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On the Computation of Flat Outputs for Nonlinear Control Systems

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***Abstract*— A simple procedure for the systematic computation of flat outputs for nonlinear control systems is presented. After symbolic linearization of the system equations, a so-called tangent flat output is determined. For this purpose, a new algorithm which is based on the calculation of nullspaces and generalized inverses is given. If some integrability conditions are satisfied, the flat output of the nonlinear system can be determined by integration. The procedure is illustrated in detail by computational examples.**

I. INTRODUCTION

Since the introduction of differential flatness about twenty years ago [1], [2], it has been shown to be one of the most powerful methods in the theory of nonlinear control systems by an impressive number of applications (see e. g. the references in [3]). Once a flat output of a system is known, the trajectory planning and feedback controller design are simplified significantly. However, deciding whether a system is flat and furthermore calculating a flat output are nontrivial tasks. Even though a considerable amount of work has been done in this direction, see e. g. [4], [5], [6], [7], [8], [9] for different approaches, it has been solved for some special system classes only, see [10], [11] for an overview. In general, this problem still remains open.

In this contribution, we consider control systems described by implicit systems

$$0 = F(x, \dot{x}), \quad x(t) \in \mathbb{R}^n \tag{1}$$

of $n - m$ equations. These systems may be the result of an input elimination an explicit differential equation

$$\dot{x} = f(x, u), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \tag{2}$$

with state x, its time derivative $\dot{x} = \frac{d}{dt}x$, and control input u. By the implicit function theorem (see [12, Chapter 7], for instance), this elimination is always possible provided that $\frac{\partial f}{\partial u}$ has full column rank m. We recall that the set of variables

$$y = (y^1, \ldots, y^m) = h(x, \dot{x}, \ddot{x}, \ldots)$$

is called a flat output of system (1) if its components are differentially independent (i.e., there is no differential equation in y) and if x can be calculated from y and a finite number of its derivatives [13].

An adapted version of the approach given in [14], [15] is proposed here for the determination of flat outputs of systems described by (1). The presented approach is also strongly related to [16], where the explicit description (2) is considered. The idea is roughly sketched as follows. At first, the differential equations are (symbolically) linearized. For the resulting tangent system a so-called tangent flat output is calculated. This is done by a new algorithm, which can easily be implemented in a computer algebra system. Subsequently, one tries to integrate this linear flat output to get a flat output of the nonlinear system. The main obstacle is that there is an infinite amount of possible flat outputs of the tangent system that need to be taken into account when checking integrability conditions. Nevertheless, the procedure proves to be successful in many applications. This is illustrated by three examples here.

The present paper is structured as follows. Section II deals with the linearized system and gives an algorithm for the determination of the tangent flat output. Subsequently, flatness conditions are discussed in terms of integrability conditions of the tangent flat output in section III. Finally, three examples are presented in Section IV.

II. TANGENT FLAT OUTPUTS

The linearization of (1) gives

$$0 = \mathrm{d}F = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x^i} \mathrm{d}x^i + \frac{\partial F}{\partial \dot{x}^i} \mathrm{d}\dot{x}^i \right), \tag{3}$$

with the exterior derivative d. We are interested in a flat output of the linearized system (3), i. e. a system of m independent one-forms ω^j, $j = 1, \ldots, m$, from which the coordinates $\mathrm{d}x$ can be calculated. Such a system of one-forms is called tangent flat output here. In the following, we will give an algorithm for calculating a tangent flat output ω^j, $j = 1, \ldots, m$, posessing the special structure

$$\omega^j = \sum_{i=1}^{n} \omega_i^j \mathrm{d}x^i, \tag{4}$$

with smooth functions ω_i^j depending on x and its derivatives. The procedure is based on an iteration which in every step consists of a coordinate transformation followed by an elimination of some variables. Thus, the dimension of the system can be reduced successively, which ends up with a trivial system (no equations) of flat outputs – provided that the tangent system is controllable. For convenience of notation, we introduce the $(n - m) \times n$ matrices

$$P_{0,[0]} = \frac{\partial F}{\partial x}(x, \dot{x}), \qquad P_{1,[0]} = \frac{\partial F}{\partial \dot{x}}(x, \dot{x}). \tag{5}$$

We use $v_{[0]}$ instead of $\mathrm{d}x$ as coordinates for the tangent system such that (3) reads

$$0 = P_{0,[0]} v_{[0]} + P_{1,[0]} \dot{v}_{[0]}, \tag{6}$$

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which is the starting point of the following algorithm. The subscript in brackets denotes the iteration counter. The following two steps have to be done repeatedly ($i = 0, 1, \ldots$):

1) The variables $v_{[i]}$ in
$$0 = P_{0,[i]} v_{[i]} + P_{1,[i]} \dot{v}_{[i]} \tag{7}$$
are replaced by $v_{[i+1]}$ and $w_{[i+1]}$ according to
$$v_{[i]} = P^{+}_{1,[i]} v_{[i+1]} + P^{\perp}_{1,[i]} w_{[i+1]}. \tag{8}$$
and its time derivative:
$$\dot{v}_{[i]} = P^{+}_{1,[i]} \dot{v}_{[i+1]} + \dot{P}^{+}_{1,[i]} v_{[i+1]} + \\ + \dot{P}^{\perp}_{1,[i]} w_{[i+1]} + P^{\perp}_{1,[i]} \dot{w}_{[i+1]}.$$
The matrices $P^{+}_{1,[i]}$ and $P^{\perp}_{1,[i]}$ are chosen such that $P_{1,[i]} P^{+}_{1,[i]} = I$ (identity) and $P_{1,[i]} P^{\perp}_{1,[i]} = 0$ hold[1] true. Thus, they clearly depend on x and its time derivatives (of order at most $i + 1$). In the new coordinates, Equation (7) reads as
$$0 = \dot{v}_{[i+1]} + A_{[i]} v_{[i+1]} + B_{[i]} w_{[i+1]} \tag{9}$$
with matrices
$$A_{[i]} = \left(P_{0,[i]} - \dot{P}_{1,[i]}\right) P^{+}_{1,[i]},$$
$$B_{[i]} = \left(P_{0,[i]} - \dot{P}_{1,[i]}\right) P^{\perp}_{1,[i]}.$$
Here, the identities $P_{1,[i]} \dot{P}^{+}_{1,[i]} = -\dot{P}_{1,[i]} P^{+}_{1,[i]}$ and $P_{1,[i]} \dot{P}^{\perp}_{1,[i]} = -\dot{P}_{1,[i]} P^{\perp}_{1,[i]}$ have been exploited. Note that the time derivatives $\dot{w}_{[i+1]}$ do not appear in Equation (9). If the matrix $B_{[i]}$ does not have full column rank, Remark 1 given below has to be taken into account.

2) The variables $w_{[i+1]}$ are eliminated from (9). For this purpose $B^{\perp}_{[i]}$ of maximal rank is calculated, such that $B^{\perp}_{[i]} B_{[i]} = 0$. This leads to
$$0 = B^{\perp}_{[i]} A_{[i]} v_{[i+1]} + B^{\perp}_{[i]} \dot{v}_{[i+1]} \\ = P_{0,[i+1]} v_{[i+1]} + P_{1,[i+1]} \dot{v}_{[i+1]}, \tag{10}$$
which is of the same form as (7) but of reduced dimension.

The Moore-Penrose inverse $B^{+}_{[i]}$ of $B_{[i]}$ can be employed together with (9) to determine $w_{[i+1]}$ from $v_{[i+1]}$ and its derivative $\dot{v}_{[i+1]}$:
$$w_{[i+1]} = -B^{+}_{[i]} \dot{v}_{[i+1]} - B^{+}_{[i]} A_{[i]} v_{[i+1]}. \tag{11}$$
The relationship between successive coordinates $v_{[i+1]}$ and $v_{[i]}$ can be calculated from (8) together with (11) as
$$v_{[i]} = P^{+}_{1,[i]} v_{[i+1]} - P^{\perp}_{1,[i]} \left(B^{+}_{[i]} \dot{v}_{[i+1]} + B^{+}_{[i]} A_{[i]} v_{[i+1]}\right) \\ = \underbrace{\left(P^{+}_{1,[i]} - P^{\perp}_{1,[i]} \left(B^{+}_{[i]} \tfrac{d}{dt} + B^{+}_{[i]} A_{[i]}\right)\right)}_{G_{[i]}(\frac{d}{dt})} v_{[i+1]}, \tag{12a}$$
$$v_{[i+1]} = P_{1,[i]} v_{[i]}. \tag{12b}$$

[1]The matrix $P^{\perp}_{1,[i]}$ should be of maximal rank, i. e., its columns span the right nullspace of the matrix $P_{1,[i]}$.

The latter equation has been obtained by a multiplication of (8) by $P_{1,[i]}$.

In every iteration step, the dimension of the system (7) is reduced. The procedure finishes when, in the k-th step, the matrix $B_{[k]}$ reaches full row rank. In this case, $v_{[k+1]}$ does not need to fulfill any equations like (10) since $B^{\perp}_{[k]}$ does not exist. Thus, $v_{[k+1]}$ can be freely chosen and hence it constitutes flat output. The original coordinates $v_{[0]}$ can be computed from $v_{[k+1]}$ and vice versa by means of $k+1$ repeated evaluations of (12):
$$v_{[0]} = G_{[0]}(\tfrac{d}{dt}) G_{[1]}(\tfrac{d}{dt}) \cdots G_{[k]}(\tfrac{d}{dt}) v_{[k+1]} \\ = G(\tfrac{d}{dt}) v_{[k+1]}, \tag{13a}$$
$$v_{[k+1]} = P_{1,[k]} P_{1,[k-1]} \cdots P_{1,[0]} v_{[0]} = Q v_{[0]}. \tag{13b}$$

Remark 1: If the matrix $B_{[i]} = (P_{0,[i]} - \dot{P}_{1,[i]}) P^{\perp}_{1,[i]}$ in (9) has linear dependent columns, the transformation (8) should be replaced by
$$v_{[i]} = P^{+}_{1,[i]} v_{[i+1]} + \tilde{P}^{\perp}_{1,[i]} w_{[i+1]} + Z_{[i]} z_{[i+1]}, \tag{14}$$
such that $\tilde{B}_{[i]} = (P_{0,[i]} - \dot{P}_{1,[i]}) \tilde{P}^{\perp}_{1,[i]}$ has full column rank. Additionally for $Z_{[i]}$, it should hold $(P_{0,[i]} - \dot{P}_{1,[i]}) Z_{[i]} = 0$ as well as $P_{1,[i]} Z_{[i]} = 0$. When applying the transformation (14), the coordinates $z_{[i+1]}$ do not appear in (7). The result is the same as in (9) where $B_{[i]}$ has to be replaced by $\tilde{B}_{[i]} = (P_{0,[i]} - \dot{P}_{1,[i]}) \tilde{P}^{\perp}_{1,[i]}$. Consequently, $z_{[i+1]}$ can be chosen freely and constitute a part of the flat output of the tangent system. The inverse transformation of (14) is given by
$$z_{[i+1]} = Z^{+}_{[i]} v_{[i]} \\ = Z^{+}_{[i]} P_{1,[i-1]} P_{1,[i-2]} \cdots P_{1,[0]} v_{[0]} \tag{15}$$
where the matrix $Z^{+}_{[i]}$ is uniquely determined by the conditions
$$Z^{+}_{[i]} Z_{[i]} = I, \qquad Z^{+}_{[i]} \tilde{P}^{\perp}_{1,[i]} = 0, \qquad Z^{+}_{[i]} P^{+}_{1,[i]} = 0.$$

Remark 2: If the matrix $B_{[i]}$ in (9) is actually a zero matrix, the coordinates $w_{[i+1]}$ disappear and (9) becomes a completely determined differential equation in the coordinates $v_{[i+1]}$, namely
$$0 = \dot{v}_{[i+1]} + A_{[i]} v_{[i+1]}.$$
This differential equation corresponds to a noncontrollable subsystem of (3). Hence, the nonlinear system (1) cannot be locally reachable and thus is not flat. Consequently, if we assume local reachability of (1), the case $B_{[i]} = 0$ cannot occur.

Remark 3: To reduce the computational effort, the matrix $P^{\perp}_{1,[i]}$ ($i = 1, 2, \ldots$) in the transformation (8) can be chosen to equal $B_{[i-1]}$ of the last iteration step provided that Remark 1 does not apply. This can be done since $P_{1,[i]} = B^{\perp}_{[i-1]}$ holds. In summary, the coordinates of a flat output (4) of the tangent system are given by (13b) and by (15). The latter equation applies for all i with $B_{[i]}$ of nonfull column rank. In both equations $v_{[0]}$ shall be replaced by $\mathrm{d}x$. Obviously, the one forms ω^j have only $\mathrm{d}x$ terms.

III. FLATNESS AND INTEGRABILITY CONDITIONS

So far, a tangent flat output (ω^j), $j = 1, \ldots, m$ of the linearized system has been calculated. Every other tangent flat output $(\tilde{\omega}^j)$, $j = 1, \ldots, m$ can be calculated by means of a unimodular $m \times m$ polynomial[2] operator matrix $U(\frac{d}{dt})$:

$$(\tilde{\omega}^j) = U(\tfrac{d}{dt})(\omega^j).$$

By unimodularity of $U(\frac{d}{dt})$, the inverse

$$(\omega^j) = \bar{U}(\tfrac{d}{dt})(\tilde{\omega}^j)$$

of this transformation exists and furthermore only ω and its derivatives are needed to calculate $\tilde{\omega}$ (no integrals have to be solved). It is well known (see [16], [6]) that a system is flat, if and only if there exist such $\tilde{\omega}$ that is integrable, i. e. there are m functions h^j, such that

$$\mathrm{d}h^j = \tilde{\omega}^j, \quad j = 1, \ldots, m.$$

This condition may lead to practically intractable calculations since the order of the differential operators in $U(\frac{d}{dt})$ is in general not bounded a priori. Thus, it is not useful as a necessary condition. Nevertheless, checking the condition for low order polynomials in $U(\frac{d}{dt})$ may be enough in some applications. The easiest case of zeroth order is covered by the classical Frobenius theorem, see e. g. [19], which reads as

$$\mathrm{d}\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^m = 0, \quad j = 1, \ldots, m, \tag{16}$$

where $\wedge$ is the exterior product. This is the dual version of the well-known formulation of Frobenius' theorem in terms of Lie brackets [12, Proposition 19.7]. If (16) is satisfied, there exists a matrix $U = (U_i^j)$ such that

$$\tilde{\omega}^j = \sum_{i=1}^{m} U_i^j \omega^i$$

is integrable. The flat output h is then given by integration of $\mathrm{d}h^j = \tilde{\omega}^j$. Since we can furthermore restrict the considerations to $\mathrm{d}F = 0$ (and time derivatives of this expression) the condition (16) can be further relaxed to

$$\mathrm{d}\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^m \wedge \mathrm{d}F^1 \cdots \wedge \mathrm{d}F^{n-m} = 0, \tag{17}$$

for $j = 1, \ldots, m$.

[2]To be more precise, we consider matrices of skew polynomials ([17], also called Ore polynomials [18]) $U(\frac{d}{dt}) \in \mathcal{K}[\frac{d}{dt}]^{m\times m}$ here. The field of coefficients $\mathcal{K}$ is given by the meromorphic functions in x and its derivatives and the multiplication rule is determined by $\frac{d}{dt}a = \dot{a} + a\frac{d}{dt}$ with $a \in \mathcal{K}$. Since the classical notions for polynomial matrices can be generalized to the skew polynomial case, we do not stress the non-commutativity in the remainder of the paper. In particular, a (skew) polynomial matrix $U(\frac{d}{dt})$ is called unimodular, if it possesses a (skew) polynomial inverse $\bar{U}(\frac{d}{dt})$ such that $\bar{U}(\frac{d}{dt})U(\frac{d}{dt}) = U(\frac{d}{dt})\bar{U}(\frac{d}{dt}) = I_m$, see [18].

IV. EXAMPLES

A. A Satellite Model

The movement of a satellite is assumed to be described by the Euler equations of rigid body motion, cf. also [20]:

$$\begin{aligned} \dot{x}^1 &= a_1 x^2 x^3 + u^1, \\ \dot{x}^2 &= a_2 x^1 x^3 + u^2 \\ \dot{x}^3 &= a_3 x^1 x^2. \end{aligned}$$

The coordinates x^i, $i = 1, 2, 3$, represent the angular velocities around the i-th principal axes of inertia. The inertia parameters a_1, a_2, and a_3 are assumed to be positive. The torques applied around the first and second principal axis by gas jet actuators are represented by u^1 and u^2. The last equation is free of inputs and thus can be taken as the implicit system description:

$$0 = F(x, \dot{x}) = \dot{x}^3 - a_3 x^1 x^2.$$

We are looking for a flat output of this system. For this purpose, the implicit desription is linearized

$$0 = \mathrm{d}F = -a_3 x^2 \mathrm{d}x^1 - a_3 x^1 \mathrm{d}x^2 + \mathrm{d}\dot{x}^3. \tag{18}$$

The matrices corresponding to (5) are

$$\begin{aligned} P_{0,[0]} &= \begin{pmatrix} -a_3 x^2 & -a_3 x^1 & 0 \end{pmatrix}, \\ P_{1,[0]} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In the first step, we calculate

$$P_{1,[0]}^{+} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad P_{1,[0]}^{\perp} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which gives

$$\begin{aligned} A_{[0]} &= (P_{0,[0]} - \dot{P}_{1,[0]})P_{1,[0]}^{+} = \begin{pmatrix} 0 \end{pmatrix}, \\ B_{[0]} &= (P_{0,[0]} - \dot{P}_{1,[0]})P_{1,[0]}^{\perp} = \begin{pmatrix} -a_3 x^2 & -a_3 x^1 \end{pmatrix}. \end{aligned}$$

Since the matrix $B_{[0]}$ is not of full column rank, the case described in Remark 1 has to be considered. We are looking for a matrix $Z_{[0]}$ such that $(P_{0,[0]} - \dot{P}_{1,[0]})Z_{[0]} = 0$ and $P_{1,[0]}Z_{[0]} = 0$. This holds true for

$$Z_{[0]} = \begin{pmatrix} x^1 \\ -x^2 \\ 0 \end{pmatrix}.$$

We choose

$$\tilde{P}_{1,[0]}^{\perp} = \begin{pmatrix} x^2 \\ x^1 \\ 0 \end{pmatrix}$$

and it follows

$$\tilde{B}_{[0]} = (P_{0,[0]} - \dot{P}_{1,[0]})\tilde{P}_{1,[0]}^{\perp} = \begin{pmatrix} -a_3(x^2)^2 - a_3(x^1)^2 \end{pmatrix}.$$

The algorithm finishes since $B_{[0]}$ has full row rank. The tangent flat output consists of $v_{[1]} = P_{1,[0]}v_{[0]}$ and $z_{[1]} = Z_{[0]}^{+}v_{[0]}$, where

$$Z_{[0]}^{+} = \begin{pmatrix} \frac{x^1}{(x^1)^2+(x^2)^2} & -\frac{x^2}{(x^1)^2+(x^2)^2} & 0 \end{pmatrix}.$$

Using the coordinates $\mathrm{d}x$ we have

$$\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ \frac{x^1}{(x^1)^2+(x^2)^2} & -\frac{x^2}{(x^1)^2+(x^2)^2} & 0\end{pmatrix}\begin{pmatrix}\mathrm{d}x^1\\ \mathrm{d}x^2\\ \mathrm{d}x^3\end{pmatrix}. \quad (19)$$

For the sake of completeness, the inverse transformation is also stated here:

$$\begin{aligned}\mathrm{d}x &= \left(P^+_{1,[0]} - \tilde{P}^\perp_{1,[0]}\left(B^+_{[0]}\tfrac{d}{dt} + B^+_{[0]}A_{[0]}\right)\right)v_{[1]} + Z_{[0]}z_{[1]}\\ &= \begin{pmatrix}\frac{x^2}{a_3((x^1)^2+(x^2)^2)}\frac{d}{dt} & x^1\\ \frac{x^1}{a_3((x^1)^2+(x^2)^2)}\frac{d}{dt} & -x^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},\end{aligned}$$

with $B^+_{[0]} = \frac{-1}{a_3((x^2)^2+(x^1)^2)}$. By putting this into (18) it is easily seen that ω^1 and ω^2 can be chosen freely. The singularity of the transformation at $0 = x^1 = x^2$ is due to the noncontrollability of the model at this point.

To get a flat output of the nonlinear system, one can try to integrate the system of one-forms (19)

$$\begin{aligned}\omega^1 &= \mathrm{d}x^3\\ \omega^2 &= \frac{x^1}{(x^1)^2+(x^2)^2}\mathrm{d}x^1 - \frac{x^2}{(x^1)^2+(x^2)^2}\mathrm{d}x^2.\end{aligned}$$

Since we have $\mathrm{d}\omega^1 = 0$ and $\mathrm{d}\omega^2 \wedge \omega^2 = 0$ the system is integrable. Indeed, by multiplying ω^2 by $((x^1)^2+(x^2)^2)$ we can solve the integral. A corresponding flat output is

$$\begin{aligned}y^1 &= x^3\\ y^2 &= \frac{1}{2}((x^1)^2-(x^2)^2).\end{aligned}$$

B. The Rolling Disc

The kinematic model of a disc rolling without slipping on a horizontal plane is described by

$$\begin{aligned}0 = F^1 &= \dot{x}^1\cos\psi + \dot{x}^2\sin\psi + a(\dot{\psi}\cos\theta + \dot{\phi})\\ 0 = F^2 &= -\dot{x}^1\sin\psi + \dot{x}^2\cos\psi + a\dot{\theta}\sin\theta\end{aligned} \quad (20)$$

where (x^1, x^2) are the coordinates of the projection of the center of the disc on the plane, ψ is the angle of rotation around the vertical axis, θ is the tilt angle of the axle, and ϕ is the rotation angle of the disc about the axle, cf. Fig. 1. We remark that, in contrast to the example studied for instance in [21], tilting of the disc is explicitly permitted here.

The rolling without slipping condition is represented by (20), see also [22], [23]. Since there are two restricting equations for this system described in the five coordinates $(x^1, x^2, \theta, \phi, \psi)$, a flat output will be of dimension three. It was shown in [24] that this system is flat but no flat output was given there.

At first the linearization of (20) is calculated:

$$\begin{aligned}\mathrm{d}F^1 &= \cos\psi\mathrm{d}\dot{x}^1 + \sin\psi\mathrm{d}\dot{x}^2 - a\dot{\psi}\sin\theta\mathrm{d}\theta +\\ &\quad + \left(\dot{x}^2\cos\psi - \dot{x}^1\sin\psi\right)\mathrm{d}\psi + a\mathrm{d}\dot{\phi} + a\cos\psi\mathrm{d}\dot{\psi},\\ \mathrm{d}F^2 &= -\sin\psi\mathrm{d}\dot{x}^1 + \cos\psi\mathrm{d}\dot{x}^2 + a\dot{\theta}\cos\theta\mathrm{d}\theta +\\ &\quad - \left(\dot{x}^1\cos\psi + \dot{x}^2\sin\psi\right)\mathrm{d}\psi + a\sin\theta\mathrm{d}\dot{\theta}.\end{aligned}$$

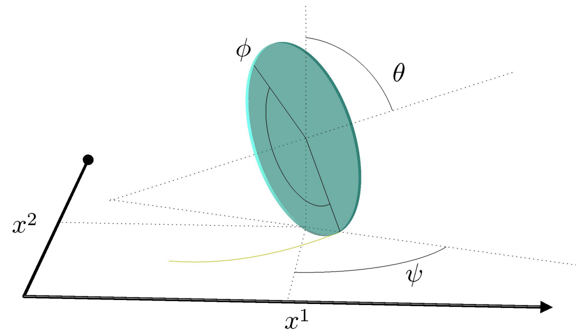

Fig. 1. Coordinates of the disc rolling on a plane.

The matrices according to (5) are

$$\begin{aligned}P_{0,[0]} &= \begin{pmatrix}0 & 0 & -a\dot{\psi}\sin\theta & 0 & \dot{x}^2\cos\psi - \dot{x}^1\sin\psi\\ 0 & 0 & a\dot{\theta}\cos\theta & 0 & \dot{x}^1\cos\psi + \dot{x}^2\sin\psi\end{pmatrix}\\ P_{1,[0]} &= \begin{pmatrix}\cos\psi & \sin\psi & 0 & a & a\cos\theta\\ -\sin\psi & \cos\psi & a\sin\theta & 0 & 0\end{pmatrix}.\end{aligned}$$

The choice

$$P^\perp_{1,[0]} = \begin{pmatrix}-a\cos\psi\cos\theta & -a\cos\psi & a\sin\psi\sin\theta\\ -a\sin\psi\cos\theta & -a\sin\psi & -a\cos\psi\sin\theta\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$P^+_{1,[0]} = \begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}$$

gives

$$\begin{aligned}A_{[0]} &= \left(P_{0,[0]} - \dot{P}_{1,[0]}\right)P^+_{1,[0]} = \begin{pmatrix}0 & -\dot{\psi}\\ \dot{\psi} & 0\end{pmatrix}\\ B_{[0]} &= \left(P_{0,[0]} - \dot{P}_{1,[0]}\right)P^\perp_{1,[0]} = \begin{pmatrix}0 & 0 & 0\\ a\dot{\phi} & -a\dot{\psi} & 0\end{pmatrix},\end{aligned}$$

Since $B_{[0]}$ is not of full column rank (cf. Remark 1), a matrix $Z_{[0]}$ is to be chosen such that $P_{1,[0]}Z_{[0]} = 0$ and $\left(P_{0,[0]} - \dot{P}_{1,[0]}\right)Z_{[0]} = 0$. Take for instance

$$\begin{aligned}Z_{[0]} &= P^\perp_{1,[0]}\begin{pmatrix}0 & 1\\ 0 & \frac{\dot{\phi}}{\dot{\psi}}\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}a\sin\psi\sin\theta & -a\left(\cos\psi\cos\theta + \frac{\dot{\phi}}{\dot{\psi}}\cos\psi\right)\\ -a\cos\psi\sin\theta & -a\left(\sin\psi\cos\theta + \frac{\dot{\phi}}{\dot{\psi}}\sin\psi\right)\\ 1 & 0\\ 0 & \frac{\dot{\phi}}{\dot{\psi}}\\ 0 & 1\end{pmatrix}.\end{aligned}$$

To complete the transformation, we single out a column of $P_{1,[0]}^{\perp}$ which is independent of the columns of $Z_{[0]}$. This column generates the matrix

$$\tilde{P}_{1,[0]}^{\perp} = P_{1,[0]}^{\perp} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a\cos\psi \\ -a\sin\psi \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,

$$\tilde{B}_{[0]} = \left(P_{0,[0]} - \dot{P}_{1,[0]}\right) \tilde{P}_{1,[0]}^{\perp} = \begin{pmatrix} 0 \\ -a\dot{\psi} \end{pmatrix}$$

is specified as well. In the sequel, the matrix $Z_{[0]}^{+}$ is uniquely determined as

$$Z_{[0]}^{+} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the second step, one calculates the left nullspace of $B_{[0]}$ spanned by

$$\tilde{B}_{[0]}^{\perp} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Hence, we have

$$P_{1,[1]} = B_{[0]}^{\perp} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and

$$P_{0,[1]} = B_{[0]}^{\perp} A_{[0]} = \begin{pmatrix} 0 & -\dot{\psi} \end{pmatrix}.$$

With these two matrices, the next iteration step starts. We choose

$$P_{1,[1]}^{\perp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$P_{1,[1]}^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and get

$$A_{[1]} = \left(P_{0,[1]} - \dot{P}_{1,[1]}\right) P_{1,[1]}^{+} = \begin{pmatrix} 0 \end{pmatrix}$$
$$B_{[1]} = \left(P_{0,[1]} - \dot{P}_{1,[1]}\right) P_{1,[1]}^{\perp} = \begin{pmatrix} -\dot{\psi} \end{pmatrix}.$$

The iteration finishes since $B_{[1]}$ has full (row) rank. The tangent flat output is given by

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} P_{1,[1]} P_{1,[0]} \\ Z_{[0]}^{+} \end{pmatrix} \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}\theta \\ \mathrm{d}\phi \\ \mathrm{d}\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 & a & a\cos\theta \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}\theta \\ \mathrm{d}\phi \\ \mathrm{d}\psi \end{pmatrix}.$$

The system of one-forms is integrable due to the fact that $\mathrm{d}\omega^2 = 0 = \mathrm{d}\omega^3$ and $\mathrm{d}\omega^1 \wedge \omega^3 = 0$. We need to calculate an 'integrating factor', which is facilitated by the ansatz

$$\tilde{\omega}^1 = \omega^1 + \mu(x^1, x^2, \theta, \phi, \psi)\omega^3.$$

After a few calculations, which are not given here, this results in

$$\tilde{\omega}^1 = \cos\psi \mathrm{d}x^1 + \sin\psi \mathrm{d}x^2 + a\mathrm{d}\phi + (x^2\cos\psi - x^1\sin\psi)\mathrm{d}\psi.$$

This one-form is integrable since $\mathrm{d}\tilde{\omega}^1 = 0$. A flat output y of the nonlinear system is obtained by integration

$$y^1 = x^1\cos\psi + x^2\sin\psi + a\phi,$$
$$y^2 = \theta,$$
$$y^3 = \psi.$$

For the sake of completeness, we give the inverse transformation here as well:

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos y^3 & -\sin y^3 \\ \sin y^3 & \cos y^3 \end{pmatrix} \begin{pmatrix} \frac{\ddot{y}^1 - \frac{\ddot{y}^3}{\dot{y}^3}\dot{y}^1}{(\dot{y}^3)^2} \\ \frac{\dot{y}^1}{\dot{y}^3} \end{pmatrix},$$

$$\theta = y^2,$$

$$\phi = \frac{1}{a}\left(\frac{\ddot{y}^1 - \frac{\ddot{y}^3}{\dot{y}^3}\dot{y}^1}{(\dot{y}^3)^2} + y^1\right),$$

$$\psi = y^3.$$

C. The Brockett Integrator

The so-called Brockett integrator given by

$$\dot{x}^1 = u^1$$
$$\dot{x}^2 = u^2$$
$$\dot{x}^3 = u^1x^2 - u^2x^1$$

is considered, see [25]. We are looking for flat output. After elminiating the inputs u^1, u^2, the system is described by the implicit representation

$$0 = F(x, \dot{x}) = \dot{x}^3 + x^1\dot{x}^2 - x^2\dot{x}^1.$$

The exterior derivative leads to the tangent system

$$\mathrm{d}F = x^2\mathrm{d}\dot{x}^1 - \dot{x}^2\mathrm{d}x^1 - x^1\dot{x}^2 + \dot{x}^1\mathrm{d}x^2 + \mathrm{d}\dot{x}^3.$$

The matrices according to (5) are

$$P_{1,[0]} = \begin{pmatrix} x^2 & -x^1 & 1 \end{pmatrix} \quad P_{0,[0]} = \begin{pmatrix} -\dot{x}^2 & \dot{x}^1 & 0 \end{pmatrix}.$$

We calculate a basis of the nullspace and a generalized inverse of $P_{1,[0]}$

$$P_{1,[0]}^{\perp} = \begin{pmatrix} -1 & x^1 \\ 0 & x^2 \\ x^2 & 0 \end{pmatrix},$$

$$P_{1,[0]}^{+} = \begin{pmatrix} \frac{x^2}{1+(x^1)^2+(x^2)^2} \\ \frac{-x^1}{1+(x^1)^2+(x^2)^2} \\ \frac{1}{1+(x^1)^2+(x^2)^2} \end{pmatrix}.$$

Subsequently, we get the matrices

$$A_{[0]} = \left(P_{0,[0]} - \dot{P}_{1,[0]}\right) P_{1,[0]}^{+} = \begin{pmatrix} \frac{-2(x^1\dot{x}^1 + x^2\dot{x}^2)}{1+(x^1)^2+(x^2)^2} \end{pmatrix}$$

as well as

$$B_{[0]} = \left(P_{0,[0]} - \dot{P}_{1,[0]}\right) P_{1,[i]}^{\perp} = \begin{pmatrix} 2\dot{x}^2 & 2(\dot{x}^1x^2 - x^1\dot{x}^2) \end{pmatrix}.$$

Since $B_{[0]}$ does not have full column rank, the calculation of a matrix $Z_{[0]}$, such that $\left(P_{0,[0]} - \dot{P}_{1,[0]}\right) Z_{[0]} = 0$ $P_{1,[0]} Z_{[0]} = 0$ is necessary. Here,

$$Z_{[0]} = \begin{pmatrix} \frac{\dot{x}^1}{\dot{x}^1 x^2 - x^1 \dot{x}^2} \\ \frac{\dot{x}^2}{\dot{x}^1 x^2 - x^1 \dot{x}^2} \\ -1 \end{pmatrix}$$

is chosen. By deletion of the first column of $P^{\perp}_{1,[0]}$, the matrix

$$\tilde{P}^{\perp}_{1,[0]} = \begin{pmatrix} x^1 \\ x^2 \\ 0 \end{pmatrix}$$

is constructed. The generalized left inverse of $Z_{[0]}$ which also fulfills the conditions according to Remark 1 is given by

$$Z^{+}_{[0]} = \begin{pmatrix} x^2 & x^1 & -(x^1)^2 - (x^2)^2 \end{pmatrix},$$

Since $B_{[0]}$ has already full row rank, the algorithm is finished. The tangent flat output is given by

$$\begin{aligned} \omega^1 &= P_{[0]} \mathrm{d}x = x^2 \mathrm{d}x^1 - x^1 \mathrm{d}x^2 + \mathrm{d}x^3 \\ \omega^2 &= Z^{+}_{[0]} \mathrm{d}x = x^2 \mathrm{d}x^1 - x^1 \mathrm{d}x^2 - \left((x^1)^2 + (x^2)^2\right) \mathrm{d}x^3. \end{aligned}$$

For this system of one-forms, an 'integrating factor' exists since $\mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^2 = 0$ and $\mathrm{d}\omega^2 \wedge \omega^1 \wedge \omega^2 = 0$. Indeed, the multiplication

$$\begin{aligned} \begin{pmatrix} \tilde{\omega}^1 \\ \tilde{\omega}^2 \end{pmatrix} &= \frac{1}{1 + (x^1)^2 + (x^2)^2} \begin{pmatrix} 1 & -1 \\ \frac{(x^1)^2 + (x^2)^2}{x^1 x^2} & \frac{1}{x^1 x^2} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{d}x^3 \\ \frac{1}{x^1} \mathrm{d}x^1 - \frac{1}{x^2} \mathrm{d}x^2 \end{pmatrix} \end{aligned}$$

results in two directly integrable one-forms. The corresponding flat output of the nonlinear system are computed by integration of $\mathrm{d}y^1 = \tilde{\omega}^1$ and $\mathrm{d}y^2 = \tilde{\omega}^2$:

$$y^1 = x^3, \qquad y^2 = \ln \frac{x^1}{x^2}.$$

The original coordinates x can be calculated from:

$$\begin{aligned} \left(x^1\right)^2 &= \frac{\dot{y}^2}{\dot{y}^1} \exp(y^1), \\ \left(x^2\right)^2 &= \frac{\dot{y}^2}{\dot{y}^1} \exp(-y^1), \\ x^3 &= y^2. \end{aligned}$$

An alternative flat output can be obtained from $\bar{\omega}^2 = \frac{x^1}{x^2} \tilde{\omega}^2$. Integration leads to $\bar{y}^2 = \frac{x^1}{x^2}$.

V. CONCLUSIONS

A simple procedure for calculating flat outputs of nonlinear control systems has been presented. The main contribution is a new algorithm, which is useful for the determination of so-called tangent flat outputs. It is based on simple linear algebraic operations, which can easily be implemented in a computer algebra system. Although the problem of giving a finitely checkable integrability condition for the tangent flat output still remains open, the presented approach has proven to be successful in numerous examples – three of them have been given here.

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REFERENCES

[1] P. Martin, *Contribution à l'étude des systèmes différentiellement plats*. Diss., École des Mines de Paris, 1992.

[2] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of nonlinear systems: introductory theory and examples," *Int. Journal of Control*, vol. 61, pp. 1327–1361, 1995.

[3] ——, "A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 44(5), pp. 922–937, 1999.

[4] K. Schlacher and M. Schöberl, "Construction of flat outputs by reduction and elimination," in *Proc. IFAC Symposium in Nonlinear Control Systems, NOLCOS, Pretoria*, vol. 7, 2007.

[5] M. Schöberl and K. Schlacher, "On parametrizations for a special class of nonlinear systems," in *Proc. 8th IFAC Symposium on Nonlinear Control Systems*, 2010.

[6] J. Lévine, "Analysis and control of nonlinear systems," in *Mathematical Engineering*. Springer, 2009.

[7] V. Chetverikov, "New flatness conditions for control systems," in *Proceedings of NOLCOS*, 2001, pp. 168–173.

[8] P. S. Pereira da Silva, "Flatness of nonlinear control systems: a cartan-kähler approach," in *Proc. Mathematical Theory of Networks and Systems*, 2000.

[9] F. Antritter and G. G. Verhoeven, "On symbolic computation of flat outputs for differentially flat systems," in *Proc. of 8th IFAC Symbposium on Nonlinear Control Systems*, 2010.

[10] P. Martin, R. Murray, and P. Rouchon, *Flat systems: open problems, infinite dimensional extension, symmetries and catalog*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin / Heidelberg, 2001, vol. 264, pp. 33–57.

[11] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Some open questions related to flat nonlinear systems," in *Open Problems in Mathematical Systems and Control Theory*, V. Blondel, E. Sontag, M. Vidyasagar, and J. Willems, Eds. Springer, 1999, pp. 99–103.

[12] J. M. Lee, *Introduction to smooth manifolds*. Springer, 2002.

[13] J. Rudolph and E. Delaleau, "Some examples and remarks on quasi-static feedback of generalized states," *Automatica*, vol. 34, no. 8, pp. 993–999, 1998.

[14] J. Lévine, *Flatness Necessary and Sufficinet Conditions: A Polynomial Matrices Approach*. Editorial Lagares, México, 2003, ch. 2, pp. 22–54.

[15] ——, "On necessary and sufficient conditions for differential flatness," *Applicable Algebra Eng. Commun. Comput.*, vol. 22, no. 1, pp. 47–90, 2011.

[16] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet, "A linear algebraic framework for dynamic feedback linearization," *IEEE Trans. on Automatic Control*, vol. 40, pp. 127–132, 1995.

[17] P. M. Cohn, *Algebra*. John Wiley & Sons, 1991, vol. 3.

[18] B. Beckermann, H. Cheng, and G. Labahn, "Fraction-free row reduction of matrices of Ore polynomials," *Journal of Symbolic Computation*, vol. 41, pp. 513–543, 2006.

[19] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. Griffiths, *Exterior Differential Systems*, ser. Mathematical Sciences Research Institute Publications. Springer, 1991.

[20] H. Nijmeijer and A. van der Schaft, *Nonlinear dynamical systems*. Springer, 1990.

[21] M. van Nieuwstadt, M. Rathinam, and R. Murray, "Differential flatness and absolute equivalence," in *Proc. 33rd IEEE Conf. Decision Control, Lake Buena Vista, FL*, 1994.

[22] J. Neimark and N. Fufaev, *Dynamics of Nonholonomic Systems*, ser. Translations of mathematical monographs. American Mathematical Society, 1972.

[23] A. M. Bloch, *Nonholonomic Mechanics and Control*, ser. Interdisciplinary applied mathematics. Springer, 2003, vol. 24.

[24] P. Martin and P. Rouchon, "Feedback linearization and driftless systems," *Math. Control Signal Systems*, vol. 7, pp. 235–254, 1994.

[25] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*. Birkhäuser, 1983.