x_p-\alpha_p))^\top \mathbb{C}_1^i\right)\frac{a\cdot \mathbb{C}_2^i}{r} + \tilde{k}(a\cdot \mathbb{C}_1^i + k_\phi \phi(L^i(x_a-\alpha_a))(L^i(x_a-\alpha_a)))(a\cdot \mathbb{C}_3^i)$$

$$w_3^i(\chi) = \left(u_i(\chi) + \bar{k}\phi((L^i_{(3)}(x_p-\alpha_p))^\top \mathbb{C}_1^i)(L^i_{(3)}(x_p-\alpha_p))^\top \mathbb{C}_1^i\right)\frac{a\cdot \mathbb{C}_3^i}{r} - \tilde{k}(a\cdot \mathbb{C}_1^i + k_\phi \phi(L^i(x_a-\alpha_a))(L^i(x_a-\alpha_a)))(a\cdot \mathbb{C}_2^i) \tag{17}$$

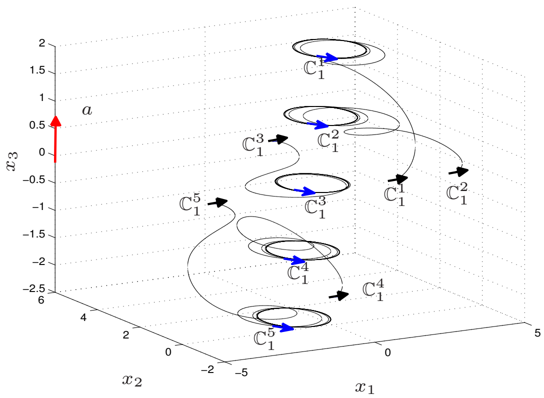

Fig. 3. Simulation - Case A

Case B. Consider the formation given by $a = [\frac{1}{\sqrt{(2)}}\ 0\ \frac{1}{\sqrt{(2)}}]^\top$, $p_1 = p_2 = p_3 = p_4 = [0\ 0\ 0]^\top$, i.e. it is required to make the particles follow a common circle, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{2\pi}{5}$, i.e. uniformly distributed around the circle, $r = 1$ and $v = 1$. The simulation result is given in Figure 4.

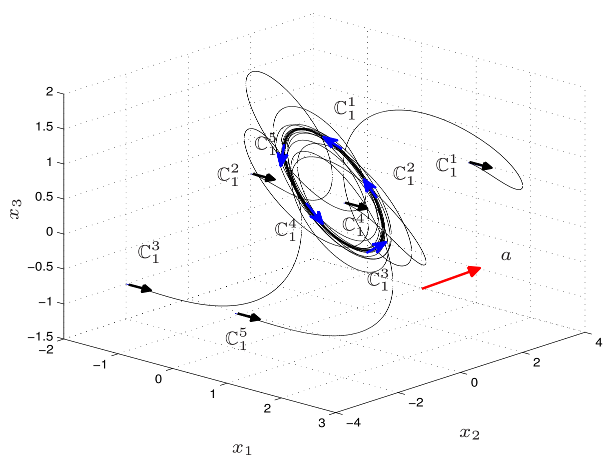

Fig. 4. Simulation - Case B

VII. Conclusions

The paper investigates the application of recent results in set stabilization and hierarchical control design in solving a circular formation control problem for a group of kinematic particles in $\mathbb{R}^3$. The problem is solved in three hierarchical steps, and the solution is obtain using a corollary to a recently developed reduction principle for asymptotic stability of closed sets. The result is proved for the case were the information graph is bidirectional. It is conjectured here that the same feedback solves the problem for directed graphs. The concepts and methods used here have been used before to solve for planar circular formations, and the results provided here suggest that those methods could be advantageous in addressing control more general problems of multi-agent systems.

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