

Extending Affine Control Policies to Hybrid Systems: Robust Control of a DC-DC Buck Converter

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Abstract—Robust model predictive control involves determining constrained sequences of inputs to dynamical systems that maximize a certain objective under uncertain dynamics or constraints. Performing an optimization over open loop control inputs in such circumstances is known to lead to conservative input choices, leading to poor performance or even a failure to find a feasible input sequence. However, this conservativeness can be reduced by including recourse in the problem formulation. For linear systems with bounded disturbances, optimization over control laws that are affine in the disturbance measurements has emerged as a valid trade-off between performance and computational expense. In this paper we extend the idea of affine recourse to *hybrid* systems, and introduce an approach that chooses integer decision variables over the control horizon in conjunction with continuous decision variables that are subject to recourse. Using the example of a buck power converter, we illustrate the efficacy of our approach in comparison to an implementation of affine control policies based on a linearized model.

I. INTRODUCTION

Robust Model Predictive Control (MPC) is concerned with controlling a system subject to external disturbances and/or parameter uncertainties, while ensuring that hard constraints on states and inputs are satisfied for any possible disturbance from some bounded set. Control decisions are planned on a receding horizon basis, in which at every time step a sequence of actions is chosen. It is well established that for Robust MPC the possibility of recourse on the decisions planned within the optimization horizon has to be included in order to avoid excessive (and unnecessary) conservatism or, even worse, feasibility problems [6]. Parameterizing such recourse using causal, affine functions of disturbances measured during the time horizon has emerged as a good compromise between performance and tractability [8], [6]. Those results were facilitated by advances in solving robust optimization problems where the minimizer was allowed to be a function of the uncertain problem data [2].

Such affine recourse functions have so far only been studied in detail for the control of linear systems. In this paper we extend the idea of using affine policies to *hybrid* systems, by proposing a robust controller that includes recourse.

In order to demonstrate the efficacy of our proposed controller on a switched dynamical system, we test it on the DC-DC buck power converter (also known as the *step-down* DC-DC converter). For this application, both linear as well as hybrid models exist, and both have been used in a predictive control context [10], [5], [4]. Using the linear

model, in Section IV we apply a standard robust controller based on the affine policy approach of [6] and highlight its limitation, namely that the linear model is a rather coarse approximation and does not capture the switched behaviour of the system. This leads to violations of hard constraints despite the use of a robust controller. This problem motivates the use of hybrid models.

We demonstrate a new controller that chooses integer decision variables taking into account that recourse will be taken on the continuous variables, thereby eliminating problems arising from the use of a standard averaged model. Results from the two controllers are compared, and significant performance improvement is reported for the new controller.

II. ROBUST HYBRID OPTIMAL CONTROL PROBLEM

In this section we pose a general robust hybrid optimal control problem. We consider Mixed Logical Dynamical (MLD) systems, which were introduced in [1]. MLD systems encompass a wide variety of systems, including switched systems such as the buck converter. The discrete-time dynamics at a time step k are described by

$$x_{k+1} = Ax_k + Bu_k + B_2\delta_k + B_3z_k + Gw_k. \quad (\text{II.1})$$

The variables $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $w_k \in \mathbb{R}^{n_w}$ are respectively the states, inputs, and disturbances. The variables $z_k \in \mathbb{R}^{n_z}$ and $\delta_k \in \{0, 1\}^{n_\delta}$ are *auxiliary* variables characterizing the hybrid behaviour of the system. They enter both in the dynamics, where they can express for instance switching between multiple *modes*, as well as in the constraints, described below, where they can be used in order to express logical conditions. In general, both continuous (z_k) as well as discrete variables (δ_k) are necessary to express such conditions.

We restrict our attention to systems which at each time step k are perturbed by a disturbance w_k bounded in a polytopic set \mathcal{W}_k , which is described by the matrix $S_k \in \mathbb{R}^{n_h \times n_w}$ and the vector $h_k \in \mathbb{R}^{n_h}$ as follows:

$$\mathcal{W}_k = \{w_k \in \mathbb{R}^{n_w} \mid S_k w_k \leq h_k\}. \quad (\text{II.2})$$

Hard state and input constraints are to be satisfied for any possible disturbance in \mathcal{W}_k at every time timestep k . They are defined as follows:

$$\left. \begin{array}{l} E_x x_k + E_u u_k + E_z z_k + E_\delta \delta_k \leq e \\ E_f x_N \leq e_f \end{array} \right\} \quad \forall w_k \in \mathcal{W}_k \quad (\text{II.3})$$

where we have introduced vectors $e \in \mathbb{R}^{n_e}$ and $e_f \in \mathbb{R}^{n_f}$, and the matrices E_x , E_u , E_z , E_δ and E_f of appropriate

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dimensions. Note that since $\delta \in \{0, 1\}^{n_\delta}$ is integer, the feasible region for the optimal control problem is in general non-convex.

We now express the equations (II.1) and (II.3) in stacked form in order to formulate a finite-horizon optimal control problem. We first introduce the variables

$$\begin{aligned} \mathbf{x} &\doteq [x_0^T, \dots, x_N^T]^T & \mathbf{u} &\doteq [u_0^T, \dots, u_{N-1}^T]^T \\ \mathbf{z} &\doteq [z_0^T, \dots, z_{N-1}^T]^T & \boldsymbol{\delta} &\doteq [\delta_0^T, \dots, \delta_{N-1}^T]^T, \\ \mathbf{w} &\doteq [w_0^T, \dots, w_{N-1}^T]^T \end{aligned} \quad (\text{II.4})$$

the matrices

$$\mathbf{A} \doteq \begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \quad \mathbf{B} \doteq \begin{pmatrix} 0 & \dots & 0 \\ B & 0 & \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \dots & AB & B \end{pmatrix}, \quad (\text{II.5})$$

and the matrices \mathbf{B}_2 , \mathbf{B}_3 , and \mathbf{G} , constructed exactly as for \mathbf{B} but with B replaced by B_2 , B_3 , and G respectively. The state-update equations (II.1) in stacked form then become

$$\mathbf{x} = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u} + \mathbf{B}_2\boldsymbol{\delta} + \mathbf{B}_3\mathbf{z} + \mathbf{G}\mathbf{w} \quad (\text{II.6})$$

For the constraints, we introduce the matrices \mathbf{E}_x , \mathbf{E}_u , and the vector \mathbf{e} defined by

$$\mathbf{E}_x = \begin{pmatrix} I_N \otimes E_x & 0 \\ 0 & E_f \end{pmatrix}, \quad \mathbf{E}_u = \begin{pmatrix} I_N \otimes E_u \\ 0 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \mathbf{1}_N \otimes e \\ e_f \end{pmatrix} \quad (\text{II.7})$$

where the operation \otimes indicates the Kronecker product, and E_f and e_f are used to define *terminal state constraints*, i.e. constraints on x_N . The matrices \mathbf{E}_δ and \mathbf{E}_z are defined analogously to \mathbf{E}_u .

We also define the stacked uncertainty set $\mathcal{W} \doteq \{\mathbf{w} \in \mathbb{R}^{Nn_w} \mid \mathbf{S}\mathbf{w} \leq \mathbf{h}\}$, where $\mathbf{S} \doteq \text{blkdiag}(S_0, S_1, \dots, S_{N-1})$ and $\mathbf{h} \doteq (h_0^T, h_1^T, \dots, h_{N-1}^T)^T$.

Constraint (II.3) can then be expressed in stacked form as

$$\mathbf{E}_x\mathbf{x} + \mathbf{E}_u\mathbf{u} + \mathbf{E}_\delta\boldsymbol{\delta} + \mathbf{E}_z\mathbf{z} \leq \mathbf{e} \quad \forall \mathbf{w} \in \mathcal{W}. \quad (\text{II.8})$$

In this paper, we consider the objective of minimizing the expected value of a quadratic state and input cost function over the horizon. This cost is defined relative to some desired set point \mathbf{x}_{ref} . Let $P \in \mathbb{R}^{n_x \times n_x}$, $P_N \in \mathbb{R}^{n_x \times n_x}$ and $Q \in \mathbb{R}^{n_u \times n_u}$ be positive semidefinite symmetric matrices defining the state, terminal state and input stage costs respectively. Let $\mathbf{P} \doteq \text{blkdiag}(P, \dots, P, P_N)$ and $\mathbf{Q} \doteq \text{blkdiag}(Q, \dots, Q)$ be stacked versions of these costs matrices. We can then write the resulting finite horizon robust optimal control problem:

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}, \mathbf{z}} \quad & \mathbb{E} [(\mathbf{x} - \mathbf{x}_{\text{ref}})^T \mathbf{P}(\mathbf{x} - \mathbf{x}_{\text{ref}}) + \mathbf{u}^T \mathbf{Q} \mathbf{u}] \\ \text{s.t.} \quad & \mathbf{x} = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u} + \mathbf{B}_2\boldsymbol{\delta} + \mathbf{B}_3\mathbf{z} + \mathbf{G}\mathbf{w} \\ & \mathbf{E}_x\mathbf{x} + \mathbf{E}_u\mathbf{u} + \mathbf{E}_\delta\boldsymbol{\delta} + \mathbf{E}_z\mathbf{z} \leq \mathbf{e}, & \forall \mathbf{w} \in \mathcal{W} \\ & \boldsymbol{\delta} \in \{0, 1\}^{N \cdot n_\delta}. \end{aligned} \quad (\text{ROCP})$$

We will now investigate different ways of choosing inputs \mathbf{u} in order to achieve good performance and constraint satisfaction for the specific example of a buck converter.

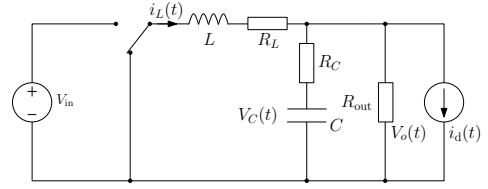


Fig. 1. Buck converter circuit

TABLE I
BUCK CONVERTER PARAMETER VALUES

Parameter	Value
Load	$R_{\text{out}} = 100 \Omega$
Parasitic Resistances	$R_L, R_C = 0.1 \Omega$
Capacitor	$C = 5 \mu F$
Inductor	$L = 10 \text{ mH}$
Supply voltage	$V_{\text{in}} = 100V$
Switching frequency	$f = 10 \text{ kHz}$
Output voltage reference	$V_o^{\text{ref}} = 33V$
Inductor current reference	$i_L^{\text{ref}} = 0.34A$
Disturbance	$ i_d \leq 0.5 \cdot i_L^{\text{ref}}$

III. MODELLING THE BUCK CONVERTER

Fig. 1 shows the topology of a buck converter (BC), which converts a DC supply voltage V_{in} to a lower DC output voltage $V_o(t)$. The purpose of this circuit, as considered in this paper, is to provide a stable output voltage under variations in the amount of power drawn at the output stage. This is achieved by operating the switch cyclically. We model the load on the output stage as a resistor plus a disturbance on the current drawn. This current represents the uncertain part of the control problem. The dynamic equations in continuous time for this system can then be readily obtained from first principles:

$$\dot{x}(t) = A^c x(t) + B^c \delta(t) + G^c w(t) \quad (\text{III.1})$$

where

$$x(t) \doteq (i_L(t) \quad V_C(t))^T, \quad w(t) = i_d(t), \quad (\text{III.2})$$

and with

$$\begin{aligned} A^c &= \begin{pmatrix} -\frac{1}{L} \cdot (R_L + \frac{R_C R_{\text{out}}}{R_C + R_{\text{out}}}) & -\frac{1}{L} \cdot (\frac{R_{\text{out}}}{R_C + R_{\text{out}}}) \\ \frac{1}{C} \cdot (\frac{R_{\text{out}}}{R_C + R_{\text{out}}}) & -\frac{1}{C} \cdot (\frac{1}{R_C + R_{\text{out}}}) \end{pmatrix} \\ B^c &= \begin{pmatrix} \frac{V_{\text{in}}}{L} \\ 0 \end{pmatrix} \quad G^c = \begin{pmatrix} \frac{1}{L} \cdot (\frac{R_C R_{\text{out}}}{R_C + R_{\text{out}}}) \\ -\frac{1}{C} \cdot (\frac{R_{\text{out}}}{R_C + R_{\text{out}}}) \end{pmatrix}. \end{aligned} \quad (\text{III.3})$$

The switch position $\delta(t) \in \{0, 1\} \forall t$ is the control input. The parameters for the system considered in this paper are given in Table I.

A. Averaged Model

By far, the simplest and commonest way of tackling the difficulty associated with the logic constraint $\delta(t) \in \{0, 1\} \forall t$ is to derive a so-called averaged model for the plant [3] and to apply a Pulse Width Modulation (PWM) scheme at the input. This is the standard way of controlling not only the buck converter but also other switched systems.

This approach is represented in Fig. 2 and can be described as follows. Instead of insisting that $\delta(t) \in \{0, 1\}$ at all times, one relaxes this constraint, divides the time horizon into

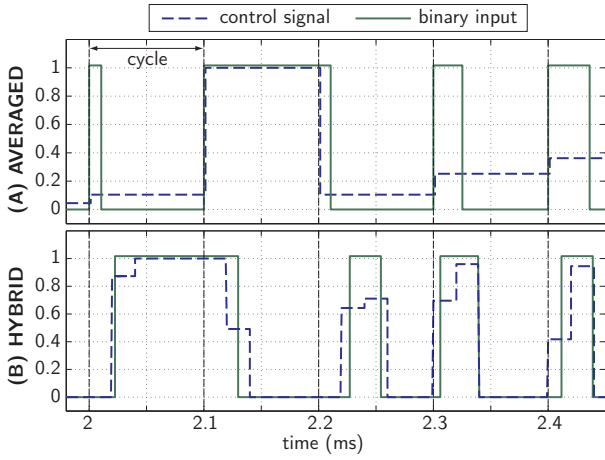


Fig. 2. The averaged model (A) assumes a constant control signal over one cycle (dashed curve), while the hybrid model (B) generates a better approximation of the on/off control signal (dashed curve). Both have to be transformed into a 0/1 signal (solid curve) by an appropriate block.

cycles (or switching periods) and allows the controller to select a δ which is constant over the cycle, and that can take any value in the range $[0, 1]$. This fractional value is called the *duty cycle of the PWM*, because this input signal is used to generate a switch on-off control pulse with a width corresponding to the duty cycle. The discrete-time dynamics can then be approximated linearly:

$$\begin{aligned} x_{k+1} &= Ax_k + B\delta_k + Gw_k \\ 0 &\leq \delta_k \leq 1 \end{aligned} \quad (\text{III.4})$$

where matrices A, B and G are obtained by discretizing (III.3). This linear approximation provides a simpler model that captures the average values of the states (in this case, voltages and currents within the circuit) over one cycle. However, the model only “sees” averages, so the true values of these signals in the real system at some time instant within the cycle may be quite different. This problem is well known [4] and results in two main drawbacks. Firstly, hard constraints on the states can be violated between two samples (we give an illustrative example of this problem in Section IV). Secondly, switching and resistive losses become hard to optimize for within the optimal control problem formulation because these depend on the ripples of the states within the cycle, which are unknown to the averaged model.

One approach to this problem that has received attention lately [10], [5], [4] is to refine this PWM scheme by dividing the cycle into samples and making an averaged approximation over these samples. A feasible pulse pattern, one that switches only once per cycle, is then recovered by introducing logic constraints on the set of admissible controls. The resulting optimal control problem is a mixed integer optimization problem instead of a convex one. We describe this procedure more in detail in the next section.

B. Hybrid Model

We obtain a more refined approximation of the hybrid nature of the system by applying an approach similar to the one described in [4]. Fig. 2 illustrates the procedure.

Each cycle of length T_{switch} is divided into M smaller time intervals of length T_{sample} called *samples*, so that $T_{\text{switch}} = M \cdot T_{\text{sample}}$. The switch turns on and off at most once within any cycle.

We discretize the continuous time system (III.1) with a sampling time of T_{sample} , and introduce new variables $\delta_k^- \in \{0, 1\}$ and $\delta_k^+ \in \{0, 1\}$ denoting the sampling times at which the switch is opened resp. closed. A $\delta_k^- = 1$ denotes that the switch is closed at sample k , $\delta_k^+ = 1$ means that it is opened. We furthermore introduce a new continuous input u_c and we restrict it to $-\delta^- \leq u_c \leq \delta^+$, i.e. the continuous signal can be different from 0 only during samples when a decision to switch on or off is made. The continuous variable u_c represents the exact time when the switch is operated between sample k and $k+1$. The resulting dynamics of the system are thus given by

$$\begin{aligned} \begin{pmatrix} i_L \\ V_C \\ s \end{pmatrix}^{k+1} &= \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_L \\ V_C \\ s \end{pmatrix}^k + \\ &+ \begin{pmatrix} B_1 & 0 & 0 \\ B_2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_c \\ \delta_+ \\ \delta_- \end{pmatrix}^k + \begin{pmatrix} G_1 \\ G_2 \\ 0 \end{pmatrix} w_k \end{aligned} \quad (\text{III.5})$$

where the new state s_k , $k \in \mathbb{N}_{[0, \dots, N]}$ keeps track of the position of the switch at each sampling time. The values $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2 \in \mathbb{R}$ are obtained by transforming the continuous time system (III.1) to a discrete time system with sampling time T_{sample} .

We ensure that the switch is turned on or off at most once per cycle by introducing constraints on the binary inputs δ^+ and δ^- . Collecting all the logic constraints, we have

$$\begin{aligned} \text{Binary inputs} & \quad \delta_k^+, \delta_k^- \in \{0, 1\} \\ \text{Binary state} & \quad 0 \leq s_k \leq 1 \\ \text{Limited input} & \quad 0 \leq \delta_k + u_{c,k} \leq 1 \\ \text{Switching time} & \quad -\delta_k^- \leq u_{c,k} \leq \delta_k^+ \\ \text{Switching constr.} & \quad \left. \begin{aligned} \sum_{i=0}^{M-1} \delta_{k+i}^+ &\leq 1 \\ \sum_{i=0}^{M-1} \delta_{k+i}^- &\leq 1 \end{aligned} \right\} \forall k \text{ s.t.} \\ & \quad \text{cycle} \\ & \quad \text{begins} \end{aligned} \quad (\text{III.6})$$

We further allow for polyhedral constraints on the states. It should be noted that while the hybrid model is sampled faster than a corresponding averaged one, the maximal switching frequency is the same as in the averaged model (once “on” and “off” per cycle).

C. Control objective

As usual we consider a quadratic objective on the state and the input. For the BC, we will evaluate the performance of the controllers in terms of how well (in expectation) they can track a reference output voltage. The output voltage is a linear function of the state x_k and the disturbances w_k .

$$V_o = \frac{R_C R_{\text{out}}}{R_C + R_{\text{out}}} i_L + \frac{R_{\text{out}}}{R_C + R_{\text{out}}} V_C + \frac{R_C R_{\text{out}}}{R_C + R_{\text{out}}} i_d \quad (\text{III.7})$$

Assuming the disturbance is zero mean and iid.

$$\begin{aligned} & \mathbb{E} \left[\begin{pmatrix} (x_k - x_k^{ref})^T & w_k^T \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \begin{pmatrix} (x_k - x_k^{ref}) \\ w_k \end{pmatrix} \right] \\ &= \mathbb{E} \left[\begin{aligned} & (x_k - x_k^{ref})^T P_{11} (x_k - x_k^{ref}) \\ & + 2 \cdot \underbrace{\mathbb{E}[(x_k - x_k^{ref})^T P_{12} w_k]}_{=0} + \underbrace{\mathbb{E}[w_k^T P_{22} w_k]}_{=const.} \end{aligned} \right] \end{aligned} \quad (\text{III.8})$$

Since we assume a causal system, the state x_k depends only on previous disturbances w_i , $i < k$ which are independent of the current disturbance w_k . Thus we can formulate our actual control objective $\min (V_o - V_o^{ref})^2$ in the BC example as a quadratic expected value objective on the state for a suitably chosen state cost $P := P_{11} \succeq 0$ and reference x_k^{ref} . No costs are imposed on the inputs ($Q = 0$) in the simulations.

The system dynamics (III.5), together with the switching constraints (III.6) and possible state constraints and the objective (III.8), can be combined to give a robust hybrid optimal control problem of the form (ROCP).

IV. AFFINE POLICIES BASED ON THE AVERAGED MODEL

In this section we highlight the limitations of the controller based on affine policies as it is applied to the linear model of the buck converter (the averaged model). The results obtained are unsatisfactory in that currents and voltages within the circuit do not satisfy the hard constraints.

We start by introducing the idea of affine recourse by parameterizing the sequence of controls \mathbf{u} by the disturbances observable up to the time when each input is applied:

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \quad i = 1, \dots, N-1. \quad (\text{IV.1})$$

This can be rewritten in stacked form as

$$\mathbf{u} = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ M_{1,0} & 0 & 0 \\ \vdots & \ddots & \vdots \\ M_{N-1,0} & \dots & M_{N-1,N-2} & 0 \end{pmatrix}}_{\doteq \mathbf{M}} \begin{pmatrix} w_0 \\ \vdots \\ w_N \end{pmatrix} + \underbrace{\begin{pmatrix} v_0 \\ \vdots \\ v_{N-1} \end{pmatrix}}_{\doteq \mathbf{v}}$$

The robust optimal control problem (ROCP) specialized to a linear system, and with the substitution above becomes

$$\begin{aligned} & \min_{\mathbf{M}, \mathbf{v}} \mathbb{E} \left[(\mathbf{x} - \mathbf{x}_{ref})^T \mathbf{P} (\mathbf{x} - \mathbf{x}_{ref}) + \mathbf{u}^T \mathbf{Q} \mathbf{u} \right] \\ & \text{s.t. } \mathbf{x} = \mathbf{A}x_0 + \mathbf{B}(\mathbf{M}\mathbf{w} + \mathbf{v}) + \mathbf{G}\mathbf{w} \\ & \quad \mathbf{E}_x \mathbf{x} + \mathbf{E}_u \mathbf{B}(\mathbf{M}\mathbf{w} + \mathbf{v}) \leq \mathbf{e} \quad \forall \mathbf{w} \in \mathcal{W} \end{aligned} \quad (\text{IV.2})$$

In Example 7 of [6], Goulart et al. borrow an argument based on duality to show that a solution to this semi-infinite problem (the requirement that the constraints are satisfied $\forall \mathbf{w} \in \mathcal{W}$ implies an infinite number of constraints) can be obtained by solving the following *finite* and *convex*

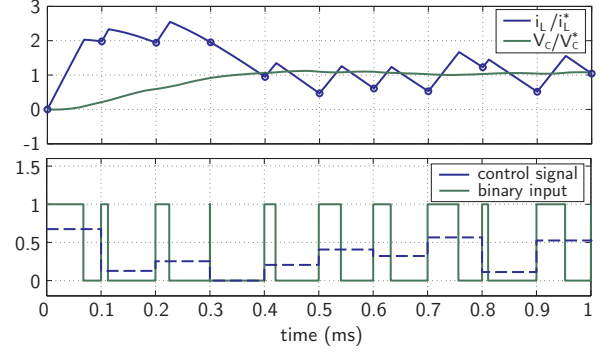


Fig. 3. Step response of the controller proposed in Section IV based on the averaged model of the BC. During the initial transient, hard constraints on the current ($i_L(k) \leq 2i_{L,ref}$) are violated. The controller also performs poorly in terms of tracking performance, as highlighted in Table II.

optimization problem

$$\begin{aligned} & \mathbf{E}_u \min_{\mathbf{M}, \mathbf{v}, \Lambda} \mathbb{E} \left[(\mathbf{x} - \mathbf{x}_{ref})^T \mathbf{P} (\mathbf{x} - \mathbf{x}_{ref}) + \mathbf{u}^T \mathbf{Q} \mathbf{u} \right] \\ & \text{s.t. } \mathbf{E}_x \mathbf{A}x_0 + \mathbf{E}_x \mathbf{B}\mathbf{v} + \mathbf{E}_u \mathbf{v} - \mathbf{e} \leq -\Lambda^T \mathbf{h} \\ & \quad \Lambda^T \mathbf{S} = (\mathbf{E}_x \mathbf{G} + \mathbf{E}_x \mathbf{B}\mathbf{M} + \mathbf{E}_u \mathbf{M}) \\ & \quad \Lambda \geq 0 \end{aligned}$$

(MPC_{avg})

in which all inequalities are component-wise and in which the new optimization variable $\Lambda \in \mathbb{R}^{(N \cdot n_h) \times (N \cdot n_e + n_f)}$ has been introduced.

We use solutions to (MPC_{avg}) in a receding horizon fashion in order to control the buck converter. These optimization problems are constructed using Yalmip [9] and solved using CPLEX 12.1 [7]. A realistic response of the converter is obtained using PLECS [11], and Fig. 3 reports the results of a simple step response (scenario 1). The disturbances in the output current that the controller has to counteract are relatively large - up to 50% of the nominal output current. For the simulations, we use uniform, uncorrelated disturbances. In Fig. 3 we can directly see the problems involved with the linear model of the plant. Despite it being controlled by a robust controller, the constraint $i_L \leq i_L^{\max} = 2i_{nom}$ is met in general only at the sampling times, marked “o” in the plot, while between the sampling times the current exceeds the bound. This motivates our investigation of robust controllers for hybrid models.

V. MIXING OPEN-LOOP DECISIONS WITH RECOURSE

We now return to the hybrid model of the buck converter and develop a robust controller that still incorporates recourse. Affine policies are an appropriate choice in a context in which state and input constraints are convex, because they can map disturbances \mathbf{w} (which are drawn from a convex polytopic set) to control inputs \mathbf{u} (which are also constrained in a convex polytopic set). But if integrality conditions are imposed on some of the variables (in our case, $\delta_k^+, \delta_k^- \in \{0, 1\}$), an affine mapping from disturbances to the controls will in general produce controls that do not satisfy these integrality constraints.

In order to overcome this, we divide the generic control inputs \mathbf{u} appearing in (II.8) into binary inputs $d_k \in \{0, 1\}^{n_d}$

and continuous inputs $u_k \in \mathbb{R}^{n_u}$. Recourse is then allowed only for the continuous part \mathbf{u} of the controls. We thus rewrite the state update equation such that the inputs are split into a continuous part u_k and binary part d_k

$$x_{k+1} = Ax_k + B_u u_k + B_d d_k + G w_k, \quad (\text{V.1})$$

in which A_k, x_k, G and w_k are as before and the new input and input matrices are

$$B_u := \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}, \quad B_d := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} \quad (\text{V.2})$$

$$u_k := u_c(k), \quad d_k := \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_k$$

The stacked matrices \mathbf{B}_u and \mathbf{B}_d are defined analogously to \mathbf{B} in (II.5). Using this notation, the following optimization problem, referred to as the MPC_{ol/cl} problem, can be defined:

$$\begin{aligned} \min_{\mathbf{v}, \mathbf{M}, \mathbf{d}, \boldsymbol{\delta}, \mathbf{z}} \quad & \mathbb{E} \left[(\mathbf{x} - \mathbf{x}_{\text{ref}})^T \mathbf{P} (\mathbf{x} - \mathbf{x}_{\text{ref}}) + (\mathbf{u}^T \mathbf{d}^T) \mathbf{Q} \begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} \right] \\ \text{s.t.} \quad & \mathbf{x} = \mathbf{A}x_0 + \mathbf{B}_u \mathbf{u} + \mathbf{B}_d \mathbf{d} + \mathbf{B}_2 \boldsymbol{\delta} + \mathbf{B}_3 \mathbf{z} + \mathbf{G} \mathbf{w} \\ & \mathbf{E}_x \mathbf{x} + \mathbf{E}_u \mathbf{u} + \mathbf{E}_d \mathbf{d} + \mathbf{E}_\delta \boldsymbol{\delta} + \mathbf{E}_z \mathbf{z} \leq \mathbf{e}, \quad \forall \mathbf{w} \in \mathcal{W} \\ & \mathbf{u} = \mathbf{M} \mathbf{w} + \mathbf{v} \\ & \boldsymbol{\delta} \in \{0, 1\}^{N \cdot n_\delta}, \quad \mathbf{d} \in \{0, 1\}^{N \cdot n_d}. \end{aligned} \quad (\text{V.3})$$

As we wish to parameterize the continuous inputs \mathbf{u} by affine functions, we chose $\mathbf{u} = \mathbf{M} \cdot \mathbf{w} + \mathbf{v}$. Feasible values for the binary variables $\mathbf{d}, \boldsymbol{\delta}$ have to be found without recourse, thus we leave them as optimization variables in the equations. Note that since we only use recourse on the continuous variables, this optimization problem does not necessarily have a solution even though a feasible solution to the finite horizon problem in terms of a general nonlinear control policy as found e.g. by dynamic programming, may exist. To simplify notation, we introduce

$$\begin{aligned} \hat{\mathbf{x}} &\doteq \mathbf{A}x_0 + \mathbf{B}_u \mathbf{v} + \mathbf{B}_d \mathbf{d} + \mathbf{B}_2 \boldsymbol{\delta} + \mathbf{B}_3 \mathbf{z} \\ \hat{\mathbf{e}} &\doteq \mathbf{E}_x \hat{\mathbf{x}} + \mathbf{E}_u \mathbf{v} + \mathbf{E}_d \mathbf{d} + \mathbf{E}_\delta \boldsymbol{\delta} + \mathbf{E}_z \mathbf{z} - \mathbf{e} \\ \hat{f} &\doteq (\hat{\mathbf{x}} - \mathbf{x}_{\text{ref}})^T \mathbf{P} (\hat{\mathbf{x}} - \mathbf{x}_{\text{ref}}) + (\mathbf{v}^T \mathbf{d}^T) \mathbf{Q} \begin{pmatrix} \mathbf{v} \\ \mathbf{d} \end{pmatrix} \end{aligned} \quad (\text{V.4})$$

Vector $\hat{\mathbf{x}}$ is the predicted state in case of zero disturbances, while $\hat{\mathbf{e}}$ equals the RHS of the system constraints and \hat{f} denotes the objective in this case. Note that they all depend on the nominal inputs \mathbf{v} , which are chosen open loop, but not on the disturbance dependent recourse $\mathbf{M} \cdot \mathbf{w}$. We can thus rewrite V.3 as

$$\begin{aligned} \min_{\mathbf{v}, \mathbf{M}, \mathbf{d}, \boldsymbol{\delta}, \mathbf{z}} \quad & \mathbb{E} \left[(\mathbf{x} - \mathbf{x}_{\text{ref}})^T \mathbf{P} (\mathbf{x} - \mathbf{x}_{\text{ref}}) + (\mathbf{u}^T \mathbf{d}^T) \mathbf{Q} \begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} \right] \\ \text{s.t.} \quad & \hat{\mathbf{e}} + (\mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M}) \mathbf{w} \leq 0, \quad \forall \mathbf{w} \in \mathcal{W} \\ & \hat{\mathbf{x}}, \hat{\mathbf{e}} \text{ as defined above} \\ & \boldsymbol{\delta} \in \{0, 1\}^{N \cdot n_\delta}, \quad \mathbf{d} \in \{0, 1\}^{N \cdot n_d}. \end{aligned} \quad (\text{V.5})$$

As in the case of the controller derived in Section IV, this problem must be rewritten in tractable form by replacing the condition $\forall \mathbf{w} \in \mathcal{W}$ with a finite one. This is achieved

starting with the following equivalences:

$$\begin{aligned} \hat{\mathbf{e}} + (\mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M}) \mathbf{w} &\leq 0, \quad \forall \mathbf{w} \in \mathcal{W} \\ \Leftrightarrow \hat{\mathbf{e}} &\leq - \max_{\mathbf{w} \in \mathcal{W}} (\mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M}) \mathbf{w} \end{aligned} \quad (\text{V.6})$$

in which the maximization is row-wise. Following the idea in [6], strong duality in linear programming gives the following equivalence:

$$\begin{aligned} \max_{\mathbf{w} \in \mathcal{W}} \quad & (\mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M}) \mathbf{w} \quad \text{s.t.} \quad \mathbf{S} \mathbf{w} \leq \mathbf{h} \\ = \min_{\Lambda \geq 0} \quad & \Lambda^T \mathbf{h} \quad \text{s.t.} \quad \Lambda^T \mathbf{S} = \mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M} \end{aligned} \quad (\text{V.7})$$

where Λ is a matrix of new optimization variables whose entries are the Lagrange multipliers associated with the row-wise maximizations; (V.6) is thus equivalent to

$$\begin{aligned} \hat{\mathbf{e}} &\leq -\Lambda^T \mathbf{h}, \\ \Lambda^T \mathbf{S} &= \mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M}, \\ \Lambda &\geq 0 \quad \text{element-wise.} \end{aligned} \quad (\text{V.8})$$

Having rewritten the constraint vector in finite form, we note that the objective function can be rewritten as follows, assuming zero mean disturbances $\mathbb{E}[w_k] = 0$:

$$\begin{aligned} & \mathbb{E} \left[(\mathbf{x} - \mathbf{x}_{\text{ref}})^T \mathbf{P} (\mathbf{x} - \mathbf{x}_{\text{ref}}) + (\mathbf{u}^T \mathbf{d}^T) \mathbf{Q} \begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} \right] \\ &= (\hat{\mathbf{x}} - \mathbf{x}_{\text{ref}})^T \mathbf{P} (\hat{\mathbf{x}} - \mathbf{x}_{\text{ref}}) + (\mathbf{v}^T \mathbf{d}^T) \mathbf{Q} \begin{pmatrix} \mathbf{v} \\ \mathbf{d} \end{pmatrix} \\ & \quad \dots + \mathbb{E}[\mathbf{w}^T \mathbf{D} \mathbf{w}] \\ &= \hat{f} + \text{trace}(\mathbf{D} \cdot \mathbf{C}_w) \end{aligned} \quad (\text{V.9})$$

where $\mathbf{C}_w \doteq \mathbb{E}[\mathbf{w} \mathbf{w}^T]$, $\hat{\mathbf{x}}$ as before and $\mathbf{D} \doteq (\mathbf{B}_u \mathbf{M} + \mathbf{G})^T \mathbf{P} (\mathbf{B}_u \mathbf{M} + \mathbf{G}) + \mathbf{M}^T \mathbf{Q}_v \mathbf{M}$ and $\mathbf{Q}_v \doteq \mathbf{Q}_{(1:N \cdot n_u, 1:N \cdot n_u)}$. Collecting constraints and objective, we can now pose the MPC_{ol/cl} optimization problem for the buck converter in tractable form:

$$\begin{aligned} \min_{\mathbf{v}, \mathbf{M}, \mathbf{d}, \boldsymbol{\delta}, \mathbf{z}, \Lambda} \quad & \hat{f} + \text{trace}(\mathbf{D} \cdot \mathbf{C}_w) \\ \text{s.t.} \quad & \hat{\mathbf{e}} \leq -\Lambda^T \mathbf{h}, \\ & \Lambda^T \mathbf{S} = \mathbf{E}_x \mathbf{B}_u \mathbf{M} + \mathbf{E}_x \mathbf{G} + \mathbf{E}_u \mathbf{M} \\ & \Lambda \geq 0 \quad \text{element-wise} \\ & \boldsymbol{\delta} \in \{0, 1\}^{N \cdot n_\delta}, \quad \mathbf{d} \in \{0, 1\}^{N \cdot n_d} \\ & \hat{f}, \hat{\mathbf{x}}, \hat{\mathbf{e}}, \mathbf{D} \text{ as defined above.} \end{aligned}$$

(MPC_{ol/cl})

VI. RESULTS

In this section we compare three controllers; MPC_{avg} based on the average model, MPC_{ol/cl} based on the hybrid model, and MPC_{ol} where recourse is excluded (i.e. $\mathbf{M} = 0$) from the hybrid controller, and only an open-loop sequence is computed at each time step. In order to make this comparison, we let the two controllers run under two different experiments.

The first experiment is a step response, in which the reference voltage $V_{o,\text{ref}}$ is raised to the desired reference of 33V. The box constraints are the same as those used to test MPC_{avg} in Section IV, i.e. box constraints on the inductor current $0 \leq i_L(k) \leq 2 \cdot i_{L,\text{ref}}$ but no constraint on the capacitor voltage, such that the comparison is fair. The results

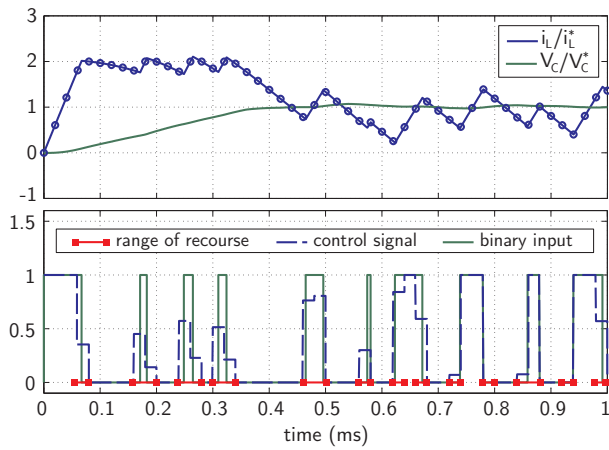


Fig. 4. Step response using MPC_{ol/cl}, based on the hybrid model of the converter. Hard constraints on the current $i_L(k) \leq 2i_{L,\text{ref}}$ are satisfied, and the controller also has better tracking performance (see Table II).

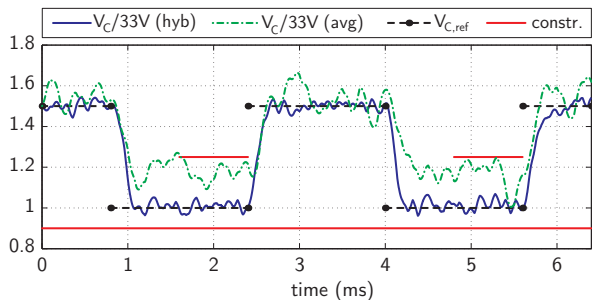


Fig. 5. Comparison of reference tracking performance of the two controllers studied in this paper. The new proposed controller has access to a more refined model of the system, and thus achieves better tracking performance. Even for such a simple experiment, without recourse the optimization problem involved in the computation of the robust controller is infeasible.

of this experiment are depicted in Fig. 4. We verify that MPC_{ol/cl} ensures hard constraints satisfaction on the current. The picture also shows the recourse margins that can be used by the controller to modify its decision on when to operate the switch, depending on the disturbances measured. Note that the controller requires measurements taken at a higher frequency than the robust controller based on the averaged model, but the frequency at which the switch is operated is the same. This is crucial to limit switching losses.

In the second test, the task is to track a periodic trajectory with some added constraints to test the transient performance. The results are shown in Fig. 5, in which the response of both MPC_{avg} as well as MPC_{ol/cl} are depicted. The open-loop hybrid MPC MPC_{ol}, on the other hand, has been tried but does not appear in the figure: the excessive conservativeness of this controller leads to infeasibility in the presence of reference changes and constraints. Note that this experiment is not designed to be unreasonably hard since even the simple linearized MPC_{avg} is able to handle it. Table II summarizes the performance results.

VII. CONCLUSIONS

In this paper we have extended the idea of affine recourse policies to hybrid systems. Recourse is crucial in robust

TABLE II
PERFORMANCE OF THE CONTROLLERS.

Controller	Steady state: RMS deviation of $V_o(t)$	Reference tracking: RMS deviation of $V_o(t)$, 2 nd test
MPC _{avg}	2.09V	15.2[V]
MPC _{ol}	0.89V	Infeasibility encountered
MPC _{ol/cl}	0.88V	9.74[V]

control to avoid excessive conservativeness and infeasibility issues, in the presence of state constraints. This paper introduced a controller where integer decision variables were chosen in conjunction with continuous variables incorporating recourse. The new controller was shown to have advantages both in constraint satisfaction and in performance, when compared to a standard robust control implementation reliant on a linearized system model.

PWM system models are common in power electronics and other applications where a duty cycle is used. The method described here, although only demonstrated on a simple system, shows some promise for extension to more complex systems, such as converter topologies with multiple inputs and outputs. A key attraction is that the robustification of the hybrid controller requires the introduction of additional continuous recourse variables M , but does not change the number of binary variables. This ensures that the computational complexity does not explode when the robustness requirement is incorporated into the controller.

Lastly, the use of robust hybrid MPC with recourse for power converters makes control objectives dependent on the hybrid dynamics (e.g. switching losses) possible, which may lead to further energy efficiency improvements.

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